Eulerian integrals of multivariable Gimel-function

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ABSTRACT
The main of the present paper is to evaluate a general Eulerian integral involving a product of two multivariable Gimel-function. By specializing the involved functions in this general Eulerian integral, we derived many Eulerian integrals.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, Eulerian integral.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\qquad\cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^r\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^2\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^2\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^2\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^3\alpha^{(k)}_{3j}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^3\alpha^{(k)}_{3ji_3}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^3\beta^{(k)}_{3ji3}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2),\Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r),\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)}=\sum_{j=1}^{m^{(k)}}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n^{(k)}}C_j^{(k)}\gamma_j^{(k)}-\tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}}D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)}+\sum_{j=n^{(k)}+1}^{p_i^{(k)}}C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right)+$$

$$-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}}A_{2ji_2}\alpha_{2ji_2}^{(k)}+\sum_{j=1}^{q_{i_2}}B_{2ji_2}\beta_{2ji_2}^{(k)}\right)-\cdots-\tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}}A_{rji_r}\alpha_{rji_r}^{(k)}+\sum_{j=1}^{q_{i_r}}B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,)\,,max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,)\,,min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty \text{ where } i=1,\cdots,r:$$

$$\alpha_i=\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] \text{ and } \beta_i=\max_{1\leqslant j\leqslant n^{(i)}}Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$$

Remark 1.
If $n_2=\cdots=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$ $A_{rj}=A_{rji_r}=B_{rji_r}=1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2=\cdots=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [6].

Remark 3.
If$A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [5].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [9,10].

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha^{(1)}_{(r-1)ji_{r-1}},\cdots,\alpha^{(r-1)}_{(r-1)ji_{r-1}};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}];\cdots;$$

$$[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta^{(1)}_{(r-1)ji_{r-1}},\cdots,\beta^{(r-1)}_{(r-1)ji_{r-1}};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}^{(1)}_j,\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}];\cdots;$$

$$[(\mathrm{d}^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{m^{(r)}+1,q^{(r)}_i}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

In this section, we give several results. In the next section, we shall use these formulae.

Lemma 1.

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.1}$$

provided $Re(\alpha),Re(\beta) > 0$

Lemma 2.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.2}$$

provided $Re(\alpha),Re(\beta) > 0, a \neq b$.

Lemma 3.

$$(ut+v)^\gamma = (au+v)^\gamma \sum_{m=0}^{\infty}\frac{(-\gamma)_m}{m!}\left[-\frac{(t-a)u}{au+v}\right]^m \tag{2.3}$$

provided $\left[-\frac{(t-a)u}{au+v}\right] < 1, t \in [a,b]$

Lemma 4. ([7], p.301 (2.2.6)).

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\mathrm{d}t = (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}{}_2F_1\left[\alpha,-\gamma;\alpha+\beta;-\frac{(b-a)u}{au+v}\right] \tag{2.4}$$

Provided $Re(\alpha), Re(\beta) > 0, a \neq b$, $\left|arg\left(\frac{bu+v}{au+v}\right)\right| < \pi$

The fractional derivative of a function $f(x)$ of complex order μ is defined by

Lemma 5. ([4], p.49 and [3] p. 181)

$${}_{\alpha}D_x^{\mu}[f(x)] = \left[\begin{array}{l} \frac{1}{\Gamma(-u)}\int_{\alpha}^{x}(x-y)^{-\mu-1}f(y)\mathrm{d}y; Re(\mu)<0, \alpha\in\mathbb{R} \\ . \\ \frac{d^m}{dx^m}{}_{\alpha}D_x^{\mu-m}[f(x)]; 0\leqslant Re(\mu)<m, m\in\mathbb{N}\end{array}\right. \tag{2.5}$$

The special cases of the fractional derivarive operator ${}_{\alpha}D_x^{\mu}$ when $\alpha = 0$ will be written as D_x^{μ}.
In this present paper we shall derive several Eulerian integrals involving the product of two generalized multivariable Gimel-function which will be represented as above. In your investigation, we shall adopt the contracted notations cited above.

3. Main integrals.

In this section, we shall see several general Eulerian integrals

Theorem 1.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta}\,\gimel(z_1(ut+v)^{a_1},\cdots,z_r(ut+v)^{a_r})$$

$$\gimel'(z_1(yt+z)^{b_1},\cdots,z_s(yt+z)^{b_s})\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(au+v)^{\gamma}(by+z)^{\delta}\sum_{l,m=0}^{\infty}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}$$

$$\left[\frac{(b-a)u}{au+v}\right]^l\left[-\frac{(b-a)y}{by+z}\right]^m \gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1(au+v)^{a_1} & \mathbb{A};(-\gamma;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r(au+v)^{a_r} & \mathbb{B};\mathbf{B},(-\gamma+l;a_1,\cdots,a_r;1):B\end{array}\right)$$

$$\gimel^{U';0,n'_r+1:V'}_{X';p'_{i_r}+1,q'_{i_r}+1,\tau'_{i_r}:R'_r:Y'}\left(\begin{array}{c|c} \mathrm{z}_1(by+z)^{b_1} & \mathbb{A}';(-\delta;b_1,\cdots,b_s;1),\mathbf{A}':A' \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_s(by+z)^{b_s} & \mathbb{B}';\mathbf{B}',(-\delta+m;b_1,\cdots,b_s;1):B'\end{array}\right) \tag{3.1}$$

provided

$a_i > 0; i = 1,\cdots,r, b_j > 0(j = 1,\cdots,s); min\{Re(\alpha), Re(\beta)\} > 0; b \neq a$;

$max\left\{\left|\left|\frac{(b-a)u}{au+v}\right|,\left|\frac{(b-a)y}{by+z}\right|\right\}\right. < 1$

$Re\,(\delta) + \sum_{i=1}^{s} b_i \min_{1\leqslant j\leqslant m'^{(i)}} Re\left[D_j'^{(i)}\left(\frac{d_j'^{(i)}}{\delta_j'^{(i)}}\right)\right] > -1$; $Re\,(\gamma) + \sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1$

$|arg\,(z_i(au+v)^{a_i})\,| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

$|arg\,(z_i(by+z)^{b_i})\,| < \frac{1}{2}A_i'^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof
To prove the theorem 1, expressing the two multivariable Gimel-functions in multiple integrals contour with the help of (1.1), interchanging the order of integrations in (3.1) which is justifiable due to the absolute convergence of the integrals involved in the process. Now collecting the power of $(au+v)$ and $(by+z)$, using the lemmae 3 and 2 respectvely and interpreting the resulting multiple integrals contour with the help of (1.1) about the gimel-function and r-variables and s-variables, we obtain the desired theorem 1.

Taking $a \rightarrow -a$ and $y \rightarrow -y$, we obtain

Theorem 2.

$$\int_{-a}^{b}(t+a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(z-ty)^{\delta}\,\gimel(z_1(ut+v)^{a_1},\cdots,z_r(ut+v)^{a_r})$$

$$\gimel'(z_1(z-yt)^{b_1},\cdots,z_s(z-yt)^{b_s})\mathrm{d}t=(b+a)^{\alpha+\beta-1}B(\alpha,\beta)(bu+v)^{\gamma}(ay+z)^{\delta}\sum_{l,m=0}^{\infty}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}$$

$$\left[-\frac{(b+a)u}{bu+v}\right]^{l}\left[-\frac{(b+a)y}{ay+z}\right]^{m}\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\mathrm{z}_1(bu+v)^{a_1} & \mathbb{A};(-\gamma;a_1,\cdots,a_r;1),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r(bu+v)^{a_r} & \mathbb{B};\mathbf{B},(-\gamma+l;a_1,\cdots,a_r;1):B\end{array}\right)$$

$$\gimel^{U';0,n'_r+1:V'}_{X';p'_{i_r}+1,q'_{i_r}+1,\tau'_{i_r}:R'_r:Y'}\left(\begin{array}{c|c}\mathrm{z}_1(ay+z)^{b_1} & \mathbb{A}';(-\delta;b_1,\cdots,b_s;1),\mathbf{A}':A'\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_s(ay+z)^{b_s} & \mathbb{B}';\mathbf{B}',(-\delta+m;b_1,\cdots,b_s;1):B'\end{array}\right) \tag{3.1}$$

provided

$a_i>0; i=1,\cdots,r, b_j>0(j=1,\cdots,s); min\{Re(\alpha),Re(\beta)\}>0; b+a\neq 0$;

$$max\left\{\left|\left|\frac{(b+a)u}{bu+v}\right|,\left|\frac{(b+a)y}{ay+z}\right|\right\}<1$$

$$Re\,(\delta)+\sum_{i=1}^{s}b_i\min_{1\leqslant j\leqslant m'^{(i)}}Re\left[D_j'^{(i)}\left(\frac{d_j'^{(i)}}{\delta_j'^{(i)}}\right)\right]>-1\,;\,Re\,(\gamma)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1$$

$|arg\,(z_i(bu+v)^{a_i})\,|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

$|arg\,(z_i(ay+z)^{b_i})\,|<\frac{1}{2}A_i'^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Now, consider the theorems 1 and 2 and replacing a_i,b_j by $-a_i,-b_j$ respectively, we get the following results.

Theorem 3.

$$\int_{a}^{b}(t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta}\,\gimel(z_1(ut+v)^{-a_1},\cdots,z_r(ut+v)^{-a_r})$$

$$\gimel'(z_1(yt+z)^{-b_1},\cdots,z_s(yt+z)^{-b_s})\mathrm{d}t=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)(au+v)^{\gamma}(by+z)^{\delta}\sum_{l,m=0}^{\infty}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}$$

$$\left[\frac{(b-a)u}{au+v}\right]^{l}\left[-\frac{(b-a)y}{by+z}\right]^{m}\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\mathrm{z}_1(au+v)^{-a_1} & \mathbb{A};(1+\gamma-l;a_1,\cdots,a_r;1),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r(au+v)^{-a_r} & \mathbb{B};\mathbf{B},(1+\gamma;a_1,\cdots,a_r;1):B\end{array}\right)$$

$$\beth^{U';0,n'_r+1:V'}_{X';p'_{i_r}+1,q'_{i_r}+1,\tau'_{i_r}:R'_r:Y'}\left(\begin{array}{c|c} \mathrm{z}_1(by+z)^{-b_1} & \mathbb{A}';(1+\delta-m;b_1,\cdots,b_s;1),\mathbf{A}':A' \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_s(by+z)^{-b_s} & \mathbb{B}';\mathbf{B}',(1-\delta;b_1,\cdots,b_s;1):B' \end{array}\right) \quad (3.1)$$

provided

$a_i > 0; i = 1,\cdots,r, b_j > 0(j = 1,\cdots,s); min\{Re(\alpha), Re(\beta)\} > 0; b \neq a$;

$max\left\{\left|\left|\frac{(b-a)u}{au+v}\right|,\left|\frac{(b-a)y}{by+z}\right|\right\}\right. < 1$

$Re\,(\delta) - \sum_{i=1}^{s} b_i \min_{1\leqslant j\leqslant m'^{(i)}} Re\left[D_j'^{(i)}\left(\frac{d_j'^{(i)}}{\delta_j'^{(i)}}\right)\right] > -1$; $Re\,(\gamma) - \sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1$

$|arg\left(z_i(au+v)^{-a_i}\right)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

$|arg\left(z_i(by+z)^{-b_i}\right)| < \frac{1}{2}A_i'^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 4.

$$\int_{-a}^{b}(t+a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(z-ty)^{\delta}\,\beth(z_1(ut+v)^{-a_1},\cdots,z_r(ut+v)^{-a_r})$$

$$\beth'(z_1(z-yt)^{-b_1},\cdots,z_s(z-yt)^{-b_s})\mathrm{d}t = (b+a)^{\alpha+\beta-1}B(\alpha,\beta)(bu+v)^{\gamma}(ay+z)^{\delta}\sum_{l,m=0}^{\infty}\frac{(\alpha)_l(\beta)_m}{(\alpha+\beta)_{l+m}l!m!}$$

$$\left[-\frac{(b+a)u}{bu+v}\right]^{l}\left[-\frac{(b+a)y}{ay+z}\right]^{m}\beth^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1(bu+v)^{-a_1} & \mathbb{A};(1+\gamma-l;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r(bu+v)^{-a_r} & \mathbb{B};\mathbf{B},(1+\gamma;a_1,\cdots,a_r;1):B \end{array}\right)$$

$$\beth^{U';0,n'_r+1:V'}_{X';p'_{i_r}+1,q'_{i_r}+1,\tau'_{i_r}:R'_r:Y'}\left(\begin{array}{c|c} \mathrm{z}_1(ay+z)^{-b_1} & \mathbb{A}';(1+\delta-m;b_1,\cdots,b_s;1),\mathbf{A}':A' \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_s(ay+z)^{-b_s} & \mathbb{B}';\mathbf{B}',(1-\delta;b_1,\cdots,b_s;1):B' \end{array}\right) \quad (3.1)$$

provided

$a_i > 0; i = 1,\cdots,r, b_j > 0(j = 1,\cdots,s); min\{Re(\alpha), Re(\beta)\} > 0; b+a \neq 0$;

$max\left\{\left|\left|\frac{(b+a)u}{bu+v}\right|,\left|\frac{(b+a)y}{ay+z}\right|\right\}\right. < 1$

$Re\,(\delta) - \sum_{i=1}^{s} b_i \min_{1\leqslant j\leqslant m'^{(i)}} Re\left[D_j'^{(i)}\left(\frac{d_j'^{(i)}}{\delta_j'^{(i)}}\right)\right] > -1$; $Re\,(\gamma) - \sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1$

$|arg\left(z_i(bu+v)^{-a_i}\right)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

$|arg\left(z_i(ay+z)^{-b_i}\right)| < \frac{1}{2}A_i'^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Remark 6.

If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then we can obtain the same Eulerian integrals formulae in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1])

Remark 7.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same Eulerian integrals formulae in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [6]).

Remark 8.
If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same Eulerian integrals formulae in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [5]).

Remark 9.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [9,10] and then we can obtain the same Eulerian integrals formulae, see Shrivastava [8] for more details.

4. Conclusion.

The Eulerian integral formulae involving in this paper are double fold generality in term of variables. By specializing the various parameters and variables involved, these formulae can suitably be applied to derive the corresponding results involving wide variety of useful functions (or product of several such functions) which can be expressed in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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