Fractional integral formulae involving the Srivastava-Daoust functions

and the multivariable Gimel-function

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ABSTRACT
In the present paperwe derive two fractional integral formulae involving the product of t wo generalized Srivasttava-Doust functions and a generalized multivariable Gimel-function. Since these functions includes a large number of special functions as its particular cases, therefore, the results established here will serve as key formulae.

Keywords : Generalized multivariable Gimel-function, Riemann-Liouville operator, Erdethe lyi-Kober operator, Srivastava-Doust function.

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1.Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r) = \gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c|} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right.$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, n_k); (k = 1, \cdots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, m_k); (k = 1, \cdots, r).$

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$ to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j} - \sum_{k=1}^{2} \beta_{2j}^{(k)} s_k\right) (j = 1, \cdots, m_2), \Gamma^{B_{3j}}\left(b_{3j} - \sum_{k=1}^{3} \beta_{3j}^{(k)} s_k\right) (j = 1, \cdots, m_3)$ $, \cdots, \Gamma^{B_{rj}}\left(b_{rj} - \sum_{i=1}^{r} \beta_{rj}^{(i)}\right) (j = 1, \cdots, m_r), \Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) +$$

$$\sum_{j=1}^{n_2} A_{2j} \alpha_{2j}^{(k)} + \sum_{j=1}^{m_2} B_{2j} \beta_{2j}^{(k)} - \tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) + \cdots +$$

$$\sum_{j=1}^{n_r} A_{rj} \alpha_{rj}^{(k)} + \sum_{j=1}^{m_r} B_{rj} \beta_{rj}^{(k)} - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$$\alpha_i = \min_{\substack{1 \leqslant k \leqslant m_i \\ 1 \leqslant j \leqslant m^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)} \frac{d_k^{(i)}}{\delta_k^{(i)}} \right) \text{ and } \beta_i = \max_{\substack{1 \leqslant k \leqslant n_i \\ 1 \leqslant j \leqslant n^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} A_{hj} \frac{a_{hj} - 1}{\alpha_{hj}^{h'}} + C_k^{(i)} \frac{c_k^{(i)} - 1}{\gamma_k^{(i)}} \right)$$

Remark 1.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [4]).

Remark 3.
If $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [3].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [8,9].
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In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

The Srivastava-Daoust function is defined by (see [6]):

$$F_{\bar{C}:D';\cdots;D^{(r)}}^{\bar{A}:B';\cdots;B^{(r)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdots\\ \cdots\\ \mathrm{z}_r\end{array}\middle|\begin{array}{l}[(\mathrm{a});\theta',\cdots,\theta^{(r)}]:[(b');\phi'];\cdots;[(b^{(r)});\phi^{(r)}]\\ [(\mathrm{c});\psi',\cdots,\psi^{(r)}]:[(d');\delta'];\cdots;[(d^{(r)});\delta^{(r)}]\end{array}\right)$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} A z_1^{m_1}\cdots z_r^{m_r} \tag{1.13}$$

where

$$A_1 = \frac{\prod_{j=1}^{\bar{A}}(a_j)_{m_1\theta_j'+\cdots+m_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b_j')_{m_1\phi_j'}\cdots\prod_{j=1}^{B^{(r)}}(b_j^{(r)})_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{m_1\psi_j'+\cdots+m_r\psi_j^{(r)}}\prod_{j=1}^{D'}(d_j')_{m_1\delta_j'}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{m_r\delta_j^{(r)}}} \tag{1.14}$$

The series given by (1.13) converges absolutely if

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)} > 0; i,j = 1,\cdots,r \tag{1.15}$$

For more details, see Srivastava and Daoust ([7], 1969).

2. Required results

The familiar fractional integral operator is defined and represented in the present paper as :

Lemma 1.

$${}_cI_x^{\mu}\{f(t)\} = \frac{1}{\Gamma(\mu)}\int_c^x (x-t)^{\mu-1}f(t)\mathrm{d}t\ , Re(\mu)>0 \tag{2.1}$$

the special case of the above operator (when $c=0$) is well known in the literature as Riemann-Liouville fractional integral operator and is written as $I_x^{\mu}\{f(t)\}$.

Also the fractional integral operator investigated by Erdelyi-Kober is defined and represented as Ross ([5],1975).

Lemma 2.

$$I_x^{\eta,\mu}\{f(t)\} = \frac{x^{-\eta-\mu+1}}{\Gamma(\mu)}\int_c^x (x-t)^{\mu-1}t^{\eta-1}f(t)\mathrm{d}t\ , Re(\mu)>0, \eta>0 \tag{2.2}$$

which is obviously a generalization of the Riemann-Liouville fractional integral operator.

Lemma 3.

The binomial expansion is given by

$$(x+\zeta)^{\lambda} = \zeta^{\lambda}\sum_{m=0}\binom{\lambda}{m}\left(\frac{x}{\zeta}\right)^m, \left|\frac{x}{\zeta}\right| < 1 \tag{1.14}$$

Lemma 4.

$${}_cI_x^{\mu}\{x^{\lambda}\} = \sum_{s=0}^{\infty}(-)^s\frac{\Gamma(\lambda+1)(x-c)^{s+\mu}x^{\lambda-s}}{\Gamma(\mu)\Gamma(\lambda-s+1)(s+\mu)s!}, Re(\lambda)>-1 \tag{1.15}$$

Lemma 5.

$$I_x^{\eta,\mu}\{x^{\lambda}\} = \frac{\Gamma(c+\lambda)}{\Gamma(c\lambda+\mu)}x^{\lambda}, Re(\lambda)>-c \tag{1.16}$$

2. Results

We also use the following short notations.

$$L'' = \beta^{\mu} a^{\sum_{i=1}^r m_i} b^{\sum_{i=1}^r m'_i} \prod_{i=1}^r \frac{Z_i^{m_i} Z_i'^{m'_i}}{m_i! m'_i! l! m!}$$

$$\gamma_1 = -\rho - \sum_{i=1}^r m_i u_i - \sum_{i=1}^r m'_i u'_i - l - m ; \alpha_1 = -\sigma - \sum_{i=1}^r m_i v_i + \sum_{i=1}^r m'_i v'_i ; \beta_1 = -\mu - \sum_{i=1}^r m_i w_i + \sum_{i=1}^r m'_i w'_i$$

$$Z_i'' = Z_i x^{U_i} \alpha^{V_i} \beta^{W_i}, i = 1, \cdots, r$$

$$A_1 = \frac{\prod_{j=1}^{\bar{A}} (a_j)_{m_1\theta_j^{(1)} + \cdots + m_r\theta_j^{(r)}} \prod_{j=1}^{B^{(1)}} (b_j^{(1)})_{m_1\phi_j^{(1)}} \cdots \prod_{j=1}^{B^{(r)}} (b_j^{(r)})_{m_r\phi_j^{(r)}}}{\prod_{j=1}^{\bar{C}} (c_j)_{m_1\psi_j^{(1)} + \cdots + m_r\psi_j^{(r)}} \prod_{j=1}^{D^{(1)}} (d_j^{(1)})_{m_1\delta_j^{(1)}} \cdots \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_{m_r\delta_j^{(r)}}}$$

$$A'_1 = \frac{\prod_{j=1}^{\bar{A}'} (a'_j)_{m'_1\theta_j'^{(1]} + \cdots + m'_r\theta_j'^{(r)}} \prod_{j=1}^{B'^{(1)}} (b_j'^{(1)})_{m'_1\phi_j'^{(1)}} \cdots \prod_{j=1}^{B'^{(r)}} (b_j'^{(r)})_{m'_r\phi_j'^{(r)}}}{\prod_{j=1}^{\bar{C}'} (c'_j)_{m'_1\psi'_j + \cdots + m'_r\psi_j'^{(r)}} \prod_{j=1}^{D'^{(1)}} (d_j'^{(1)})_{m'_1\delta_j'^{(1)}} \cdots \prod_{j=1}^{D'^{(r)}} (d_j'^{(r)})_{m'_r\delta_j'^{(r)}}}$$

$$F = F_{\bar{C}:D^{(1)};\cdots;D^{(r)}}^{\bar{A}:B^{(1)};\cdots;B^{(r)}} ; F' = F_{\bar{C}':D'^{(1)};\cdots;D'^{(r)}}^{\bar{A}':B'^{(1)};\cdots;B'^{(r)}}$$

We shall prove the following fractional integral formulae involving the product of two generalized Srivastava-Daoust functions and a generalized multivariable Gimel-function.

Theorem 1.

$$ {}_cI_x^v \Bigg\{ x^{\rho} (x+\alpha)^{\sigma} (x+\beta)^{\mu} F \begin{pmatrix} Z_1 x^{\mu_1} (x+\alpha)^{v_1} (x+\beta)^{w_1} \\ \cdots \\ \cdots \\ Z_r x^{\mu_r} (x+\alpha)^{v_r} (x+\beta)^{w_r} \end{pmatrix} F' \begin{pmatrix} Z_1 x^{\mu'_1} (x+\alpha)^{v'_1} (x+\beta)^{w'_1} \\ \cdots \\ \cdots \\ Z_r x^{\mu'_r} (x+\alpha)^{v'_r} (x+\beta)^{w'_r} \end{pmatrix} $$

$$ \beth \begin{pmatrix} Z_1 x^{U_1} (x+\alpha)^{V_1} (x+\beta)^{W_1} \\ \cdots \\ \cdots \\ Z_r x^{U_r} (x+\alpha)^{V_r} (x+\beta)^{W_r} \end{pmatrix} \Bigg\} = \sum_{\substack{m_1, \cdots, m_r, l, m, n = 0 \\ m'_1, \cdots, m'_r = 0}}^{\infty} \frac{A_1 A'_1 L'' (x-c)^{n+\delta}}{\alpha^{l+\alpha_1} \beta^{m+\beta_1} x^{n+\gamma_1} \Gamma(\delta) \Gamma(n+\delta) n!} \beth_{X;p_{i_r}+3, q_{i_r}+3, \tau_{i_r}:R_r:Y}^{U;m_r+1, n_r+2:V} $$

$$ \left(\begin{array}{c|c} Z_1 x^{U_1} \alpha^{V_1} \beta^{W_1} & \mathbb{A}; (\alpha_1; U_1, \cdots, U_r), (\beta_1; V_1, \cdots, V_r), \mathbf{A}, (\gamma_1; W_1, \cdots, W_r) : A \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r x^{U_r} \alpha^{V_r} \beta^{W_r} & \mathbb{B}; (\gamma_1 + n; U_1, \cdots, U_r), \mathbf{B}, (l + \alpha_1; V_1, \cdots, V_r), (\beta_1 + m; W_1, \cdots, W_r) : \end{array} \right) \quad \textbf{(2.1)} $$

Provided

$$ min\{u_i, v_i, w_i, u'_i, v'_i; w'_i, U_i, V_i, W_i\} > 0, i = 1, \cdots, r \;\; ; \left| \frac{x}{\alpha} \right| < \pi, \left| \frac{x}{\beta} \right| < \pi; $$

$$ Re\,(1 - \alpha_1 - l) + \sum_{i=1}^r V_i \min_{\substack{1 \leqslant k \leqslant m_i \\ 1 \leqslant j \leqslant m^{(i)}}} Re \left(\sum_{h=2}^r \sum_{h'=1}^h B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)} \frac{d_k^{(i)}}{\delta_k^{(i)}} \right) > 0 $$

$$Re\left(1-\beta_1-m\right)+\sum_{i=1}^{r}W_i \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\left(1-\gamma_1-n\right)+\sum_{i=1}^{r}U_i \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant \leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$\left|arg\left(x^{U_i}(x+\alpha)^{V_i}(x+\beta)^{W_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi \ , \quad \text{where } A_i^{(k)} \text{ is defined by (1.4)}$$

Theorem 2 .

$$I_x^{c,v}\left\{x^{\rho}(x+\alpha)^{\sigma}(x+\beta)^{\mu}F\begin{pmatrix}Z_1x^{\mu_1}(x+\alpha)^{v_1}(x+\beta)^{w_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ Z_rx^{\mu_r}(x+\alpha)^{v_r}(x+\beta)^{w_r}\end{pmatrix}F'\begin{pmatrix}Z'_1x^{\mu'_1}(x+\alpha)^{v'_1}(x+\beta)^{w'_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ Z'_rx^{\mu'_r}(x+\alpha)^{v'_r}(x+\beta)^{w'_r}\end{pmatrix}\right.$$

$$\left.\gimel\begin{pmatrix}Z_1x^{U_1}(x+\alpha)^{V_1}(x+\beta)^{W_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ Z_rx^{\mu_r}(x+\alpha)^{V_r}(x+\beta)^{W_r}\end{pmatrix}\right\}=\sum_{\substack{m_1,\cdots,m_r,l,m=0\\ m'_1,\cdots,m'_r=0}}^{\infty}\frac{A_1A'_1Z''}{\alpha^{\alpha_1+l}\beta^{\beta_1+m}x^{\gamma_1}}\gimel^{U;m_r+1,n_r+2:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c}Z_1x^{U_1}\alpha^{V_1}\beta^{W_1} & \mathbb{A};\ (\alpha_1;U_1,\cdots,U_r),(\beta_1;V_1,\cdots,V_r),\mathbf{A},(1-c+\gamma_1;W_1,\cdots,W_r):A\\ \cdot & \cdot\\ \cdot & \cdot\\ Z_rx^{U_r}\alpha^{V_r}\beta^{W_r} & \mathbb{B};\ (1\text{-c-}\delta+\gamma_1+n;U_1,\cdots,U_r),\mathbf{B},(l+\alpha_1;V_1,\cdots,V_r),(\beta_1+m;W_1,\cdots,W_r):\end{array}\right) \quad \textbf{(2.2)}$$

Provided that

$$Re(v)>0; min\{u_i,v_i,w_i,u'_i,v'_i;w'_i,U_i,V_i,W_i\}>0, i=1,\cdots,r \ ;\left|\frac{x}{\alpha}\right|<\pi,\left|\frac{x}{\beta}\right|<\pi;$$

$$Re\left(1-\alpha_1-l\right)+\sum_{i=1}^{r}V_i \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\left(1-\beta_1-m\right)+\sum_{i=1}^{r}W_i \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\left(1-\gamma_1\right)+\sum_{i=1}^{r}U_i \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$\left|arg\left(x^{U_i}(x+\alpha)^{V_i}(x+\beta)^{W_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi \ , \quad \text{where } A_i^{(k)} \text{ is defined by (1.4)}$$

Proof

To establish (2.1), we first express the Srivastava-Daoust functions occurring on the left-hand side in series form given by (1.13) and replace the generalized multivariable Gimel-function by its Mellin-Barnes integral contour (1.1), collecting the power of $x,(x+\alpha)$ and $(x+\beta)$ and applying the binomial expansion several times with the help of lemma 3. Further, making use of the lemma 4 and interpreting the resulting Mellin-Barnes multiple integrals contour as the generalized multivariable Gimel-function, we obtain the result (2.1).

Following the procedure (2.1) and using the lemma 5 instead of lemma4, we obtain the result (2.2) .

Remark 6.
If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then we can obtain the same fractional integrals in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1])

Remark 7.
If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2= =\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then we can obtain the same fractional integrals in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [4]).

Remark 8.
If$A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2 = \cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then we can obtain the same fractional integrals in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [3]).

Remark 9.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [8,9] and then we can obtain the same fractional integrals.

3. Conclusion.

The importance of our fractional formulae lies in their manifold generality. Firstly, in view of the generality of the generalized Srivastava-Daoust function and making every use by Srivastava and Daout [6,7], our formulae can be reduced to a large simpler special functions. Secondly, by specializing the various parameters and variables involved in the generalized multivariable Gimel-function, we get a several fractional integral formulae involving in remarkably wide variety of useful function (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

REFERENCES.

[1] F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

[2] B.L.J. Braaksma, Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[3] Y.N. Prasad, Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , 231-237.

[4] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 1-12.

[5] B. Ross, Fractional calculus and its applications, Lecture notes in maths, New York, Springer Verlaq 45(1975).

[6] H.M. Srivastava, and M.C. Daoust, Certain generalized Newman expansions associated with Kampe de Feriet function, Nedel. Akad. Wetensch. Proc. Ser. A 72, Indiga math. 31 (1969), 449-457

[7] H.M. Srivastava, and M.C. Daoust, A note on the convergence of Kampe de Feriet double hypergeometric series, Math Nach 53 (1972), 151-159.

[8] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[9] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.