Survey of Impulsive Differential Equations with Continuous Delay

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Abstract:

In this study, we investigate first order impulsive differential equations with continuous time dependent delays and revisit some of the fundamental concepts in literature. Consequently, we analyse the change in the formulation of impulsive differential equation problems from studying piecewise continuously differentiable trajectories to working with piecewise absolute continuous trajectories and determine a suitable solution space for them.

Keywords: *Impulsive, differential equation, continuous delay, trajectories, impulse points*

I. INTRODUCTION

Impulsive differential equations arise often in studying dynamical systems or evolution processes. Those determinstic dynamical systems or phenomena whose evolution processes are characterised by sudden or abrupt changes (duration negligible or short compared to total time of evolution process) are said to under go impulses and are modelled by impulsive differential equations. On the other hand, a system whose present state is influenced by the past history is modelled by delay differential equations. The fussion of these two areas produces a more qualitative and versatile area called impulsive differential equations ([3], [4], [1], [5]).

Physical systems dynamics and lots of every day phenomena in medical and natural Sciences, Economics, Engineering, Space sciences, and control systems are modelled by impulsive delay differential equations ([29], [4], [1], [6], [8], [9]). Most dynamical systems which should actually be modelled by impulsive delay differential equations are assumed to be modelled by ordinary differential equations, delay differential equations or impulsive differential equations, but in reality, these have been simplified due to some technical difficulties in handling the problems ([1], [9]). Thus, any evolution process which exhibits both impulsive and delay traits/characteristics should usually be modelled by Impulsive Differential Equations with Delays.

This research work is on impulsive differential equations with continuous delays, an area of differential equations which is lately drawing some attention ([1], [2]). It is a coupling of two areas of differential equations - delayand impulsive differential equations. Illustrative instances are seen in a patient being treated for an aiment will require for proper assement, not only the situation also past history . The effect of treatment certainly depends on the previous history of drug intake and the present drug effect. If the treatment is effective, there should be a jump in the patient's state of health within a very short time (impulse). Again, consider a car which hits a solid object, the effect may not automatically stop the car from functioning properly, but after a while the car may develop a fault as a result of the past impact which can lead abrupt change in performance.

The usual practice in formulating an Impulsive Delay Differential Equations is to either get an Impulsive Differential Equation (IDE) and then add delays, or get a Delay Differential Equation (DDE) and add impulses. In the traditional case (non-impulse), the solution trajectories for delay equations are $C_1[a,b]$ (continuously differentiable functions on the closed and bounded interval $[a,b]$). The conventional impulsive differential equations without delays studied, maintain that the trajectories are at least piecewise continuously differentiable once ($PC_1[a,b]$) ([10], [4], [11]). The impulse moments or points where jumps occur can be finite or infinite in number, while delays (previous history) can occur at discrete points (constants), or continuous ([10], [4], [12], [9]). Any combination of these cases presents its own challenges that require unique discussion. Specifically, in this study,we consider systems of first order impulsive differential equations with continuous time dependent delays and fixed moments of impulses..

Since it may not be possible or even desirable to find an explicit solution for most given differential equations, the analysis of the basic qualitative properties of solutions of these equations has been of fundamental importance ([4], [5]). For any given differential equation, the question of existence and uniqueness of solutions, boundedness, dependence of solution on the initial value or function, and stability/asymptotic stability analysis usually pose some interesting challenges which often open up new horizon for further research. For any system modeled by such a differential equation, the ability to represent reasonably (in functional or tabular form) the measure of response to impact is synonymous to existence of solution, how future response depends on initial state is continuous dependence of solution on initial data, and whether small changes leads to small response, or little change now leads to small future changes are addressed by stability and asymptotic stability of solutions. Proper understanding of these concepts ensure that in applications, whichever system is involved, the problem can be well-managed. A proper understanding of these concepts is therefore crucial in applications, as lack of understanding of structural stability of systems led to a good number of system failures in the past ([30] - [33]).

On the strength of the above, our main focus will be to examine the nature of solution space or trajectories, formulate problems associated with some qualitative properties of systems of impulsive differential equations with continuous time dependent delays.

II. RESULTS

In the existing literature on impulsive delay differential equations (which are relatively few in number compared to those on delay or impulsive differential equations), most discussions are centred on discrete delays ([4], [13] -[20]). Those who have tried to discuss some of these qualitative properties for equations with impulses and continuous delays, mostly base their discussions on piecewise continuously differentiable spaces with illustrations using discrete delays. The solutions of impulsive differential equations with the general continuous delays are however, fundamentally different.

Both impulsive differential equations, and delay differential equations have received fairly good attention over the years, but however, the more general problems of Impulsive Delay Differential Equations is, relatively speaking, in its infancy ([4], [1], [21], [9]). That is, Impulsive Differential Equations with Delays or retarded Arguments is still a nearly virgin or untapped area of impulsive differential equations. On the other hand, compared to delay and impulsive differential equations, it is assumed that the corresponding theories of impulsive delay differential equations are less developed. This has been attributed to significant technical and theoretical difficulties associated with such problems ([29], [1]).

Impulsive delay differential equations usually model dynamic or evolution processes where jumps occur in the state variable $x(t)$ at fixed time $t = t_j,\ j \in Z$ and which also require some previous history usually referred to as delays ([29], [4]). Impulsive differential equations with delays are classified based on the nature of the delays. There are those with single delay; several but discrete delays, state dependent delays, and those with continuous delays. The ones with continuous delays are considered the more general type ([4], [2], [22])

Currently, in the literature, Impulsive differential equations are usually defined by a pair of equations - an ordinary differential equation to be satisfied during the continuous phase of the evolution and difference equations defining the change of state at the discrete impulsive points. This is the main formulation of early scholars such as Bainov, Simeonov, Lakshminkatham, Gopalsamy, Zhang, among others ([10], [11], [23]). Solutions are usually considered to be piecewise continuously differentiable functions with discontinuities occurring at the impulsive times ([4], [23]).

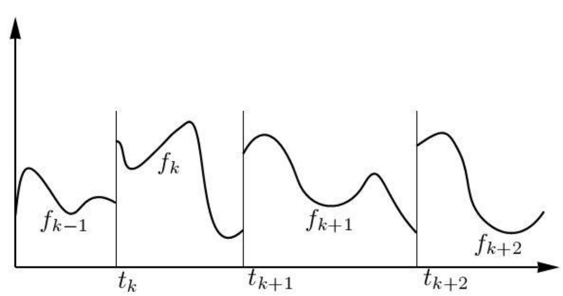

Figure 1: Impulse Dynamics

For figure 1, after the impulse point at t_k, the control is switched from f_{k-1} to f_k. Similarly, after t_{k+1} control is again switched to f_{k+1} and so on and so forth. In this case, the jumps are such that $f(t_k) = f(t-0)$ (i.e left continuous at t_k) and $f(t_k-0)>f(t_k+0)$ for all $k\in N$. It is possible to have $f(t_k-0)<(t_k+0)$ or a mixture of these cases at the impulse points.

On the other hand, delay differential equations are equations in which the right hand side depends on the value of the state variable x not only at time t but also at time $t-s$ previously, where $s..0$. Delay equations are solved by method of steps or successive approximations in a continuous domain ([10], [4], [24], [25]).

A delay differential equation can then be regarded as an ordinary differential equation whose solution trajectories depend on both the present state $x(t)$ and the state previously $x(t-s)$ for some $s>0$, where s is called the delay and may be a set of constants or continuous functions of time or even the state. In delay equation the past history has significant influence on the present state of the system and can be viewed as a projection of past event into the present [26]. View figure (2) on the next page for the dynamics of delay system with a single delay starting at time $t-\tau^{[j]}$ and progress sucessively with the smallest delay $\tau_{[j]}$

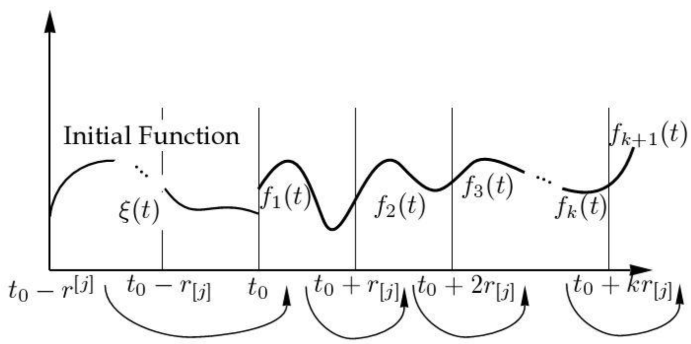

Figure 2: Delay Dynamics

In figure 2, the effect of the smallest delay $\tau_{[j]}$ on the forcing function. This can be viewed as a projection of past events into the future. The process starts at time $t-\tau^{[j]}$ and progresses sucessively with the smallest delay $\tau_{[j]}$ up to the last possible point in the interval $[a,b]$.

In what follows, we start by briefly discussing concepts in Impulsive Differential Equations without delays; and Delay Differential Equations without impulses to enhance understanding of the coupled problem of impulsive delay differential equations. Thereafter, the main concepts in Impulsive Delay Differential Equations (IDDE) are then considered. Like we mentioned earlier, the usual practice in formulating an IDDE is to either get an impulsive differential equation (IDE) and the add delay(s); or get a delay differential equation(DDE) and then add impulse(s) [4]. This seemingly easy formulation poses some challenges, some of which this work is set out to address.

For easier and compact presentation of the tasks ahead, we introduce some of the following definitions/notations which we shall use throughout this work: Let $T\subset R$ be a set of time points and let our processes take place in R^n described by $x:T\to R^n$ state functions and the processes may be influenced by past events defined by delay functions $h_i:T\to R^+$. The properties of these functions will be specified later. Let $\bar{g}:=(g_1,g_2,\dots,g_m)$, $\hat{g}:=(g,g,\dots,g)\in R^m$ and $\hat{f}\circ\bar{x}:=(f(x_1),f(x_2),\dots,f(x_m))$ it then follows that:

$$f(t,x(t),x(t-h_1(t)),x(t-h_2(t)),\dots,x(t-h_m(t)))=f(t,x(t),\hat{x}\circ(\hat{t}-\bar{h}\circ\hat{t})) \qquad (1.1)$$

In the course of this work, we shall assume $(a,b):=T\subset R$ is a non-empty open subset of R. Let $S:=\{t_k\}_{k=1}^{\infty}$ or $S:=\{t_k\}_{k=-\infty}^{\infty}$ be an increasing sequence of numbers (to be referred to as impulse times or points) with at most two condensation points. Let $D^*:=S\times R^n$ and $D:=T\times R^{(m+1)\times n}$. Also, let $f:D\to R^n$ and $f^*:D^*\to R^n$ be continuous functions fulfilling Lipschitz condition in $x\in D$ and $x\in D^*$ respectively at each fixed $t\in T$. Let $h_i:T\to R^+$ be continuous ascending delay function such that $h_i(t)\le t-a$, $\forall t\in T$, $\forall 1\le i\le m$. Then from the notation in equation (1.1), a system of first order impulsive differention equation with continuous delay is of the form:

$$\begin{cases} x'(t) = f(t, x(t), x(t-h_1(t)), \ldots, x(t-h_m(t))), \forall t \in T \setminus S \\ \Delta x(t_k) = x(t_k+0) - x(t_k-0) = f^*(t_k, x(t_k)), \forall t_k \in S. \end{cases} \qquad (1.2)$$

From the notation in (1.1), it can be written in a more compact form as:

$$\begin{cases} x'(t) = f(t, x(t), x \circ (\hat{t} - \bar{h} \circ \hat{t})), \forall t \in T \setminus S \\ \Delta x(t_k) = f^*(t_k, x(t_k)), \forall t_k \in S. \end{cases} \qquad (1.3)$$

Let $f : \Omega \to R^n, \Omega \subset T \times R^n$, be continuous and x at least once differentiable. Let $h_j \in C(t)$ be continuous delay functions, $1 \leq j \leq m$ and $\bar{h} \circ \hat{t} = (h_1(t), h_2(t), \ldots, h_m(t))$ be the continuous delay vector, then $r = \sup_{t \in (a,b)} \max_{1 \leq j \leq m} \{h_j(t)\}$ is called the delay constant (where the delays are discrete $r = \max_{1 \leq j \leq m} \{h_j\}$). Let equation (1.2) or equation (1.3) be given subject to the initial or history function:

$$x(t) = \varphi(t), t_0 - r \leq t \leq t_0. \qquad (1.4)$$

where $x(t_k) = x(t_k - 0)$ and $f^*(t_k, x(t_k))$ prescribes the jump at each impulse point $t_k \in S$, then equation (1.2) or (1.3) is called a first order impulsive delayed differential equation with continuous delays. Subject to equation (1.4) it is called an initial value or function problem. Let A, B be real *n* by *n* matrix functions with components in $C(a,b)$; let $\vec{g}$ be a vector with n components in $C(a,b)$, and $\tilde{A}_k$ be an n by n matrix function on *S*, then a system of linear impulsive differential equation with continuous delays is defined as:

$$\begin{cases} x'(t) = A(t) + B(t)\hat{x} \circ (\hat{t} - \bar{h}\hat{t}) + \vec{g}, \ \forall t \in T \setminus S \\ \Delta x(t_k) = \tilde{A}_k x(t_k), \forall t_k \in S. \end{cases} \qquad (1.5)$$

If $\vec{g}$ is identically zero, equation (1.5) is called a homogeneous equation and is given by:

$$\begin{cases} x'(t) = A(t)x(t) + B(t)\hat{x} \circ (\hat{t} - \bar{h}\hat{t}), \forall t \in T \setminus S \\ \Delta x(t_k) = \tilde{A}_k x(t_k), \forall t_k \in S. \end{cases} \qquad (1.6)$$

To formulate the problem of impulsive differential with delays or discuss the nature of its solutions, we must take into cognizance certain perculiarity of the model. We assume that for $t \in T \setminus S$, the solution $x(t)$ is determined by the delay differential equation $x'(t) = f(t, x(t), x(\hat{t} - \bar{h} \circ (\hat{t})))$ and for $t \in S$, a change by jump of the solution $x(t)$ occur so that $x(t_k^-) = x(t_k)$ and $x(t_k^+) = x(t_k) + f^*(t_k, x(t_k))$ ([10], [29], [24], [25], [26]).

It is necessary to state here that, existence of solutions and other qualitative properties of solutions for some special cases of equations (1.2) and (1.4) or equations (1.3) and (1.4) with discrete and some special cases of continuous delays have been discussed under a piecewise continuously differentiable space. We would like to state that using continuous functions as delay functions; we can find continuous delays for which some of the assumptions made on the *PC* solution space fail to hold. Thus the solution of impulsive differential equations with the general concepts of general delays are however, fundamentally different. We observe that the derivative of the solution becomes discontinuous even at non-impulse points and so ceases to be differentiable. Also, the forcing function may have no limit even when the delays are strictly ascending and continuously differentiable functions. We see that it is possible to define a continuous ascending delay function whose derivative exist everywhere on an interval I_k extendable to *T*, whereas the right hand side of equation (1.2) does not have limits at certain other points which are not even impulse points. The issues raised here are attested to by theillustrative figures presented. On the basis of this, it is necessary to discuss impulsive problems involving continuous delays under piecewise absolutely continuous space where measurable functions are continuously differentiable almost everywhere. In figure 3 through 5 below we demostrate geometrically some of the dynamics observed whose demonstration is one of the objectives of this work.

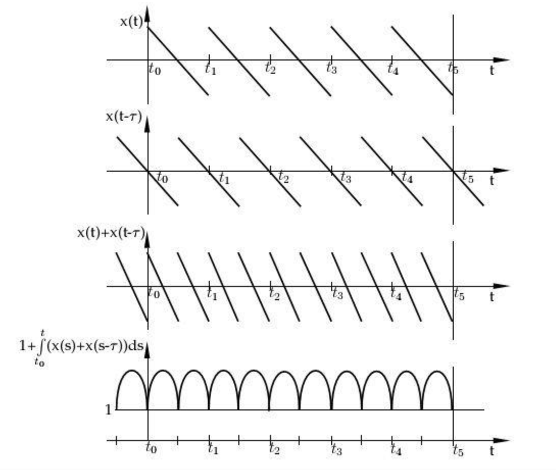

Figure 3: Delay Effect on Continuity of $f\left(t, x(s), x(s-\tau)\right)$

Figure 3 shows the interplay between $x(t)$ and $x(t-\tau)$, where $x \in PC[T,R]$ and $x(t-\tau)$ is not in $PC[T,R]$ category. What is more, the function $f(x(t), x(t-\tau)) = x(t) + x(t-\tau)$ is not in $PC[T,R]$ hence the integral function can only be $PC[T,R]$. The derivative is not a continuous function in the intervals $(t_j, t_{j+1}]\ \forall t_j \in S$ as shown at the bottom of figure 3.

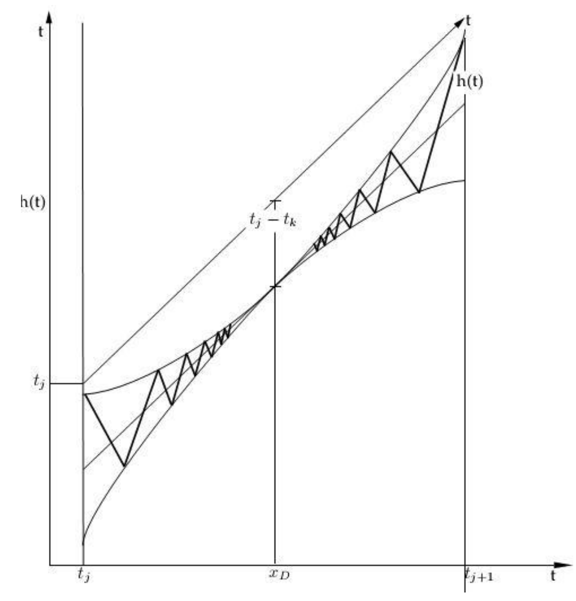

Figure 4: Continuous ascending delay dynamics

Figure 4 shows interplay between impulse points and continuous delays. The upper figure shows the position of the impulse points. The lower figure gives the construction of a continuous delay function $h(t)$. To show the reference, the time function *t* is shown and around the point of discontinuity $\rho = x_D$. The delay function*h(t)* is shown to wind around a constant between t_j and t_{j+1}.

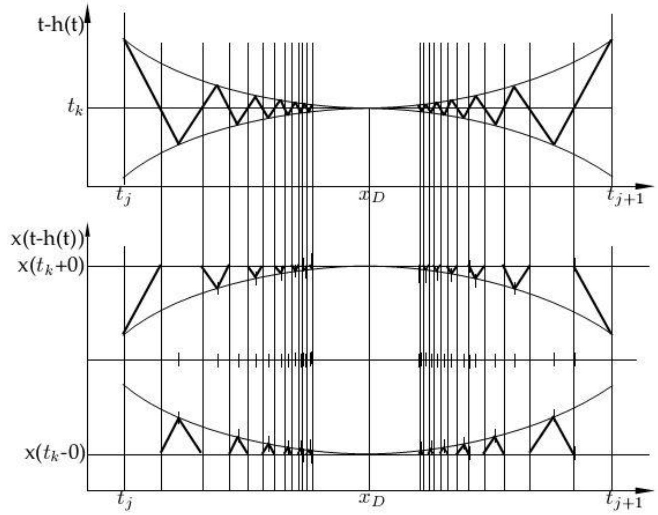

Figure 5: Dynamics of Interplay between Impulses and Continuous Delays

These two curves in figure 5 show that wherever the first curve passes through t_k, the second curve has a jump. These jumps accumulates at x_D and, hence, in x_D, there is no limits so no continuity, meaning that $f(t,x(t),x(t-h(t)))$ is not continuous and the equation $x'(t)=f(t,x(t),x(t-h(t)))$ is to be solved with measurable right side.

III. CONCLUSION

Given the impulsive differential equation with continuous delays defined in equation (1.2) subject to the initial function (1.4), the question of its qualitative properties becomes challenging. In most of the existing literature,these concepts are discussed on a piecewise continuously differentiable solution space with the exception of Benchora et al [34], where a semi-normed space is used in their discussion of stability/asymptotic stability of certain classes of impulsive delay differential equations. With continuous delays, the piecewise continuously differentiable solutions cannot be upheld since the trajectories may not be continuously differentiable even at non-impulse points. We found however,that for continuous delays, the space of piecewise absolutely continuous functions can serve successfully as solution space.

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