Some Properties of Fibonacci Numbers

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Abstract

Fibonacci numbers are well known for some of its interesting properties [1]. Golden ratio is one of the amazing property. Fibonacci numbers and Golden ratio have applications in physics, astrophysics, biology, chemistry and technology [2]. This article proves property of determinant of Fibonacci numbers , geometric consideration for Golden ratio and construction of Fibonacci subsequence from a Fibonacci sequence. The determinant of first n^2 $n >= 2$ of a Fibonacci numbers is zero. The golden ratio is shown to be sequence of lines converging to a line with slope as golden ratio. Method of constructing a subsequence from a Fibonacci sequence is presented. Examples presented in [2] is not exhaustive list of applications. One may find other applications in different domains of science.

Keywords

Fibonacci , Generalized Fibonacci sequence, Golden ratio, Set of lines, rational sequence, irrational number, sequence, subsequence, convergence

I. INTRODUCTION

The well known Fibonacci sequence is defined as

$f_0 = 0$, $f_1 = 1$ and

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ (i)

The range of Fibonacci sequence is

R (f) = { 0, 1, 1, 2, 3, 5, 8, 13, … }

The generalized Fibonacci sequence is defined as :

For some integers $k_1 \leq k_2$,

$f_0 = k_1$, $f_1 = k_2$ and

$f_n = f_{n-2} + f_{n-1}$ for $n \geq 2$ (ii)

For example, if $k_1 = 4$ and $k_2 = 7$, using (ii) the range of Fibonacci sequence is

R(f) = { 4, 7, 11, 18, 29, 47, … }.

The following notations will be used in this article.

II. NOTATIONS

(a) If $A = [a_{ij}]$, $i = 1 \dots m$ and $j = 1 \dots n$, then determinant of A denoted by det(A) is written as

Det (A) = det ($a_{11}, a_{12}, \dots, a_{1n}; a_{21}, a_{22}, \dots a_{2n}; \dots; a_{m1}, a_{m2}, \dots, a_{mn}$)

Here rows are separated by ; and columns are separated by ,.

(b) Let C_j denote j th column of the matrix $A = [a_{ij}]$, $i = 1 \dots m$ and $j = 1 \dots n$ and is written as

$C_j = [\ a_{1j}\ \ a_{2j}\ \ \dots a_{mj}\ ]$.

Then matrix A can be written as $A = (C_1\ C_2\ C_3\ \dots C_n)$

Example 1: For Fibonacci sequence { 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... }

The determinant of first 9 terms is

$$D1 = \begin{vmatrix} 0 & 1 & 1 \\ 2 & 3 & 5 \\ 8 & 13 & 21 \end{vmatrix}$$

can be represented by D1 = det(0, 1, 1 ; 2, 3, 5 ; 8, 13, 21). Note that rows are separated by ; and columns are separated by ,.

With this notation D2 =

det (3, 5, 8 ; 13 , 21 , 34 ; 55 , 89 , 144) means

$$D2 = \begin{vmatrix} 3 & 5 & 8 \\ 13 & 21 & 34 \\ 55 & 89 & 144 \end{vmatrix}$$

III. DETERMINANT

Proposition 1: Determinant of a real square matrix in which any row is a linear combination of other rows or columns is zero.

Theorem 1: A 3x3 determinant of any 9 consecutive terms of Fibonacci sequence is zero.

Proof: Let f_n, f_{n+1}, f_{n+2},f_{n+3},f_{n+4},f_{n+5},f_{n+6},f_{n+7},f_{n+8},f_{n+9} be 10 consecutive terms of Fibonacci sequence. For simplicity we define $a=f_n$, $b=f_{n+1}$, $c=f_{n+2}$, $p=f_{n+3}$, $q=f_{n+4}$, $r=f_{n+5}$, $x=f_{n+6}$, $y=f_{n+7}$, $z=f_{n+8}$

Thus a, b, c, p, q, r, x, y, z are 9 consecutive terms of Fibonacci sequence with

$c = a + b$, $p = b + c$, $q = c + p$, $r = p + q$, $x = q + r$,

$y = r + x$ and $z = x + y$. (1)

We can express the terms of Fibonacci sequence in terms of first two terms i.e. a and b.

Using (1),

$c = a + b,$

$p = b + c = b + (a + b) = a + 2 b$

$q = c + p = (a + b) + (a + 2 b) = 2 a + 3 b$

$r = p + q = (a + 2 b) + (2 a + 3 b) = 3 a + 5 b$

$x = q + r = (2 a + 3 b) +(3 a + 5 b) = 5 a + 8 b$

$y = r + x = (3 a + 5 b) + (5 a + 8 b) = 8 a + 13 b$

$z = x + y = (5 a + 8 b) +(8 a + 13 b) = 13 a + 21 b$ (i)

Now, Consider determinant D of order 3x3 defined by

D = det(a, b, c; p, q, r; x, y, z)

We shall show that D = 0.

Substituting the values of c, p, q, r , x, y, z from equations (i), we obtain

$$D = \begin{vmatrix} a & b & a+b \\ a+2b & 2a+3b & 3a+5b \\ 5a+8b & 8a+13b & 13a+21b \end{vmatrix}$$

Since third column is sum of first two columns by Proposition 1, D = 0.

Example 1: For Fibonacci sequence { 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... }

It can be verified that

If D1 = det (0, 1, 1; 2, 3, 5; 8, 13, 21) then D1 = 0.

If D2 = det (3, 5, 8; 13, 21, 34; 55, 89, 144) then D2 = 0.

Theorem 2: A 2x2 determinant of any four consecutive terms of Fibonacci sequence is constant upto sign.

Proof: Let a, b, c, d, e be any 5 consecutive terms of Fibonacci sequence so that

c = a+b, d=b+c = b+(a+b) = a+2b and

e = c + d = (a+b) + (a + 2b) = 2a + 3b

Let D be the determinant obtained from first 4 terms a, b, c, d.

D = det(a,b; c,d) = ad - bc = a(a + 2b) - b(a+b)

$= a^2 + 2ab - ab - b^2$

$= a^2 + ab - b^2$

Let $k = a^2 + ab - b^2$. Then D = k (i)

Let D1 be the determinant obtained from next 4 consecutive terms b, c, d, e.

D1 = det(b,c; d,e)

= be -cd = b(2a+3b) -(a+b) (a+2b)

$= 2ab + 3b^2 - (a^2 + 2ab + ab + 2b^2)$

$= b^2 - a^2 - ab = -(a^2 + ab - b^2)$

= - k (ii)

Thus from (i) and (ii) we conclude that determinant of any 4 consecutive terms of Fibonacci sequence is either + k or

- k where $k = a^2 + ab - b^2$.

This completes the proof.

The value of k can be determined from first 2 terms.

Example 2: If 0, 1, 1, 2, 3, 5, 8,13,21,34 are first 10 terms of Fibonacci sequence then

a = 0, b = 1 and $k = a^2 + ab - b^2 = -1$

D1 is determinant of first 4 terms and j =1 .

D1 = det(0,1; 1,2) = -1 = $(-1)^{j+1}$ k = k where j =1

D2 is determinant of next 4 terms starting from 2 nd term and j =2.

D2 = det(1,1; 2,3) = 1 = $(-1)^{j+1}$ k = - k where j =2

D6 is determinant of next 4 terms starting from 6 th term and j =6.

D6 = det(5,8; 13,21) = 1 = $(-1)^{j+1}$ k = - k where j = 6

D7 is determinant of next 4 terms starting from 7 th term and j = 7.

D7 = det(8,13; 21,34) = -1 = $(-1)^{j+7}$ k = k

This illustrates that determinant of any four consecutive terms of Fibonacci sequence is either +k or -k.

The following example illustrates that the result applied to any general Fibonacci sequence is true.

Example 3: If 3,7,10,17,27 ... is a general Fibonacci sequence then

a=3, b=7 and $k = a^2 + ab - b^2 = -19$

D1 is determinant of first 4 terms and j =1 .

Let D1 = de(3,7; 10,17)= -19 = $(-1)^{j+1}$ k = k where j = 1

D2 is determinant of 4 terms starting from 2 nd term and j =2 .

D2 = det(7,10; 17,27) = 19 = - k = $(-1)^{j+1}$ k where j = 2

Thus D1 = k = - 19 and D2 = - k = 19

Example 4: Let 3, 4, 7, 11, 18, ... be terms in Fibonacci sequence. Then with a = 3 and b = 4,

$k = a^2 + ab - b^2 = 5$

D1 = det(3, 4; 7, 11) = k = $(-1)^{j+1}$ k where j = 1

D2 = det(4, 7; 11, 18) = -k = $(-1)^{j+1}$ k where j = 2

IV. GOLDEN RATIO

Theorem 2: Each term of Fibonacci sequence can be expressed as a linear combination of first two terms and each term can be represented by a point in 2 dimensional Euclidean plane.

Proof: Let f_1, f_2, f_3, ... f_n, f_{n+1},f_{n+2}, f_{n+3}, f_{n+4}, ... be terms of Fibonacci sequence.

Let us denote $a f_1 + b f_2$ by ordered pair (a,b) where a and b are non negative integers.

$f_1 = 1.f_1 + 0.f_2 = (1,0) = (a_1,b_1)$

$f_2 = 0.f_1 + 1.f_2 = (0,1) = (a_2,b_2)$

$f_3 = f_1 + f_2 = 1.f_1 + 1.f_2 = (1,1) = (a_3, b_3)$

$f_4 = f_2 + f_3 = (0,1) +(1,1) = (1,2) = (a_4, b_4)$

$= f_1 + 2 f_2$

$f_5 = f_3 + f_4 = (1,1)+(1,2) = (2,3) = (a_5, b_5)$

$= 2 f_1 + 3 f_2$

$f_6 = f_4 + f_5 = (1,2) + (2,3) = (3,5) = (a_6, b_6)$

$= 3 f_1 + 5 f_2$

$f_7 = f_5 + f_6 = (2,3)+(3,5) = (5,8) = (a_7, b_7)$

$= 5 f_1 + 8 f_2$

$f_8 = f_6 + f_7 = (3,5) + (5,8) = (8,13)= (a_8, b_8)$

and so on.

Now, consider a sequences (a_n) and (b_n) , n=1,2,...

Range of sequence (a_n) = {1, 0, 1, 1, 2, 3, 5, 8,....}

Range of sequence (b_n) = {0, 1, 1, 2, 3, 5, 8,....}

Sequence (a_n) is increasing sequence for $n \geq 2$.

Sequence (b_n) is increasing sequence.

Here $a_k = b_{k-1}$ for k = 2, 3, 4,...

and $b_k = a_{k-1} + b_{k-1}$ for k = 2, 3,....

Suppose $f_k = a_k f_1 + b_k f_2 = (a_k, b_k)$

where $b_k = a_{k-1} + b_{k-1}$ for k = 3, 4, 5,...

Define L = { L_k : L_k is a straight line joining origin to the point (a_k , b_k), $k \geq 3$ }

Define S = {S_k : S_k is slope of the line L_k defined in L, $k \geq 3$ }

Denote by L_k a straight line joining (0,0) to the point (a_k , b_k) .

We shall represent this line L_k by [a_k , b_k] and its slope S_k by b_k / a_k .

Note that L = { L_3=[1,1], L_4=[1,2], L_5=[2,3],L_6=[3,5],.., L_k=[a_k, b_k], .. . } and

S = { S_3=1, S_4=2, S_5=3/2, S_6=5/3, S_7=8/5, S_8=13/8, ... , S_k= b_k / a_k , ... }

Sequence S_k is a decreasing sequence for $k \geq 4$ and is bounded below by 1 and bounded above by 2. Hence it is convergent and converges to g.

We know that $a_1 = 0$ and $b_1 = 1$ in a Fibonacci sequence and

$a_k = b_{k-1}$ for k = 2, 3, 4,...

and $b_k = a_{k-1} + b_{k-1}$ for k=2, 3,....

$f_k = (a_k, b_k)$

Now for $k \geq 3$, consider $b_k / a_k = (a_{k-1} + b_{k-1}) / b_{k-1}$

$= (a_{k-1} / b_{k-1}) + 1$ (1)

For large k , we take $x = b_k / a_k$

Equation (1) then becomes,

$x = 1/x + 1$.

Solving for x , we get $x = (1 \pm \sqrt{5})/2$

Taking positive value , x = 1.6180339874989 approximately.

This x is called golden ratio 'g' for Fibonacci sequence.

This means the set of lines defined in the set S converges to a line with slope equal to golden ratio.

Note that golden ratio is irrational number.

Corollory1: Sequence of rational numbers constructed from terms of Fibonacci sequence converges to golden ratio.

Proof : Construct sequences $\{a_n\}$ and $\{b_n\}$ as defined in Theorem 2.

Theorem 3: Given f_1 and f_2 , the n th term of Fibonacci sequence is given by

$f_n = (a_n, b_n)$ where $a_n = b_{n-1}$, $b_n = a_{n-1} + b_{n-1}$ for n = 3,4,5,...

Proof: By induction on n.

Let f_k be the statement that $f_k = a_k f1 + b_k f2$ then

$f_{k+1} = a_{k+1} f_1 + b_{k+1} f_2$ where $a_k = b_{k-1}$ and $b_k = a_{k-1} + b_{k-1}$

Let f_1 and f_2 be known i.e. $f_1 = (1, 0)$ and $f_2 = (0, 1)$.

Then,

$f_3 = f_1 + f_2 = (1, 1) = (a_1, b_1)$ where $a_1 = 1$ and $b_1 = 1$

Result is true for k=3
(1)

$f_4 = f_2 + f_3$

$= f_2 + (f_1 + f_2)$

$= f_1 + 2 f_2$

$= (1,2) = (a_2, b_2)$

Here $a_2 = b_1$ and $b_2 = 1 + 1 = a_1 + b_1$

Result is true for k = 4

$f_5 = f_3 + f_4$

$= (f_1 + f_2) + (f_1 + 2 f_2)$

$= 2 f_1 + 3 f_2$

$= (2,3) = (a_3, b_3)$

Result is true for k = 5

Assume that f_k is true for n < k.

This means if $f_1, f_2, f_3, \ldots , f_{k-1}$ are true then we shall show that f_k is true.

Consider

$f_k = f_{k-2} + f_{k-1}$

$= (a_{k-2}, b_{k-2}) + (a_{k-1}, b_{k-1})$

$= (b_{k-3}, a_{k-3} + b_{k-3}) + (b_{k-2}, a_{k-2} + b_{k-2})$

$= (b_{k-3} + b_{k-2}, a_{k-3} + b_{k-3} + a_{k-2} + b_{k-2})$

$= (b_{k-3} + b_{k-2}, (a_{k-2} + a_{k-3}) + (b_{k-3} + b_{k-2}))$

$= (b_{k-1}, a_{k-1} + b_{k-1}) = (a_k, b_k)$

$= a_k f_1 + b_k f_2$

f_k is true

Hence

if f_k is true for $k < n$ then f_k is true for $k = n$ (2)

Hence by mathematical induction proof follows from (1) and (2).

Theorem 4: Square root of difference of product of two consecutive terms of Fibonacci sequence is a term of Fibonacci sequence and it forms new Fibonacci sequence starting from second term .

Proof: Let $f_1, f_2, \ldots f_n, f_{n+1}, \ldots$be Fibonacci sequence of numbers. Assuming that the first two terms i.e. f_1, f_2 are known then other terms can be found by the recurrence relation

$f_{n+2} = f_n + f_{n+1}$, $n = 1, 2, 3, \ldots$

Define $g_n = f_n * f_{n+1}$ $n=1, 2, 3, \ldots$

Define $h_n = g_{n+1} - g_n$ $n=1, 2, 3, \ldots$

Simplifying h_n,

$h_n = g_{n+1} - g_n$

$= (f_{n+1} * f_{n+2}) - (f_n * f_{n+1})$

$= (f_{n+1} * (f_n + f_{n+1})) - (f_n * f_{n+1})$

$= f_{n+1} * f_n + f_{n+1} * f_{n+1} - f_n * f_{n+1} = f_{n+1} * f_{n+1}$

$= (f_{n+1})^2$

Thus a sequence (F_n) defined by $F_n = \sqrt{h_n}$, $n = 1, 2, \ldots$ is Fibonacci sequence whose first term is given by f_{n+1}

This proves the theorem.

Note that Fibonacci sequence so obtained i.e. $F_n = \sqrt{h_n}$ is a subsequence of Fibonacci sequence f_n.

Example 4: Consider a Fibonacci sequence $\{f_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

Define $g_n = f_n * f_{n+1}$ $n = 1,2,3,4, \ldots$

Therefore $\{g_n\} = \{0.1, 1.1, 1.2, 2.3, 3.5, 5.8, \ldots\} = \{0, 1, 2, 6, 15, 40, \ldots\}$

Define $h_n = g_{n+1} - g_n$ $n=1,2,3,4, \ldots$

$\{h_n\} = \{1-0, 2-1, 6-2, 15-6, 40-15, \ldots\} = \{1,1,4,9,25, \ldots\}$

$\{\sqrt{h_n}\} = \{1,1,2,3,5, \ldots\}$ is a Fibonacci sequence $f_2, f_3, \ldots$

and sequence ($\sqrt{h_n}$) is a subsequence of (f_n)

V. CONCLUSION

Interesting properties may be found based on general definition of Fibonacci sequence and its application in different domains of science.

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