Some Fractional Derivatives of Multivariable Gimel-Function

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ABSTRACT
In this present paper we derive a number of key formulae involving fractional derivatives of multivariable Gimel-function. Application of some of these key results provide potentially useful generalization of results in the theory of fractional calculus.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, fractional derivative formulae.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]$$
$$[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_r}]:\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]$$

$$;\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}]$$
$$\left.;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,n^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{n^{(r)}+1,q_i^{(r)}}]\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2),\Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r),\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)}\gamma_j^{(k)} - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right) +$$

$$-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2}\beta_{2ji_2}^{(k)}\right) - \cdots - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \rightarrow \infty$ where $i = 1,\cdots,r:$

$\alpha_i = \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]$ and $\beta_i = \max_{1\leqslant j\leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$

Remark 1.
If $n_2=\cdots=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$ $A_{rj}=A_{rji_r}=B_{rji_r}=1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2=\cdots=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [4].

Remark 3.
If$A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}$ $=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [3].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [7,8].

In your investigation, we shall use the following notations.

$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$

$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha^{(1)}_{(r-1)ji_{r-1}},\cdots,\alpha^{(r-1)}_{(r-1)ji_{r-1}};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}];\cdots;$$

$$[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta^{(1)}_{(r-1)ji_{r-1}},\cdots,\beta^{(r-1)}_{(r-1)ji_{r-1}};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}^{(1)}_j,\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}];\cdots;$$

$$[(\mathrm{d}^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{m^{(r)}+1,q^{(r)}_i}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

Saigo [5] defines the integral operator in terms of the Gauss's hypergeometric function as follows :

Lemma 1.

$$I^{\alpha,\beta,\eta}_{0,x}f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-t/x)f(t)\mathrm{d}t \tag{2.1}$$

Where $\alpha > 0,\beta$ and η are real numbers, $f(x)$ is a real valued and f is a continuous function defined on the interval $(0,\infty)$ having the order $0(x^k)$ near $x=0$ with $k > max(0,\beta-\eta)-1$.
When $\alpha > 0$, by letting in a positive integer such that : $0 < \alpha+n \leqslant 1$, he defines

$$I^{\alpha,\beta,\eta}_{0,x}f(x) = \frac{d^n}{dx^n}I^{\alpha+n,\beta-n,\eta-n}_{0,x}f(x) \tag{2.2}$$

provided that the R.H.S. has a definite meaning.

We have :

Lemma 2.

$$I^{\alpha,\beta,\eta}_{0,x}(x^\lambda) = \frac{\Gamma(1+\lambda)(\Gamma(1+\lambda-\beta+\eta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)}, Re(\lambda)+1-\beta > 0 \tag{2.3}$$

3. Main result.

In this section, we shall five two general formulae about the multivariable Gimel-function.

Theorem 1.

$$I_{0,x}^{\alpha,\beta,\eta}\left\{x^{k}(x^{v}+\zeta)^{\lambda}\gimel\left(z_{1}x^{a_{1}}(x^{v}+\zeta)^{-b_{1}},\cdots,z_{r}x^{a_{r}}(x^{v}+\zeta)^{-b_{r}}\right)\right\}=\zeta^{\lambda}x^{k-\beta}\sum_{m=0}^{\infty}\frac{\left(-\frac{x^{v}}{\zeta}\right)^{m}}{m!}\gimel_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}$$

$$\left(\begin{array}{c|c} z_{1}x^{a_{1}}\zeta^{-b_{1}} & \mathbb{A};(1+\lambda-m;b_{1},\cdots,b_{r};1),(-k-mv;a_{1},\cdots,a_{r};1),(-k+\beta-\eta-mv;a_{1},\cdots,a_{r};1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_{r}x^{a_{r}}\zeta^{-b_{r}} & \mathbb{B};\mathbf{B},(1+\lambda;b_{1},\cdots,b_{r};1),(-k+\beta-mv;a_{1},\cdots,a_{r};1),(-k-\alpha-\eta-mv;a_{1},\cdots,a_{r};1):B \end{array}\right) \quad (3.1)$$

provided

$v,a_{i}b_{i}>0(i=1,\cdots,r),\left|arg\left(\frac{x^{v}}{\zeta}\right)\right|<\pi,\ Re\,(k)+\sum_{i=1}^{r}a_{i}\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]+1>0$

$\left|arg\left(z_{i}x^{a_{i}}(x^{v}+\zeta)^{-b_{i}}\right)\right|<\frac{1}{2}A_{i}^{(k)}\pi$

Proof

To prove the theorem, expressing the generalized multivariable Gimel-function in multiple integrals contour with the help of (1.1), applying the following binomial expansion

$(x+a)^{\lambda}=a^{\lambda}\sum_{m=0}^{\infty}\binom{\lambda}{a}\left(\frac{x}{a}\right)^{m},\left|\frac{x}{a}\right|<1$ and the lemma 2, and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

Theorem 2 .

$$I_{0,x}^{\alpha,\beta,\eta}I_{0,y}^{\alpha',\beta',\eta'}\left\{x^{k}y^{k'}(x^{v}+\zeta)^{\lambda}(x^{w}+\delta)^{\lambda'}\gimel\left(z_{1}x^{a_{1}}y^{c_{1}}(x^{v}+\zeta)^{-b_{1}}(y^{w}+\delta)^{-d_{1}},\cdots,z_{r}x^{a_{r}}y^{c_{r}}(x^{v}+\zeta)^{-b_{r}}(y^{w}+\delta)^{-d_{r}}\right)\right\}=$$

$$\zeta^{\lambda}\delta^{\lambda'}x^{k-\beta}y^{k'-\beta'}\sum_{l,m=0}^{\infty}\frac{\left(-\frac{x^{v}}{\zeta}\right)^{l}\left(-\frac{y^{v'}}{\delta}\right)^{m}}{l!m!}\gimel_{X;p_{i_r}+6,q_{i_r}+6,\tau_{i_r}:R_r:Y}^{U;0,n_r+6:V}\left(\begin{array}{c|c} z_{1}x^{a_{1}}y^{c_{1}}\zeta^{-b_{1}}\delta^{-d_{1}} & \mathbb{A};(1+\lambda-l;b_{1},\cdots,b_{r};1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_{r}x^{a_{r}}y^{c_{r}}\zeta^{-b_{r}}\delta^{-d_{r}} & \mathbb{B};\mathbf{B},(1+\lambda;b_{1},\cdots,b_{r};1) \end{array}\right.$$

$$\begin{array}{c} (1+\lambda'-m;d_{1},\cdots,d_{r};1),(-k'-mw;c_{1},\cdots,c_{r};1),(-k'-mw+\beta'-\eta';c_{1},\cdots,c_{r};1), \\ \cdot \\ \cdot \\ (1+\lambda';d_{1},\cdots,d_{r};1),(-k'-mw+\beta';c_{1},\cdots,c_{r};1),(-k'-mw+\alpha'-\eta';c_{1},\cdots,c_{r};1), \end{array}$$

$$\left.\begin{array}{c} (-k-lv;a_{1},\cdots,a_{r};1),(-k-lv'+\beta-\eta;a_{1},\cdots,a_{r};1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (-k-lv+\beta-\eta;a_{1},\cdots,a_{r};1),(-k-lv-\alpha-\eta;a_{1},\cdots,a_{r};1) \end{array}\right) \quad (3.2)$$

provided

$v,w,a_{i},b_{i},c_{i},d_{i}>0(i=1,\cdots,r),max\left\{\left|arg\left(\frac{x^{v}}{\zeta}\right)\right|,\left|arg\left(\frac{x^{w}}{\delta}\right)\right|\right\}<\pi,,$

$Re\,(k)+\sum_{i=1}^{r}a_{i}\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]+1>0,\ Re\,(k')+\sum_{i=1}^{r}c_{i}\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_{j}^{(i)}\left(\frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right)\right]+1>0$

$\left|arg\left(z_i x^{a_i} y^{c_i}(x^v+\zeta)^{-b_i}(x^w+\delta)^{-d_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$

Proof

To prove the theorem 2, we use the theorem 1 twice with respect to the variable y , and then with respect to the variable x , here x and y are two independent variables.

4. Special cases.

In this section, we shall give the particular cases.

Consider the theorem 1, taking $b_i \to 0(i=1,\cdots,r)$, we obtain

Corollary 1.

$$I_{0,x}^{\alpha,\beta,\eta}\left\{x^k(x^v+\zeta)^{\lambda}\daleth(z_1x^{a_1},\cdots,z_rx^{a_r})\right\} = \zeta^{\lambda}x^{k-\beta}\sum_{m=0}^{\infty}\binom{\lambda}{m}\left(-\frac{x^v}{\zeta}\right)^m$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c} \mathrm{z}_1x^{a_1} & \mathbb{A};(-k-mv;a_1,\cdots,a_r;1),(-k+\beta-\eta-mv;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_rx^{a_r} & \mathbb{B};\mathbf{B},(-k+\beta-mv;a_1,\cdots,a_r;1),(-k-\alpha-\eta-mv;a_1,\cdots,a_r;1):B \end{array}\right) \quad (4.1)$$

provided

$v,a_i>0(i=1,\cdots,r),\left|arg\left(\frac{x^v}{\zeta}\right)\right|<\pi$, $Re\,(k)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]+1>0$

$|arg\,(z_ix^{a_i})| < \frac{1}{2}A_i^{(k)}\pi$

Consider the theorem 2, taking $b_i,d_i \to 0(i=1,\cdots,r)$, e get

Corollary 2.

$$I_{0,x}^{\alpha,\beta,\eta}I_{0,y}^{\alpha',\beta',\eta'}\left\{x^ky^{k'}(x^v+\zeta)^{\lambda}(x^w+\delta)^{\lambda'}\daleth(z_1x^{a_1}y^{c_1},\cdots,z_rx^{a_r}y^{c_r})\right\}=$$

$$\zeta^{\lambda}\delta^{\lambda'}x^{k-\beta}y^{k'-\beta'}\sum_{l,m=0}^{\infty}\binom{\lambda}{l}\binom{\lambda'}{m}\left(-\frac{x^v}{\zeta}\right)^l\left(-\frac{y^{v'}}{\delta}\right)^m\daleth_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}\left(\begin{array}{c|} \mathrm{z}_1x^{a_1}y^{c_1} \\ \cdot \\ \cdot \\ \mathrm{z}_rx^{a_r}y^{c_r} \end{array}\right.$$

$\mathbb{A}$: (-k'-mw;$c_1,\cdots,c_r;1),(-k'-mw+\beta'-\eta';c_1,\cdots,c_r;1)$,

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$\mathbb{B}$; **B** ,(-k'-mw+$\beta';c_1,\cdots,c_r;1),(-k'-mw+\alpha'-\eta';c_1,\cdots,c_r;1)$,

$$\left.\begin{array}{c} (\text{-k-lv};a_1,\cdots,a_r;1),(-k-lv'+\beta-\eta;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (\text{-k-lv}+\beta-\eta;a_1,\cdots,a_r;1),(-k-lv-\alpha-\eta;a_1,\cdots,a_r;1) \end{array}\right) \quad (4.2)$$

provided

$v,w,a_i,c_i,>0(i=1,\cdots,r),max\left\{\left|arg\left(\frac{x^v}{\zeta}\right)\right|,\left|arg\left(\frac{x^w}{\delta}\right)\right|\right\}<\pi,,$

$$Re\,(k)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]+1>0,\ Re\,(k')+\sum_{i=1}^{r}c_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]+1>0$$

$$|arg\,(z_i x^{a_i} y^{c_i})|<\frac{1}{2}A_i^{(k)}\pi$$

Consider the corollary , taking $\lambda=0$, we have the following formula, with $m=0$

Corollary 3.

$$I_{0,x}^{\alpha,\beta,\eta}\left\{x^k\daleth\,(z_1x^{a_1},\cdots,z_rx^{a_r})\right\}=x^{k-\beta}$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c}\mathrm{z}_1x^{a_1} & \mathbb{A};(-k-;a_1,\cdots,a_r;1),(-k+\beta-\eta;a_1,\cdots,a_r;1),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_rx^{a_r} & \mathbb{B};\mathbf{B},(-k+\beta;a_1,\cdots,a_r;1),(-k-\alpha-\eta;a_1,\cdots,a_r;1):B\end{array}\right)\quad(4.1)$$

provided

$$v,a_i>0(i=1,\cdots,r),\left|arg\left(\frac{x^v}{\zeta}\right)\right|<\pi,\ Re\,(k)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]+1>0$$

$$|arg\,(z_i x^{a_i})|<\frac{1}{2}A_i^{(k)}\pi$$

Consider the corollary , taking $\lambda=\lambda'=0$, we have the following formula, with $m=0=l$

Corollary 4.

$$I_{0,x}^{\alpha,\beta,\eta}I_{0,y}^{\alpha',\beta',\eta'}\left\{x^ky^{k'}\daleth\,(z_1x^{a_1}y^{c_1},\cdots,z_rx^{a_r}y^{c_r})\right\}=x^{k-\beta}y^{k'-\beta'}\ \daleth_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}\left(\begin{array}{c|}\mathrm{z}_1x^{a_1}y^{c_1}\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{a_r}y^{c_r}\end{array}\right.$$

$$\begin{array}{c}\mathbb{A}:\ (\text{-k'}c_1,\cdots,c_r;1),(-k'+\beta'-\eta';c_1,\cdots,c_r;1),\\ \cdot\\ \cdot\\ \mathbb{B}\ ;\ \mathbf{B}\ ,(\text{-k'}+\beta';c_1,\cdots,c_r;1),(-k'+\alpha'-\eta';c_1,\cdots,c_r;1),\end{array}$$

$$\left.\begin{array}{c}(\text{-k;a}_1,\cdots,a_r;1),(-k+\beta-\eta;a_1,\cdots,a_r;1),\mathbf{A}:A\\ \cdot\\ \cdot\\ (\text{-k}+\beta-\eta;a_1,\cdots,a_r;1),(-k-\alpha-\eta;a_1,\cdots,a_r;1)\end{array}\right)\quad(4.2)$$

provided

$$v,w,a_i,c_i,>0(i=1,\cdots,r),max\left\{\left|arg\left(\frac{x^v}{\zeta}\right)\right|,\left|arg\left(\frac{x^w}{\delta}\right)\right|\right\}<\pi,,$$

$$Re\,(k)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]+1>0,\ Re\,(k')+\sum_{i=1}^{r}c_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]+1>0$$

$$|arg\,(z_i x^{a_i} y^{c_i})|<\frac{1}{2}A_i^{(k)}\pi$$

Remark 6.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then we can obtain the same fractional derivatives formulae in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1])

Remark 7.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same fractional derivatives formulae in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [4]).

Remark 8.
If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same fractional derivatives formulae in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [3]).

Remark 9.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [7,8] and then we can obtain the same fractional derivatives formulae, see Shrivastava [6] for more details.

5. Conclusion.

The fractional derivative formulae involving in this paper are double fold generality in term of variables. By specializing the various parameters and variables involved, these formulae can suitably be applied to derive the corresponding results involving wide variety of useful functions (or product of several such functions) which can be expressed in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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