# Stability and Local Hopf Bifurcation Analysis in Rayleigh Price Model with Two Time Delays 

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#### Abstract

This paper mainly investigates a Rayleigh price model with two time delays. First, two time delays are introduced to the original Rayleigh price model and we establish a new model. Second, the linear stability of the model and the local Hopf bifurcation are studied and we derived the conditions for the stability and the existence of Hopf bifurcation at the equilibrium of the system. Besides, some numerical simulation results are confirmed that the feasibility of the theoretical analysis. At last, some conclusions of this paper are given.


Keywords—Rayleigh price model, Stability, Hopf bifurcation, Equilibrium, Numerical simulation.

## I. INTRODUCTION

With the wide application of differential equations in economy and other fields, many domestic and overseas scholars have established many differential equation models which can reflect the characteristics of dynamic systems. Recently, more and more people tend to realize that the price changes have an important impact on people's economic life. Price is a dynamic economic phenomenon, which is closely related to people's life and it is also affected by the supply and demand. In order to study this phenomenon exactly, many researchs have paid attention to it and proposed many price differential equation models[1-3]. Among them, the Rayleigh price model is a classical economic model. In [3], the author ignorced the effect of time delay and studied the price differential equation model which gived a dynamic system to research the dynamics properties of the Rayleigh price model. In [4] further studied on price model with delay and provided a qualitative analysis for different kinds of economic phenomenon by qualitative theory of differential equations. In [5], by using the method of $\tau-D$ partitioning approach of exponential polynomial, Lv and Liu have investigated the Reyleigh price model with time delay, they draw a conclusion that supply depends only on the price of the past. In [6], a modified Reyleigh price model with time delay is investigated by applying the method of Hopf bifurcation analysis. In real life, price changes are influenced by many factors, such as national policies, inflation and so on. In this paper, our study based on the previous Reyleigh price model and proposed a new Rayleigh price model with two delays.
In [3], the original Rayleigh price model can be described by the following nonlinear differential equations:

$$
\left\{\begin{array}{l}
\cdot  \tag{1}\\
x(\mathrm{t})=-y(\mathrm{t})+l\left(\frac{1}{3} a x^{3}(\mathrm{t})+\frac{1}{2} b x^{2}(\mathrm{t})+\mathrm{c} x(\mathrm{t})\right), \\
\cdot \\
y(\mathrm{t})=x(\mathrm{t})
\end{array}\right.
$$

where $x(\mathrm{t})$ represents the price at time $t, y(\mathrm{t})$ denotes the amount of supply at time $t, l>0$ and $a, b, c$ are the constants.

Considering the effect of time delay on the above system (1), the Eq. (1) can be changed into the following
form [5]:

$$
\left\{\begin{array}{l}
\dot{x}(\mathrm{t})=-y(\mathrm{t})+l\left(\frac{1}{3} a x^{3}(\mathrm{t})+\frac{1}{2} b x^{2}(\mathrm{t})+\mathrm{c} x(\mathrm{t})\right)  \tag{2}\\
\dot{y}(\mathrm{t})=x(\mathrm{t}-\tau)
\end{array}\right.
$$

where $\tau>0$ is the time delay and the other parameters are definition of this model (2) are the same as model (1).

Besides, in [6] it showed that the Eq. (2) can be modified by:

$$
\left\{\begin{array}{l}
\dot{x}(\mathrm{t})=-y(\mathrm{t})+l\left(\frac{1}{3} a x^{3}(\mathrm{t})+\frac{1}{2} b x^{2}(\mathrm{t})+\mathrm{c} x(\mathrm{t})\right)  \tag{3}\\
\dot{y}(\mathrm{t})=\frac{1}{2} d x(\mathrm{t})+\frac{1}{2} k x(\mathrm{t}-\tau)
\end{array}\right.
$$

In this paper, we introduced two time delays to the above model(2), Hence, we propose a new model as follows:

$$
\left\{\begin{array}{l}
\dot{x}(\mathrm{t})=-y\left(\mathrm{t}-\tau_{2}\right)+l\left(\frac{1}{3} a x^{3}(\mathrm{t})+\frac{1}{2} b x^{2}(\mathrm{t})+\mathrm{c} x(\mathrm{t})\right),  \tag{4}\\
\dot{y}(\mathrm{t})=x\left(\mathrm{t}-\tau_{1}\right) .
\end{array}\right.
$$

where $\tau_{1}>0, \tau_{2}>0$.
The rest of the paper is arranged as follows, the linear stability of the model and the local Hopf bifurcation are studied and the conditions for the stability and the existence of Hopf bifurcation at the equilibrium are derived in section 2. In section 3, the correctness of theoretical analysis are confirmed by some numerical simulation results. At last, some conlusions are obtained in section 4.

## II. STABILITY AND LOCAL HOPF BIFURCATION ANALYSIS

In this section, we only discuss the problems of the Hopf bifurcation and stability for the unique equilibrium point $(0,0)$. The linearation of system (4) at $(0,0)$ is

$$
\left\{\begin{array}{l}
\dot{x}(\mathrm{t})=-y\left(-\mathrm{t} \tau_{1}+\right) l \quad \mathfrak{a}  \tag{5}\\
\dot{y}(\mathrm{t})=x \quad\left(-\mathrm{t} \tau_{2} \quad\right)
\end{array}\right.
$$

The correspoding characteristic equation of Eq. (5) is as follows.

$$
\begin{equation*}
\lambda^{2}-l c \lambda+e^{-\lambda\left(\tau_{1}+\tau_{2}\right)}=0 \tag{6}
\end{equation*}
$$

Lemma 1. When $\tau_{1}=\tau_{2}=0$ and $c<0$ hold, the equilibrium point $(0,0)$ of price model (4) is locally asymptotically stable.

Proof. When $\tau_{1}=\tau_{2}=0$ is met, Eq. (6) becomes

$$
\begin{equation*}
\lambda^{2}-l c \lambda+1=0 \tag{7}
\end{equation*}
$$

further, if $c<0$ is met, we have the following conditions:

$$
\begin{equation*}
D_{1}=-l c>0, D_{2}=1>0 \tag{8}
\end{equation*}
$$

According to the Routh-Hurwitz criteria[7-8], all roots of characteristic equation (7) have negative real parts. Hence, when $\tau_{1}=\tau_{2}=0$ and $c<0$ are satisfied, the equilibrium point $(0,0)$ of system (4) is locally asymptotically stable.

Lemma 2. When $\tau_{1}=0, \tau_{2}>0$, Then Eq.(6) has a pair of purely imaginary roots $\pm i \omega_{20}$ when $\tau_{2}=\tau_{20}$, where

$$
\begin{gathered}
\omega_{20}=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}}, \\
\tau_{20}=\frac{1}{\omega_{20}} \arctan \left(-\frac{l c}{\omega_{20}}\right) .
\end{gathered}
$$

Proof. When $\tau_{1}=0, \tau_{2}>0$, the characteristic equation (6) becomes

$$
\begin{equation*}
\lambda^{2}-l c \lambda+e^{-\lambda \tau_{2}}=0 \tag{9}
\end{equation*}
$$

Let $\lambda=i \omega_{2}\left(\omega_{2}>0\right)$ is a solution of the characteristic equation (6), then

$$
-\omega_{2}^{2}-i l c \omega_{2}+\left(\cos \omega_{2} \tau_{2}-i \sin \omega_{2} \tau_{2}\right)=0
$$

The separation of the real and imaginary parts, it follows

$$
\left\{\begin{array}{l}
-\omega_{2}^{2}+\cos \omega_{2} \tau_{2}=0  \tag{10}\\
-l c \omega_{2}-\sin \omega_{2} \tau_{2}=0
\end{array}\right.
$$

From (10) we obtain

$$
\begin{gathered}
\omega_{2}=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}} \\
\tau_{2 j}=\frac{1}{\omega_{2}}\left[\arctan \left(-\frac{l c}{\omega_{2}}\right)+j \pi\right], \quad j=0,1,2, \cdots .
\end{gathered}
$$

Obviously, set $j=0$, then

$$
\begin{align*}
& \omega_{20}=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}}  \tag{11}\\
& \tau_{20}=\frac{1}{\omega_{20}} \arctan \left(-\frac{l c}{\omega_{20}}\right) \tag{12}
\end{align*}
$$

As a result, when $\tau_{2}=\tau_{20}$, the equation (6) has a pair of purely imaginary roots.

Lemma 3. Let $\lambda\left(\tau_{2}\right)=\alpha\left(\tau_{2}\right)+i \omega\left(\tau_{2}\right)$ be the root of (9) with $\alpha\left(\tau_{20}\right)=0$ and $\omega\left(\tau_{20}\right)=\omega_{20}$ then we have the following transversality condition $\left.\operatorname{Re}\left(\frac{d \lambda}{d \tau_{2}}\right)^{-1}\right|_{\tau_{2}=\tau_{20}}>0$ is satisfied.

Proof. By differentiating both sides of Eq. (9) with regard to $\tau_{2}$ and applying the implicit function theorem, we have

$$
\begin{aligned}
\left.\frac{d \lambda}{d \tau_{2}}\right|_{\tau_{2}=\tau_{20}} & =\frac{\lambda e^{-\lambda \tau_{20}}}{2 \lambda-l c-\tau_{2} e^{-\lambda \tau_{20}}} \\
& =\frac{\omega_{20} \sin \omega_{20} \tau_{20}+i \omega_{20} \cos \omega_{20} \tau_{20}}{\left(-l c-\tau_{20} \cos \omega_{20} \tau_{20}\right)+i\left(2 \omega_{0}+\tau_{20} \sin \omega_{20} \tau_{20}\right)}
\end{aligned}
$$

then

$$
\left.\mathrm{Re} \frac{d \lambda}{d \tau_{2}}\right|_{\tau_{2}=\tau_{20}}=\frac{l^{2} c^{2} \omega_{2}^{2}+2 \omega_{2}^{4}}{\left(-l c-\tau_{20} \cos \omega_{20} \tau_{20}\right)^{2}+\left(2 \omega_{20}+\tau_{20} \sin \omega_{20} \tau_{20}\right)^{2}}
$$

Thus we have $\left.\operatorname{Re}\left(\frac{d \lambda}{d \tau_{2}}\right)^{-1}\right|_{\tau_{2}=\tau_{20}}>0$. The proof is completed.
By applying the Hopf bifurcation theorem for time-delayed differential equation and the above analysis [9-13], we have the following results.
Theorem 1. For Eq. (9), when $\tau_{2}<\tau_{20}$, all of his roots have negative real parts. The equilibrium ( 0,0 ) is locally asymptotically stable, and system (4) produces a Hopf bifurcation at the equilibrium ( 0,0 ) when $\tau_{2}=\tau_{20}$.

Similarly, when $\tau_{2}=0, \tau_{1}>0$, we have the same results as above and we omit it.
Lemma 4. When $\tau_{1}=\tau_{2}=\tau$, Then Eq.(6) has a pair of purely imaginary roots $\pm i \omega_{0}$ when $\tau=\tau_{0}$, where

$$
\begin{gathered}
\omega_{0}=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}}, \\
\tau_{0}=\frac{1}{2 \omega_{0}} \arctan \left(-\frac{l c}{\omega_{0}}\right) .
\end{gathered}
$$

Proof. When $\tau_{1}=\tau_{2}=\tau$, the characteristic equation (6) becomes

$$
\begin{equation*}
\lambda^{2}-l c \lambda+e^{-2 \lambda \tau}=0 \tag{13}
\end{equation*}
$$

Let $\lambda=i \omega(\omega>0)$ is a solution of the characteristic equation (6), then

$$
-\omega^{2}-i l c \omega+(\cos 2 \omega \tau-i \sin 2 \omega \tau)=0
$$

The separation of the real and imaginary parts, it follows

$$
\left\{\begin{array}{l}
-\omega^{2}+\cos 2 \omega \tau=0  \tag{14}\\
-l c \omega-\sin 2 \omega \tau=0
\end{array}\right.
$$

From (14) we obtain

$$
\begin{gathered}
\omega=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}} \\
\tau_{j}=\frac{1}{2 \omega}\left[\arctan \left(-\frac{l c}{\omega}\right)+j \pi\right], \quad j=0,1,2, \cdots .
\end{gathered}
$$

Obviously, set $j=0$, then

$$
\begin{align*}
& \omega_{0}=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}}  \tag{15}\\
& \tau_{0}=\frac{1}{2 \omega_{0}} \arctan \left(-\frac{l c}{\omega_{0}}\right) \tag{16}
\end{align*}
$$

As a result, when $\tau=\tau_{0}$, the equation (6) has a pair of purely imaginary roots.
Lemma 5. Let $\lambda(\tau)=\alpha(\tau)+i \omega(\tau)$ be the root of (13) with $\alpha\left(\tau_{0}\right)=0$ and $\omega\left(\tau_{0}\right)=\omega_{0}$ then we have the following transversality condition $\left.\operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)^{-1}\right|_{\tau=\tau_{0}}>0$ is satisfied.
Proof. By differentiating both sides of Eq. (13) with regard to $\tau$ and applying the implicit function theorem, we have

$$
\begin{aligned}
\left.\frac{d \lambda}{d \tau}\right|_{\tau=\tau_{0}} & =\frac{2 \lambda e^{-2 \lambda \tau_{0}}}{2 \lambda-l c-2 \tau e^{-2 \lambda \tau_{0}}} \\
& =\frac{2 \omega_{0} \sin 2 \omega_{0} \tau_{0}+i 2 \omega_{0} \cos 2 \omega_{0} \tau_{0}}{\left(-l c-2 \tau_{0} \cos 2 \omega_{0} \tau_{0}\right)+i\left(2 \omega_{0}+2 \tau_{0} \sin 2 \omega_{0} \tau_{0}\right)}
\end{aligned}
$$

then

$$
\left.\operatorname{Re} \frac{d \lambda}{d \tau}\right|_{\tau=\tau_{0}}=\frac{2 l^{2} c^{2} \omega_{2}^{2}+4 \omega_{0}^{4}}{\left(-l c-2 \tau_{0} \cos 2 \omega_{0} \tau_{0}\right)^{2}+\left(2 \omega_{0}+\tau_{0} \sin 2 \omega_{0} \tau_{0}\right)^{2}}
$$

Thus we have $\left.\operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)^{-1}\right|_{\tau_{2}=\tau_{20}}>0$. Similarly, we have the following results.
Theorem 2. For Eq. (13), when $\tau<\tau_{0}$, all of his roots have negative real parts. The equilibrium ( 0,0 ) is locally asymptotically stable, and system (4) produces a Hopf bifurcation at the equilibrium ( 0,0 ) when $\tau=\tau_{0}$.

Lemma 6. When $\tau_{1}>0, \tau_{2}>0$ and $\tau_{2} \in\left[0, \tau_{20}\right)$, Then Eq.(6) has a pair of purely imaginary roots $\pm i \omega_{10}^{*}$
when $\tau_{1}=\tau_{10}^{*}$, where

$$
\begin{gathered}
\omega_{10}^{*}=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}}, \\
\tau_{10}^{*}=\frac{1}{\omega_{10}^{*}} \arctan \left(-\frac{l c}{\omega_{10}^{*}}\right) .
\end{gathered}
$$

Proof. Let $\lambda=i \omega_{1}^{*}\left(\omega_{1}^{*}>0\right)$ is a solution of the characteristic equation (6), then

$$
-\omega_{1}^{* 2}-i l c \omega_{1}^{*}+\left(\cos \omega_{1}^{*}\left(\tau_{1}+\tau_{2}\right)-i \sin \omega_{1}^{*}\left(\tau_{1}+\tau_{2}\right)\right)=0
$$

The separation of the real and imaginary parts, it follows

$$
\left\{\begin{array}{l}
-\omega_{1}^{* 2}+\cos \omega_{1}^{*}\left(\tau_{1}+\tau_{2}\right)=0  \tag{17}\\
-l c \omega-\sin \omega_{1}^{*}\left(\tau_{1}+\tau_{2}\right)=0
\end{array}\right.
$$

From (17) we obtain

$$
\begin{gathered}
\omega_{1}^{*}=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}} \\
\tau_{1 j}^{*}=\frac{1}{2 \omega}\left[\arctan \left(-\frac{l c}{\omega}\right)+j \pi\right], \quad j=0,1,2, \cdots .
\end{gathered}
$$

Obviously, set $j=0$, then

$$
\begin{align*}
& \omega_{10}^{*}=\sqrt{\frac{-l^{2} c^{2}+\sqrt{\left(l^{2} c^{2}\right)^{2}+4}}{2}}  \tag{18}\\
& \tau_{10}^{*}=\frac{1}{2 \omega_{10}^{*}} \arctan \left(-\frac{l c}{\omega_{10}^{*}}\right) \tag{19}
\end{align*}
$$

As a result, when $\tau_{1}=\tau_{10}^{*}$, the equation (6) has a pair of purely imaginary roots.
Lemma 7. Let $\lambda\left(\tau_{1}\right)=\alpha\left(\tau_{1}\right)+i \omega\left(\tau_{1}\right)$ be the root of (9) with $\alpha\left(\tau_{10}\right)=0$ and $\omega\left(\tau_{10}\right)=\omega_{10}^{*}$ then we have the following transversality condition $\left.\operatorname{Re}\left(\frac{d \lambda}{d \tau_{2}}\right)^{-1}\right|_{\tau_{2}=\tau_{20}}>0$ is satisfied.

Proof. By differentiating both sides of Eq. (6) with regard to $\tau_{1}$ and applying the implicit function theorem, we have

$$
\begin{aligned}
\left.\frac{d \lambda}{d \tau_{1}}\right|_{\lambda=i \omega_{10}^{*}} & =\frac{2 \lambda e^{-\lambda\left(\tau_{1}+\tau_{2}\right)}}{2 \lambda-l c-\left(\tau_{1}+\tau_{2}\right) e^{-\lambda\left(\tau_{1}+\tau_{2}\right)}} \\
& =\frac{2 \omega_{10}^{*} \sin \omega_{10}^{*}\left(\tau_{1}+\tau_{2}\right)+i 2 \omega_{10}^{*} \cos \omega_{10}^{*}\left(\tau_{1}+\tau_{2}\right)}{\left(-l c-\left(\tau_{1}+\tau_{2}\right) \cos \omega_{10}^{*}\left(\tau_{1}+\tau_{2}\right)\right)+i\left(2 \omega_{10}^{*}+\left(\tau_{1}+\tau_{2}\right) \sin \omega_{10}^{*}\left(\tau_{1}+\tau_{2}\right)\right)}
\end{aligned}
$$

then

$$
\left.\operatorname{Re} \frac{d \lambda}{d \tau_{1}}\right|_{\tau_{1}=\tau_{10}}=\frac{2 l^{2} c^{2} \omega_{10}^{* 2}+4 \omega_{10}^{* 4}}{\left(-l c-\left(\tau_{1}+\tau_{2}\right) \cos \omega_{10}^{*}\left(\tau_{1}+\tau_{2}\right)\right)^{2}+\left(2 \omega_{10}^{*}+\left(\tau_{1}+\tau_{2}\right) \sin \omega_{10}^{*}\left(\tau_{1}+\tau_{2}\right)\right)^{2}}
$$

Hence, we have $\left.\operatorname{Re}\left(\frac{d \lambda}{d \tau_{1}}\right)^{-1}\right|_{\tau_{1}=\tau_{10}^{*}}>0$. Based on the above analysis, we have the following results.
Theorem 3. When $\tau_{1}>0, \tau_{2}>0$ and $\tau_{2} \in\left[0, \tau_{20}\right.$ ), For Eq. (6), when $\tau_{1}<\tau_{10}^{*}$, all of his roots have negative real parts. The equilibrium $(0,0)$ is locally asymptotically stable, and system (4) produces a Hopf bifurcation at the equilibrium $(0,0)$ when $\tau_{1}=\tau_{10}^{*}$.

## III. NUMERICAL SIMULATION

In this section, we present numerical results to confirm the analytical predictions obtained in the previous section. For system (4), We take the parameters $a=-1, b=-1, c=-2, l=-2$, , then the Eq. (4) can be described as follow

$$
\left\{\begin{array}{l}
\cdot  \tag{20}\\
\dot{x}(\mathrm{t})=-y\left(\mathrm{t}-\tau_{2}\right)+2\left(-\frac{1}{3} x^{3}(\mathrm{t})-\frac{1}{2} x^{2}(\mathrm{t})-2 x(\mathrm{t})\right) \\
\dot{y}(\mathrm{t})=x\left(\mathrm{t}-\tau_{1}\right)
\end{array}\right.
$$

When $\tau_{1}=0, \tau_{2}>0$, By directly computing, we get that $\omega_{20}=0.249515, \tau_{20}=6.04572$, If we choose $\tau_{2}=5<\tau_{20}$, the equilibrium point $(0,0)$ of the system (4) is asymptotically stable proved by numerical simulations. While the delay value $\tau_{2}=7>\tau_{20}$, the the equilibrium point is unstable and system (4) produces a Hopf bifurcation at the equilibrium $(0,0)$ when $\tau_{2}=\tau_{20}$ (see Figs. 1 and 2).

When $\tau_{1}=\tau_{2}=\tau$, we obtained that $\omega_{0}=0.249515, \tau_{0}=3.02286$, if $\tau=2<\tau_{0}$, the equilibrium point $(0,0)$ of the system (4) is asymptotically stable. If the delay value $\tau$ passes through the critical value $\tau_{0}$, a Hopf bifurcation occurs. Namely, the the equilibrium point is unstable and system (4) produces a Hopf bifurcation at the equilibrium $(0,0)$ when $\tau=\tau_{0}$ (see Figs. 3 and 4).

When $\tau_{1}>0, \tau_{2}=2 \in\left[0, \tau_{20}\right)$, we get that $\omega_{10}^{*}=0.249515, \tau^{*}=4.04572$, If we choose $\tau_{1}=3.5<\tau_{10}^{*}$, the equilibrium point $(0,0)$ of the system (4) is asymptotically stable. If the delay value $\tau_{1}>\tau_{10}^{*}$, the the equilibrium point is unstable. Similarly, we find that the system undergoes a Hopf bifurcation at $(0,0)$ (see Figs. 5 and 6).


Fig. 1 The system (4) is asymptotically stable at $(0,0)$ when $\tau_{2}=5<\tau_{20}$




Fig. 2 The system (4) is unstable at $(0,0)$ when $\tau_{2}=7>\tau_{20}$


Fig. 3 The system (4) is asymptotically stable at $(0,0)$ when $\tau=2<\tau_{0}$


Fig. 4 The system (4) is unstable at $(0,0)$ when $\tau=3.5<\tau_{0}$


Fig. 5 The system (4) is asymptotically stable at $(0,0)$ when $\tau_{1}=3.5<\tau^{*}{ }^{*}$


Fig. 6 The system (4) is unstable at $(0,0)$ when $\tau_{1}=4.5>\tau^{*}$

## IV. CONCLUSIONS

According to the control and bifurcation theory, a Rayleigh price model with two delays was studied by this paper. Until now, there are few results concerned the price models with two time delays and we provide an insight to unexplored aspects of them. By applying the stability and bifurcation theory analysis, We discussed the effect of the delay on the system. Moreover, we derived the conditions for the stability and the existence of Hopf bifurcation at the equilibrium of the system. Some computer simulation results have been presented to illustrate the validity of the theoretical analysis. The research of this paper enriches and develops the study on Rayleigh price model.

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## REFERENCES

[1] J. Li, W. Xu, W. Xie, Z. Ren, Research on nonlinear stochastic dynamical price model, Chaos Solitons \& Fractals, vol. 37, no. 5, pp. 1391-1396, 2008.
[2] L. Tang-hong and L.H. Zhou, "Hopf and resonant codimension two bifurcation in price rayleigh equation with delays," Journal of Northeast Normal University (Natrual Science Edition), vol. 44, no. 4, pp. 43-49,2012.
[3] W. Shuhe, Differential equation model and chaos, Journal of China Science and Technology University, pp. 312-324, 1999.
[4] Z. Xi-fan, C. Xia, and C. Yun-qing, "A qualitative analysis of price model in differential equations of price," Journal of Shenyang Institute of Aeronautical Engineering, vol. 21, no. 1, pp. 83-86, 2004.
[5] T. Lv and Z. Liu, "Hopf bifurcation of price Reyleigh equation with time delay," Journal of Jilin University, vol. 47, no. 3, pp. 441-448, 2009.
[6] Y. Zhai, H. Bai, Y. Xiong, and X. Ma, Hopf bifurcation analysis for the modifed Rayleigh price model with time delay, Abstract and Applied Analysis, vol. 2013, Article ID 290497, 6 pages, 2013.
[7] Y. Kuang, Delay Differential Equations: With Applications in population Dynamics, Acdemic Press, New York, NY, USA, 1993.
[8] E. Beretta and Y. Kuang, Geometric stability switch criteria in delay differential systems with delay dependent parameters, SIAM Journal on Mathematical Analysis, 33 (5) (2002) 1144-1165.
[9] J. Hale, Theory of Functional Differential Equations, Springer, 1977.
[10] B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge University Press, Cambridge, 1981.
[11] D. Tao, X. Liao, T. Huang, Dynamics of a congestion control model in a wireless access network, Nonlinear Analysis: Real World Applications, vol. 14, no .1, pp. 671-683, 2013.
[12] D. Ding, J. Zhu, X.S. Luo, Delay induced Hopf bifurcation in a dual model of Internet congestion, Nonlinear Analysis: Real World Applications, vol. 10, no. 1, pp. 2873-2883, 2009.
[13] Y.G. Zheng, Z.H. Wang, Stability and Hopf bifurcation of a class of TCP/AQM networks, Nonlinear Analysis: Real World Applications, vol. 11, no. 3, pp. 1552-1559, 2010.

