On Certain Expansions for Generalized Multivariable Gimel-Function II

Frédéric Ayant
Teacher in High School , France

ABSTRACT
The object of the paper is to evaluate firstly six finite integrals involving the product of Jacobi polynomials and the generalized multivariable Gimel-function. We derive six expansion formulae for the generalized multivariable Gimel-function in series involving Jacobi polynomials. We also derive some interesting expansions for different functions.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, Jacobi polynomials, expansion series, finite integrals.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j = n_k+1,\cdots,p_{i_k}); (k=2,\cdots,r).$

$b_{kji_k} \in \mathbb{C}; (j = m_k+1,\cdots,q_{i_k}); (k=2,\cdots,r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=m^{(k)}+1,\cdots,q_{i^{(k)}}); (k=1,\cdots,r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=n^{(k)}+1,\cdots,p_{i^{(k)}}); (k=1,\cdots,r).$

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k\right)(j=1,\cdots,n_2), \Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha^{(k)}_{3j}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha^{(i)}_{rj}\right)(j=1,\cdots,n_r), \Gamma^{C^{(k)}_j}\left(1-c^{(k)}_j+\gamma^{(k)}_j s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k\right)(j=1,\cdots,m_2),\Gamma^{B_{3j}}\left(b_{3j}-\sum_{k=1}^{3}\beta^{(k)}_{3j}s_k\right)(j=1,\cdots,m_3)$ $,\cdots,\Gamma^{B_{rj}}\left(b_{rj}-\sum_{i=1}^{r}\beta^{(i)}_{rj}\right)(j=1,\cdots,m_r),\Gamma^{D^{(k)}_j}\left(d^{(k)}_j-\delta^{(k)}_j s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A^{(k)}_i\pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j\delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j\gamma^{(k)}_j - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}}\delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}}\gamma^{(k)}_{ji^{(k)}}\right)+$$

$$\sum_{j=1}^{n_2} A_{2j}\alpha^{(k)}_{2j} + \sum_{j=1}^{m_2} B_{2j}\beta^{(k)}_{2j} - \tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha^{(k)}_{2ji_2} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2}\beta^{(k)}_{2ji_2}\right)+\cdots+$$

$$\sum_{j=1}^{n_r} A_{rj}\alpha^{(k)}_{rj} + \sum_{j=1}^{m_r} B_{rj}\beta^{(k)}_{rj} - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha^{(k)}_{rji_r} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r}\beta^{(k)}_{rji_r}\right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \rightarrow \infty$ where $i = 1,\cdots,r$:

$\alpha_i = \min\limits_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta^{h'}_{hj}} + D^{(i)}_k\frac{d^{(i)}_k}{\delta^{(i)}_k}\right)$ and $\beta_i = \max\limits_{\substack{1\leqslant k\leqslant n_i\\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha^{h'}_{hj}} + C^{(i)}_k\frac{k^{(i)}_j-1}{\gamma^{(i)}_k}\right)$

Remark 1.
If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.
If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2= =\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1,$ then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [4]).

Remark 3.

If$A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2$ $=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [3].

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [7,8]. In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

We shall require following integrals due to author [6] for the development of the present work :

Lemma 1.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\sigma+\rho+1}\Gamma(1+\rho)\Gamma(1+\sigma+n)\Gamma(-n-\sigma)}{n!\Gamma(2+n+\sigma+\rho)\Gamma(1+n+\rho)}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)\Gamma(1+\rho+n+k)\Gamma(\alpha+\beta+k+n-\sigma-\rho)}{k!\Gamma(1+\alpha+\beta+n+k-\sigma)\Gamma(\alpha+k-\sigma-\rho)} \tag{1.1}$$

where : $Re(\rho+1)>0, Re(\sigma+1)>0, Re(-n-\sigma)>0, Re(1+\alpha)>0, Re(\alpha+\beta+n+k-\rho-\sigma)>0$

Lemma 2.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{(-)^n 2^{\sigma+\rho+1}\Gamma(1+\sigma)\Gamma(1+\rho+n)\Gamma(-n-\rho)}{n!\Gamma(2+n+\sigma+\rho)\Gamma(1+n+\sigma)}$$
$$\sum_{k=0}^{\infty}\frac{\Gamma(1+\beta+n+k)\Gamma(1+\sigma+n+k)\Gamma(\alpha+\beta+k+n-\sigma-\rho)}{k!\Gamma(1+\alpha+\beta+n+k-\rho)\Gamma(\beta+k-\sigma-\rho)} \tag{1.2}$$

where : $Re(\rho+1)>0, Re(\sigma+1)>0, Re(-n-\rho)>0, Re(-\rho)>0, Re(1+\beta)>0, Re(\alpha+\beta+n+k-\rho-\sigma)>0$

Lemma 3.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\sigma+\rho+1}\Gamma(1+\sigma)\Gamma(1+\rho)}{n!\Gamma(1+n+\rho)}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)\Gamma(1+\rho+n+k)\Gamma(1-\beta+n+\sigma)}{k!\Gamma(2+k+\sigma+\rho)\Gamma(2+n+k+\alpha+\sigma)} \tag{1.3}$$

where : $Re(\rho+1)>0, Re(\sigma+1)>0, Re(1+\alpha)>0, Re(1-\beta+n+\sigma)>0$

Lemma 4.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\sigma+\rho+1}\Gamma(1+\sigma)\Gamma(1+\rho)}{n!\Gamma(1+n+\rho)}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1+\beta+n+k)\Gamma(1+\sigma+n+k)\Gamma(1-\alpha+k+\rho)}{k!\Gamma(2+k+\sigma+\rho)\Gamma(2+n+k+\beta+\rho)} \tag{1.4}$$

where : $Re(\rho+1)>0, Re(\sigma+1)>0, Re(1+\beta)>0, Re(1-\alpha+k+\rho)>0$

Lemma 5.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\sigma+\rho+1}\Gamma(1+\sigma)\Gamma(1+\rho+n)\Gamma(1+\rho)}{n!\Gamma(2+n+\rho+\sigma)\Gamma(-1-\rho-\sigma)}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)\Gamma(-1-\rho-\sigma+k)\Gamma(1-\beta+k+\sigma)}{k!\Gamma(-\beta-n-\rho+k)\Gamma(-\alpha-\beta-n+k+\sigma)} \tag{1.5}$$

where : $Re(1+\alpha+\beta+2n)>0, Re(-\alpha-\beta-2n)>0, Re(1+\sigma)>0, Re(1+\rho)>0, Re(-\rho-n)>0,$

$Re(-\alpha-\beta-n+k)>0, Re(-1-\rho-\sigma+k)>0, Re(1-\beta+\sigma+k)>0$

Lemma 6.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{(-)^n 2^{\sigma+\rho++1}\Gamma(1+\rho)\Gamma(1+\sigma+n)\Gamma(-\sigma-n)}{n!\Gamma(2+n+\rho+\sigma)\Gamma(-1-\rho-\sigma)}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)\Gamma(-1-\rho-\sigma+k)\Gamma(1-\beta+k+\rho)}{k!\Gamma(-\beta-n-\sigma+k)\Gamma(-\alpha-\beta-n+k+\rho)} \tag{1.6}$$

where : $Re(1+\alpha+\beta+2n)>0, Re(-\alpha-\beta-2n)>0, Re(1+\sigma)>0, Re(1+\rho)>0, Re(-\sigma+n)>0,$

$Re(-\alpha-\beta-n+k)>0, Re(-1-\rho-\sigma+k)>0, Re(1-\beta+\rho+k)>0$

3. Main integrals.

In this section, we shall prove the following six finite integrals involving product of Jacobi polynomials and generalized multivariable Gimel-function.

Theorem 1.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\daleth\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)\mathrm{d}x=\frac{2^{\sigma+\rho+1}}{n!}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)}{k!}\daleth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+5,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{a_1+b_1}z_1 & \mathbb{A};\ (\text{-}\rho;a_1,\cdots,a_r;1),(-\sigma-n;b_1,\cdots,b_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r+b_r}z_r & \mathbb{B};\ (\alpha+\beta+k+n-\rho+\sigma;a_1+b_1,\cdots,a_r+b_r;1) \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-n-k-}\rho;a_1,\cdots,a_r;1),\mathbf{A},(1+\alpha+\beta+n+k-\sigma;a_1+b_1,\cdots,a_r+b_r;1)(n+k-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1):A \\ \cdot \\ \cdot \\ (\text{-n-}\sigma;b_1,\cdots,b_r;1),\mathbf{B},(-1-\rho-\sigma-n;a_1+b_1,\cdots,a_r+b_r;1),(-n-\rho;a_1,\cdots,a_r;1):B \end{array}\right) \tag{3.1}$$

provided

$$a_i,b_i>0(i=1,\cdots,r),\ Re(1+\alpha)>0,\ Re\left(-n-\sigma\right)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\left(\alpha+\beta+k+n-\rho-\sigma\right)-\sum_{i=1}^{r}(a_i+b_i)\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\left(1+\rho\right)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\left(1+n+\sigma\right)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\left(1+n+k+\rho\right)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove (2.1), on the left hand side of (2.1), expressing the generalized multivariable Gimel-function as Mellin-Barnes multiple integrals contour with the the help of (1.1), interchanging the order of summation and integration which is justified under the conditions mentioned above, evaluating the inner integral with the help of the lemma 1 and finally interpreting the Mellin-Barnes multiple integrals contour in terms of the generalized multivariable Gimel-function, we get the desired result (3.1).

Theorem 2.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\gimel\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)\mathrm{d}x=\frac{2^{\sigma+\rho+1}}{n!}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1+\beta+n+k)}{k!}\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{a_1+b_1}z_1 & \mathbb{A};\ (\text{-}\rho;a_1,\cdots,a_r;1),(-\rho-n;a_1,\cdots,a_r;1),(-n-k-\sigma;b_1,\cdots,b_r;1),\mathbf{A}, \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r+b_r}z_r & \mathbb{B};\ (\text{-n-}\rho;a_1,\cdots,a_r;1),(\alpha+\beta+n+k-\rho;a_1+b_1,\cdots,a_r+b_r;1),\mathbf{B} \end{array}\right.$$

$$\left.\begin{array}{c} (1+\alpha+\beta+n+k-\rho;a_1,\cdots,a_r;1),(\beta+k-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1):A \\ \cdot \\ \cdot \\ (\text{-n-}\sigma;b_1,\cdots,b_r;1),(-1-n-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1):B \end{array}\right) \tag{3.2}$$

provided

$a_i,b_i>0(i=1,\cdots,r),\ Re(1+\beta)>0,$

$$Re\,(1+n+k+\sigma)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+\rho+n)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\,(-n-\rho)-\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(\alpha+\beta+n+k-\rho-\sigma)-\sum_{i=1}^{r}(a_i+b_i)\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+\sigma)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 3.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\gimel\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)\mathrm{d}x=\frac{2^{\sigma+\rho+1}}{n!}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1+\beta+n+k)}{k!}\gimel^{U;m_r,n_r+4:V}_{X;p_{i_r}+5,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{a_1+b_1}z_1 & \mathbb{A};\ (\text{-}\rho;a_1,\cdots,a_r;1),(-\sigma;b_1,\cdots,b_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r+b_r}z_r & \mathbb{B};\mathbf{B},\ (\text{-1-k-}\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),(1-\beta+k+\sigma;b_1,\cdots,b_r;1), \end{array}\right.$$

$$\left.\begin{matrix} \text{(-n-k-}\rho; a_1,\cdots,a_r;1),\beta-k-\sigma;b_1,\cdots,b_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ \text{(-1-k-}\sigma-\rho;a_1+b_1,\cdots,a_r+b_r;1),(-1-n-k-\sigma;b_1,\cdots,b_r;1):B \end{matrix}\right) \tag{3.3}$$

provided

$a_i,b_i>0(i=1,\cdots,r)$, $Re(1+\alpha)>0$

$$Re\,(1+\rho)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\,(1-\beta+k+\sigma)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+\sigma)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 4.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\daleth\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)\mathrm{d}x=\frac{2^{\sigma+\rho+1}}{n!}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1+\beta+n+k)}{k!}\daleth_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r,n_r+4:V}\left(\begin{matrix}2^{a_1+b_1}z_1\\ \cdot\\ \cdot\\ 2^{a_r+b_r}z_r\end{matrix}\ \middle|\ \begin{matrix}\mathbb{A};\text{(-}\sigma;b_1,\cdots,a_r;1),(-\rho;a_1,\cdots,a_r;1),\\ \cdot\\ \cdot\\ \mathbb{B};\mathbf{B},\text{(-n-}\rho;b_1,\cdots,b_r;1),(1+\sigma;b_1,\cdots,b_r;1),\end{matrix}\right.$$

$$\left.\begin{matrix}\text{(-n-k-}\sigma;b_1,\cdots,b_r;1),(\alpha-k-\rho;a_1,\cdots,a_r;1),\mathbf{A}:A\\ \cdot\\ \cdot\\ \text{(-1-k-}\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),(-1-\beta-n-k-\sigma;a_1,\cdots,a_r;1):B\end{matrix}\right)$$

provided

$a_i,b_i>0(i=1,\cdots,r)$, $Re(1+\beta)>0$

$$Re\,(1+\sigma)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\,(1-\alpha+k+\rho)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+\rho)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 5.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\gimel\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)\mathrm{d}x = \frac{2^{\sigma+\rho+1}}{n!}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)}{k!}\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+4,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} 2^{a_1+b_1}z_1 & \mathbb{A};\,(\beta-\sigma-k;b_1,\cdots,b_r;1),\mathbf{A},(-1-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r+b_r}z_r & \mathbb{B};\,(\text{-1-}\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),(-\beta-n+k-\rho;a_1,\cdots,a_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\beta-n+k-\rho;a_1,\cdots,a_r;1),(-1-n-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1):A \\ \cdot \\ \cdot \\ (\text{-1+k-}\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),\mathbf{B},(1+\alpha+\beta+n-k-\sigma;b_1,\cdots,b_r;1):B \end{array}\right) \quad (3.5)$$

provided

$a_i,b_i>0(i=1,\cdots,r),Re(1+\alpha+\beta)>0,Re(-\alpha-\beta-2n)>0,Re(-\alpha-\beta-n+k)>0$

$$Re(1+n+\rho)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\,(1+\sigma)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(-1+k-\rho-\sigma)-\sum_{i=1}^{r}(a_i+b_i)\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1-\beta+k+\sigma)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(-n-\rho)-\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 6.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_u^{(\alpha,\beta)}(x)\gimel\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)\mathrm{d}x = \frac{(-)^n2^{\sigma+\rho+1}}{n!}$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)}{k!}\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{a_1+b_1}z_1 & \mathbb{A};\ (\text{-n-}\sigma;b_1,\cdots,b_r;1),(-\rho;a_1,\cdots,a_r;1),(\beta-k-\rho;a_1,\cdots,a_r;1),\mathbf{A}, \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r+b_r}z_r & \mathbb{B};\ (\text{-n-}\sigma;b_1,\cdots,b_r;1),(-1+k+\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-1-}\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),(-\beta-n+k-\sigma;b_1,\cdots,b_r;1):A \\ \cdot \\ \cdot \\ (\text{-1-n-}\rho-\sigma;a_1b_1,\cdots,a_r+b_r;1),(1+\alpha+\beta+n-k-\rho;a_1,\cdots,a_r;1):B \end{array}\right) \tag{3.6}$$

provided

$a_i,b_i>0(i=1,\cdots,r), Re(1+\alpha+\beta)>0, Re(-\alpha-\beta-2n)>0, Re(-\alpha-\beta-n+k)>0,$

$$Re\,(\rho+1)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\,(-\rho-\sigma-1+k)+\sum_{i=1}^{r}(a_i+b_i)\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(-n-\sigma)-\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1-\beta+\rho+k)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+\sigma+n)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4)

To prove the theorems 2 to 6, we use the similar lines to the formula (3.1) by using the formulae (2.2), to (2.6) respectively, instead of result (2.1).

4. Expansion formulae.

In this section, we shall derive six expansion formulae.

Theorem7.

$$(1-x)^{\rho}(1+x)^{\sigma}\gimel\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)=$$

$$\sum_{n,k=0}^{\infty}\frac{2^{\sigma+\rho}\Gamma(1+\alpha+n+k)(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}{k!\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} 2^{a_1+b_1}z_1 & \mathbb{A};\ (\text{-n-}\sigma;b_1,\cdots,b_r;1),(k+n-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r+b_r}z_r & \mathbb{B};\ (\text{-n-}\beta+\sigma;b_1,\cdots,b_r;1),(k+n-\rho-\sigma;a_1-b_1,\cdots,a_r-b_r;1) \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-n-k-}\alpha-\rho;a_1,\cdots,a_r;1),\mathbf{A},(k+n-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),((k-\beta-\sigma-\rho;a_1+b_1,\cdots,a_r+b_r;1):A \\ \cdot \\ \cdot \\ \mathbf{B},(\text{-1-}\alpha-\beta-\rho-\sigma-n;a_1+b_1,\cdots,a_r+b_r;1),(-n-\alpha-\rho;a_1,\cdots,a_r;1):B \end{array}\right) \tag{4.1}$$

provided

$a_i,b_i>0(i=1,\cdots,r)$, $Re(1+\alpha)>0$, $Re(1+\alpha+\beta)>0$

$$Re\,(k+n-\rho-\sigma)-\sum_{i=1}^{r}(a_i+b_i)\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\,(-n-\sigma)-\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+\rho+\alpha)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+n+\beta+\sigma)-\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

Let

$$(1-x)^{\rho}(1+x)^{\sigma}\gimel\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r((1-x)^{a_r}(1+x)^{b_r}\right)=\sum_{R=0}^{\infty}A_RP_R^{(\alpha,\beta)}(x) \tag{4.2}$$

The above equation is valid since the expression on the left-hand side is continuous and bounded variation in the interval $(-1,1)$. Multiplying both sides of (4.2) by $(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)$ and integrating with respect to x from 1 to 1. On the left hand side using the theorem 1 and on the righthand side, changing the order of summations and integrations which is justified under the conditions mentioned above, and then using the orthogonality property of jacobi polynomials ([5], p. 258), we get

$$C_n=\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}\sum_{k=0}^{\infty}\frac{\Gamma(1+\alpha+n+k)}{k!}\gimel^{*}\left(2^{a_1+b_1}z_1,\cdots,2^{a_r+b_r}z\right) \tag{4.3}$$

where $\gimel^{*}\left(2^{a_1+b_1}z_1,\cdots,2^{a_r+b_r}z_r\right)$ is a generalized multivariable Gimel-function defined on the right hand side of (4.1). Substituting (4.3) in (4.2) , we obtain the theorem 1.

Theorem 8.

$(1-x)^{\rho}(1+x)^{\sigma}\gimel\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)=$

$$\sum_{n,k=0}^{\infty}\frac{2^{\sigma+\rho}(1+\alpha+\beta+2n)\Gamma(1+\beta+n+k)\Gamma(1+\alpha+\beta+n)}{k!\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)\daleth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+4,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|l} 2^{a_1+b_1}z_1 & \mathbb{A};\ (\text{-}\rho-\beta;b_1,\cdots,b_r;1),(-\rho-\alpha-n;a_1,\cdots,a_r;1),(-n-k-\alpha-\rho;a_1,\cdots,a_r;1),\mathbf{A}, \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r+b_r}z_r & \mathbb{B};\ (\text{-n-}\alpha;a_1,\cdots,a_r;1),(k+n-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),\mathbf{B},(-n-\beta;b_1,\cdots,b_r;1), \end{array}\right.$$

$$\left.\begin{array}{l} (1{+}\beta+n+k-\rho;a_1,\cdots,a_r;1),(k-\alpha-\sigma-\rho;a_1+b_1,\cdots,a_r-b+r;1):A \\ \cdot \\ \cdot \\ (1{+}\beta+\sigma;b_1,\cdots,b_r;1),\mathbf{B},(-1-\alpha-\beta-\rho-\sigma-n;a_1+b_1,\cdots,a_r+b_r;1):B \end{array}\right) \tag{4.4}$$

provided

$a_i,b_i>0(i=1,\cdots,r)$, $Re(1+\alpha)>0$, $Re(1+\beta)>0$

$$Re\,(k+n-\rho-\sigma)-\sum_{i=1}^{r}(a_i+b_i)\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\,(-n-\alpha-\rho)-\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+n+\alpha+\rho)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+\beta+\sigma)+\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 9.

$(1-x)^{\rho}(1+x)^{\sigma}\daleth\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right)=$

$$\sum_{n,k=0}^{\infty}\frac{2^{\sigma+\rho}(1+\alpha+\beta+2n)\Gamma(1+\alpha+n+k)\Gamma(1+\alpha+\beta+n)}{k!\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$\daleth^{U;m_r,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|l} 2^{a_1+b_1}z_1 & \mathbb{A};\ (\text{-}\sigma-\beta;b_1,\cdots,b_r;1),(-\alpha-\rho;a_1,\cdots,a_r;1),(-n-k-\rho-;a_1,\cdots,a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r+b_r}z_r & \mathbb{B};\mathbf{B},\ (\text{-n-}\alpha-\rho;a_1,\cdots,a_r;1),(-1-k-\alpha-\beta-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1), \end{array}\right.$$

$$\left.\begin{array}{l} (\text{-k-}\sigma;b_1,\cdots,b_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (\text{-1-}\alpha-\beta-n-k-\rho;b_1,\cdots,a_b;1):B \end{array}\right) \tag{4.5}$$

provided

$a_i, b_i > 0 (i = 1, \cdots, r), Re(1+\alpha) > 0, Re(1+\beta) > 0$

$$Re\,(1+\alpha+\rho) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$$Re\,(1+\beta+\sigma) + \sum_{i=1}^{r} b_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$$Re\,(1+k+\sigma) + \sum_{i=1}^{r} b_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 10.

$$(1-x)^{\rho}(1+x)^{\sigma}\beth\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right) =$$

$$\sum_{n,k=0}^{\infty}\frac{2^{\sigma+\rho}(1+\alpha+\beta+2n)\Gamma(1+\beta+n+k)\Gamma(1+\alpha+\beta+n)}{k!\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}P_n^{(\alpha,\beta)}(x)$$

$$\beth^{U;m_r,n_r+4+:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c} 2^{a_1+b_1}z_1 \\ \cdot \\ \cdot \\ 2^{a_r+b_r}z_r \end{array}\left|\begin{array}{c} \mathbb{A};\,(\text{-}\beta-\sigma;b_1,\cdots,b_r;1),(-k-\rho;a_1,\cdots,a_r;1),(-\rho-\alpha;a_1,\cdots,a_r;1), \\ \cdot \\ \cdot \\ \mathbb{B};\,\mathbf{B},\,(\text{-n-}\beta-\sigma;b_1,\cdots,b_r;1),(-1-n-\beta-\rho;a_1,\cdots,a_r;1), \end{array}\right.\right.$$

$$\left.\begin{array}{c} (\text{n-k-}\beta-\sigma;b_1,\cdots,b_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (\text{-1-k-}\alpha-\beta-\rho-\sigma;a_1+b,\cdots,a_r+b_r;1):B \end{array}\right) \tag{4.6}$$

provided

$a_i, b_i > 0 (i = 1, \cdots, r), Re(1+\alpha), Re(1+\beta) > 0$

$$Re\,(1+\alpha+\rho) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0\,,$$

$$Re\,(1+\beta+\sigma) + \sum_{i=1}^{r} b_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 11.

$$(1-x)^{\rho}(1+x)^{\sigma}\beth\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right) =$$

$$\sum_{n,k=0}^{\infty} \frac{2^{\sigma+\rho}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(-\alpha-\beta-n+k)}{k!\Gamma(1+\alpha+n)\Gamma(1+\beta+n)} P_n^{(\alpha,\beta)}(x)$$

$$\daleth_{X;p_{i_r}+5,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;m_r+3,n_r+3:V} \left(\begin{array}{c} 2^{a_1+b_1} z_1 \\ \cdot \\ \cdot \\ 2^{a_r+b_r} z_r \end{array} \middle| \begin{array}{c} \mathbb{A};\ (\text{-n-}\alpha-\rho; a_1,\cdots,a_r;1), (-\beta-\sigma; b_1,\cdots,b_r;1), (-\sigma-k; b_1,\cdots,b_r;1), \mathbf{A}, \\ \cdot\ \cdot \\ \mathbb{B};\ (\text{-}\rho-\alpha-n; a_1,\cdots,a_r;1), (-1+k-\alpha-\beta-\rho-\sigma; a_1+b_1,\cdots,a_r+b_r;1), \mathbf{B}, \end{array} \right.$$

$$\left. \begin{array}{c} (\text{-1-}\alpha-\beta-\rho-\sigma; a_1+b_1,\cdots,a_r+b_r;1), (-\alpha-\beta-\rho-n+k; a_1,\cdots,a_r;1) : A \\ \cdot \\ \cdot \\ (\text{-1-}\alpha-\beta-\rho-\sigma; a_1+b_1\cdots,a_r+b_r;1), (1+\alpha-+n-k-\sigma; b_1,\cdots,b_r;1) : B \end{array} \right) \tag{4.7}$$

provided

$a_i, b_i > 0 (i=1,\cdots,r),\ Re(1+\alpha+\beta+n) > 0, Re(-\alpha-\beta-n+k) > 0$

$$Re\,(-\alpha-n-\sigma) - \sum_{i=1}^{r} b_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)} \frac{d_k^{(i)}}{\delta_k^{(i)}} \right) > 0$$

$$Re\,(-1-k+\alpha-\beta-\rho-\sigma) - \sum_{i=1}^{r} (a_i+b_i) \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)} \frac{d_k^{(i)}}{\delta_k^{(i)}} \right) > 0\,,$$

$$Re\,(1+\beta+\sigma+n) + \sum_{i=1}^{r} b_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)} \frac{d_k^{(i)}}{\delta_k^{(i)}} \right) > 0$$

$$Re\,(1-\beta+\alpha+k+\rho) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)} \frac{d_k^{(i)}}{\delta_k^{(i)}} \right) > 0$$

$$Re\,(1+\rho+\alpha) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)} \frac{d_k^{(i)}}{\delta_k^{(i)}} \right) > 0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right| < \frac{1}{2} A_i^{(k)} \pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 12.

$(1-x)^{\rho}(1+x)^{\sigma} \daleth\left(z_1(1-x)^{a_1}(1+x)^{b_1},\cdots,z_r(1-x)^{a_r}(1+x)^{b_r}\right) =$

$$\sum_{n,k=0}^{\infty} \frac{2^{\sigma+\rho}(-)^n(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\alpha+\beta-n+k)}{k!\Gamma(1+\alpha+n)\Gamma(1+\beta+n)} P_n^{(\alpha,\beta)}(x)\ \daleth_{X;p_{i_r}+5,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+3:V}$$

$$\left(\begin{array}{c} 2^{a_1+b_1} z_1 \\ \cdot \\ \cdot \\ 2^{a_r+b_r} z_r \end{array} \middle| \begin{array}{c} \mathbb{A};\ (-\rho-\alpha; a_1,\cdots,a_r;1), (-\beta-\sigma-n; b_1,\cdots,b_r;1), (\beta-\alpha-k-\rho; a_1,\cdots,a_r;1), \mathbf{A}, \\ \cdot \\ \cdot \\ \mathbb{B};\ (\text{-1-}\alpha-n-\sigma; b_1,\cdots,b_r;1), (-1-\alpha-\beta+k-\rho-\sigma; a_1+b_1,\cdots,a_r+b_r;1), \mathbf{B}, \end{array} \right.$$

$$\left.\begin{array}{c}(\text{-}1\text{-}\alpha-\beta-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1),(-2\beta-n+k-\sigma;b_1,\cdots,b_r;1):A\\ \vdots \\ (\text{-}1\alpha-\beta-n-\rho-\sigma;a_1+b_1,\cdots,a_r+b_r;1)(1+\beta+n-k-\rho;a_1,\cdots,a_r;1):B\end{array}\right) \tag{4.8}$$

provided
$a_i,b_i>0(i=1,\cdots,r)$,

$$Re\,(\alpha+\rho+1)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0\,,$$

$$Re\,(-1-\alpha-\beta+k-\rho-\sigma)-\sum_{i=1}^{r}(a_i-b_i)\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(-n-\beta-\sigma)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+\sigma+n+\beta)-\sum_{i=1}^{r}b_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1-\beta+\alpha+k+\rho)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant k\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$\left|arg\left(z_i(1-x)^{a_i}(1+x)^{b_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and

$Re(-\alpha-\beta-n+k)>0, Re(1+\alpha+\beta)>0.$

To prove the theorems 8 to 12, we use the similar lines to the theorem 7 by using the theorems 2 to 6 respectively, instead of theorem 1.

Remark 6.
If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then we can obtain the same expansion formulae in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1])

Remark 7.
If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2==\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then we can obtain the same expansion formulae in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [4]).

Remark 8.
If$A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then we can obtain the same expansion formulae in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [3]).

Remark 9.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [7,8] and then we can obtain the same expansion formulae.

5. Conclusion.

The importance of our results lie in their manifold generality. Firstly, in view of Jacobi polynomials making use of special cases, they can be reduced to a large number of formulae involving simpler special functions (ultraspherical -Gegenbauer, Legendre, Tchebyshev, Bateman's, Hermite, Laguerre polynomials and others). Secondly, by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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