# Finie Triple Series Relation Involving the Multivariable Gimel-Function

Frédéric Ayant

Teacher in High School , France

ABSTRACT
The aim of this paper is to establish four finite triple series relation involving the multivariable Gimel-function defined here. These relations aer quite general in nature, from which a large number of relations can be obtained simply by specializing the parameters of the multivariable Gimel-function.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, finite triple series.

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## 1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad\cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha^{(k)}_{3j}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha^{(k)}_{3ji_3}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta^{(k)}_{3ji3}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

The contour $L_k$ is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where $\sigma$ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2), \Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r), \Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour $L_k$ and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour $L_k$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)}\gamma_j^{(k)} - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right) +$$

$$-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2}\beta_{2ji_2}^{(k)}\right) - \cdots - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.4}$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \to 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \to \infty$ where $i = 1,\cdots,r$ :

$$\alpha_i = \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] \text{ and } \beta_i = \max_{1\leqslant j\leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$$

**Remark 1.**
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

**Remark 2.**
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [6].

**Remark 3.**
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [5].

**Remark 4.**
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [8,9].

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \quad . \tag{1.5}$$

$$\mathbf{A} = [(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$B = [(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

## 2. Main results.

In this section, we establish four finite triple relations about the multivariable Gimel-function.

**Theorem 1.**

$$\sum_{s=0}^{M}\sum_{t=0}^{N}\sum_{u=0}^{P}\frac{(-M)_s(-N)_t(-P)_u}{s!t!u!}\gimel_{X;p_{i_r}+5,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1\text{-}a\text{-}t\text{-}u;a_1,\cdots,a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B},\ (1\text{-}c\text{-}s\text{-}t\text{-}u;\ c_1,\cdots,c_r;1), \end{array}\right.$$

$$(1\text{-}c\text{+}a\text{-}s;c_1-a_1,\cdots,c_r-a_r;1),(1-c+b-s-u;c_1-b_1,\cdots,c_r-b_r;1),(1-b-t;b_1,\cdots,b_r;1),\mathbf{A},$$

$$(d\text{+}M\text{-}s;c_1-a_1-b_1,\cdots,c_r-a_r-b_r;1),(b+\beta-a-u;a_1-b_1,\cdots,a_r-b_r;1)$$

$$\left.\begin{array}{c} (1\text{+}d\text{-}N\text{+}t;c_1-a_1-b_1,\cdots,c_r-a_r-b_r;1):A \\ \cdot \\ \cdot \\ :B \end{array}\right) = (-)^{M+N+P}\gimel_{X;p_{i_r}+5,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};(1\text{-}a\text{-}M;a_1,\cdots,a_r;1),(1-c+a-N-P;c_1-a_1,\cdots c_r-a_r;1),(1-b-M-P;b_1,\cdots,b_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B},\ (1\text{-}c\text{-}M\text{-}N\text{-}P;\ c_1,\cdots,c_r;1),(d;c_1-a_1-b_1,\cdots,c_r-a_r-b_r;1), \end{array}\right.$$

$$(1+b-c-N;c_1-b_1,\cdots,c_r-b_r;1),\mathbf{A},(1+d;c_1-a_1-b_1,\cdots,c_r-a_r-b_r;1):A \\ \cdot \\ \cdot \\ ((b-a;a_1-b_1,\cdots,a_r-b_r;1):B \Bigg) \tag{2.1}$$

Provided that

$d=a+b-c$ is not an integer, $c_i>b_i>a_i>0, c_i-a_i-b_i>0(i=1,\cdots,r)$

$|arg\,(z_i)|\;<\frac{1}{2}A_i^{(k)}\pi$ , where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove the theorem 1, expressing the multivariable Gimel-function in multiple integrals contour with the help of (1.1) , interchanging the order of summations and integrations which is justified as the triple serie involved is finite and evaluating the inner triple series with the help of result due to Pradhan ([4], p. 33) :

$$\sum_{s=0}^{M}\sum_{t=0}^{N}\sum_{u=0}^{P}\frac{(-M)_s(-N)_t(-P)_u(a)_{t+u}(c-a)_s(c-b)_{s+u}(b)_s}{s!t!u!(c)_{s+t+u}(1-D-M)_s(1+d-N)_t(1+a-b-P)_u}=\frac{(a)_M(c-a)_{N+P}(b)_{M+P}(c-b)_N}{(d)_N(b-a)_P(c)_{M+N+P}} \tag{2.2}$$

where $d=a+b-c$ is not an integer

The right-hand side of (2.1) id obtained, now interpreting the resulting Mellin-Barnes multiple integrals contour as the multivariable Gimel-function with the help of (1.1), we obtain the desired theorem 1.

**Theorem 2.**

$$\sum_{s=0}^{M}\sum_{t=0}^{N}\sum_{u=0}^{P}\frac{(-M)_s(-N)_t(-P)_u(a)_{t+u}}{s!t!u!}\gimel^{U;0,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1\text{-c+a-s};c_1,\cdots,c_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(1\text{-c-s-t-u};c_1,\cdots,c_r;1), \end{array}\right.$$

$$(1\text{-b-t};b_1,\cdots,b_r;1),(1-c+b-s-u;c_1-b_1,\cdots,c_r-b_r;1),\mathbf{A},(1+a-b+P+u;b_1,\cdots,b_r;1):A \\ \cdot \\ \cdot \\ (d+M+s;c_1-b_1,\cdots,c_r-b_r;1),(1+d-N+t;c_1-b_1,\cdots,c_r-b_r;1):B \Bigg)$$

$$=(-)^{M+N+P}\gimel^{U;0,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1\text{-c+a-N-P};c_1,\cdots c_r;1),(1-b-M-P;b_1,\cdots,b_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(1\text{-c-M-N-P};c_1,\cdots,c_r;1),(1+d-N;c_1-b_1,\cdots,c_r-b_r;1), \end{array}\right.$$

$$(1+b-c-N;c_1-b_1,\cdots,c_r-b_r;1),\mathbf{A},(1+d;c_1-b_1,\cdots,c_r-b_r;1),(1+a-b;b_1,\cdots,b_r;1):A \\ \cdot \\ \cdot \\ :B \Bigg) \tag{2.3}$$

Provided that

$d=a+b-c$ is not an integer, $a_i,b_i,c_ic_i-b_i>0(i=1,\cdots,r)$

$|arg\,(z_i)|\;<\frac{1}{2}A_i^{(k)}\pi$ , where $A_i^{(k)}$ is defined by (1.4).

**Theorem 3.**

$$\sum_{s=0}^{M}\sum_{t=0}^{N}\sum_{u=0}^{P}\frac{(-M)_s(-N)_t(-P)_u(b)_t}{s!t!u!}\gimel_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}\left(\begin{array}{c|c} z_1 & \mathbb{A};(\text{1-a-t-u};a_1,\cdots,a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(\text{1-c-s-t-u};c_1,\cdots,c_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} (\text{1-c+a-s};c_1-a_1,\cdots,c_r-a_r;1),(1-c+b-s-u;c_1,\cdots,c_r;1),\mathbf{A},(1+d-N-t;c_1-a_1,\cdots,c_r-a_r;1):A \\ \cdot \\ \cdot \\ (\text{d+M-s};c_1-a_1,\cdots,c_r-a_r;1),(b+P-a-u;c_1-b_1,\cdots,c_r-a_r;1):B \end{array}\right)$$

$$=(-)^{M+N+P}(b)_{M+P}\gimel_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}\left(\begin{array}{c|c} z_1 & \mathbb{A};(\text{1-a-M};a_1,\cdots a_r;1),(1-c+a-N-P;c_1-a_1,\cdots,c_r-a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(\text{1-c-M-N-P};c_1,\cdots,c_r;1),(d;c_1-a_1,\cdots,c_r-a_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-c+b-N};c_1,\cdots,c_r;1),\mathbf{A},(1+d;c_1-a_1,\cdots,c_r-a_r;1):A \\ \cdot \\ \cdot \\ (\text{b-a};a_1,\cdots,a_r;1):B \end{array}\right) \tag{2.4}$$

Provided that

$d=a+b-c$ is not an integer, $a_i,b_i,c_ic_i-a_i>0(i=1,\cdots,r)$

$|arg\,(z_i)|\ <\dfrac{1}{2}A_i^{(k)}\pi$ , where $A_i^{(k)}$ is defined by (1.4).

**Theorem 4.**

$$\sum_{s=0}^{M}\sum_{t=0}^{N}\sum_{u=0}^{P}\frac{(-M)_s(-N)_t(-P)_u(b)_t(c-b)_{s+u}(c-a)_s}{s!t!u!(1+a-b-P)_u}\gimel_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c} z_1 & \mathbb{A};(\text{1-a-t-u};a_1,\cdots,a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (\text{1-b-t};c_1,\cdots,c_r;1),\mathbf{A},(1-d-M+s;c_1,\cdots,c_r;1):A \\ \cdot \\ \cdot \\ (\text{1-c-s-t-u};c_1,\cdots,c_r;1),(-d+N-t;c_1,\cdots,ac_r;1):B \end{array}\right)=(-)^{M+N}\frac{(c-a)_{N+P}(c-b)_N}{(b-a)_P}\gimel_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};(\text{1-a-M};c_1,\cdots c_r;1),(1-b-M-P;c_1,\cdots,c_r;1),\mathbf{A},(1-d;c_1,\cdots,c_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(\text{1-c-M-N-P};c_1,\cdots,c_r;1),(-d;c_1,\cdots,c_r;1):B \end{array}\right) \tag{2.5}$$

Provided that

$d=a+b-c$ is not an integer, $a_i,b_i,c_ic_i-a_i>0(i=1,\cdots,r)$

$|arg\,(z_i)| < \frac{1}{2}A_i^{(k)}\pi$ , where $A_i^{(k)}$ is defined by (1.4).

To prove the theorems 2 to 4, we use the similar methods that theorem 1.

**Remarks :**
We obtain the same triple finite series relations with the functions cited in the section I.

Bohara et al. [2] have obtained the same relations concerning the multivariable H-function [8,9], Saxena et al. [7] have obtained the same relations concerning the multivariable I-function defined by Prasad [5].

## 3. Conclusion.

The importance of our triple finite series formulae lies in their manifold generality. By specializing the various parameters and variables involved in the generalized multivariable Gimel-function, we get a several fractional integral formulae involving in remarkably wide variety of useful function (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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