On Minimal Hyperconnectedness

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Abstract
We introduce and study new a type of hyperconnectedness namely, minimal hyperconnectedness(m-hyperconnectednes). Several characterizations of minimal hyperconnected spaces and conditions under which m - hyperconnectedness is preserved are provided.

Key words: *minimal space,* m *- hyperconnectedness,* $m-$ *dense , m-nowhere dense*

I. INTRODUCTION AND PRELIMINARIES

Popa and Noiri[1] introduce the notion of minimal space . Noiri [2, 3] studied hyperconnected spaces. Ekici [4] introduced and studied hyperconnectedness in generalized topological spaces which is also studied in [5]. The purpose of the present paper
is to introduce and study the notion of m - hyperconnectedness in minimal spaces.

Characterizations and properties of *m*- hyperconnectedness are investigated. Some preservation theorems are given.

Definition 1.1. [1] Let X be a nonempty set. A subset $M \subset P(X)$ is called minimal structure if $X, \phi \in M$. The pair (X, *M*) is called minimal space (*m*- space). The members of *M* are called m-open sets and the complement of *m* - open set is *m* - closed set. The collection of all *m*- closed sets will be denoted by M^c .

Definition 1.2. Let (X,τ) be a topological space and A be a subset of X. The closure and the interior of A will be denoted by cl(A) and int(A). A set A is called α - open[6] (resp. preopen[7], semi- open[8], b- open[9], β - open[10]) if A ⊂ int(cl(int(A)) (resp. A ⊂ int(cl(A)), A ⊂ cl(int(A)), A ⊂ int(cl(A)) ∪ cl(int(A)), A ⊂ cl(int(cl(A))). The complement of the each set are called it respective closed set.
The family of all α - open (resp. pre- open, semi-open, b- open, β -open) sets are denoted by $\alpha(X)$, *PO*(X), *SO*(X), *BO*(X) and β O(X) respectively, *P*(X) is the power set of X, for a nonempty set X.

Remark 1.3. The collections τ , α O(*X*), *PO*(*X*), *SO*(*X*), *BO*(*X*) and β O(*X*) are minimal structure.

Definition 1.4 [1]. Let (X, M) be an *m*- space and A is a subset of X, then the *m*-interior and *m*-closure of A are defined respectively as follows,

(1) $\text{int}^m(A) = \bigcup\{U \in M : U \subseteq A\}$.

(2) $cl^m(A) = \bigcap\{F \in M^c : A \subset F\}$.

Theorem 1.5 [11] Let (X, M) be a *m*-space and $A \subset X$, then

(1) $\text{int}^m(A) \subseteq A$.

(2) $\text{int}^m(A) = A$ if $A \in M$.

(3) $A \subseteq B \Rightarrow \text{int}^m(A) \subseteq \text{int}^m(B)$.

(4) $A \subseteq cl^m(A)$.

(5) If $A \subseteq B \Rightarrow cl^m(A) \subseteq cl^m(B)$

(6) $cl^m(A)$ =A if A is *m*- closed.

(7) $x \in cl^m(A)$ if and only if every m- open set U_x containing x, $U_x \cap A \neq \phi$.

(8) $cl^m(A^c) = (\text{int}^m(A))^c$ and $\text{int}^m(A^c) = (cl^m(A))^c$.

Now, we give the following definition.

Definitition 1.6. A subset A of a - space (X, M) is said to be

(1) $\alpha_m - open$ if A$\subset$ $\text{int}^m(cl^m(\text{int}^m$(A))).

(2) $pre_m - open$ (briefly $p_m - open$) if A$\subset$ $\text{int}^m(cl^m$(A)).

(3) $semi_m - open$ (briefly $s_m - open$) if A$\subset$ $cl^m(\text{int}^m$(A)).

(4) $b_m - open$ if A$\subset$ $cl^m(\text{int}^m$(A)) $\cup$ $\text{int}^m(cl^m$(A)).

(5) $\beta_m - open$ if A$\subset$ $cl^m(\text{int}^m(cl^m$(A))).

(6) $r_m - open$ if A= $\text{int}^m(cl^m$(A)).

The set of all $\alpha_m - open$ (resp. $p_m - open$, $s_m - open$, $b_m - open$, $\beta_m - open$, $r_m - open$) sets is denoted by $m\alpha O(X)$ (resp. $mPO(X)$, $mSO(X)$, $mBO(X)$, $m\beta O(X)$), $mRO(X)$).
The complement of each of the above sets is its respective closed set.

From Definition 1.6 we can prove that,

Proposition 1.7.

$m-$open $\Rightarrow$ $\alpha_m - open$ $\Rightarrow$ $s_m - open$ $\Rightarrow$ $b_m - open$ $\Rightarrow$ $\beta_m - open$ and

$\alpha_m - open$ $\Rightarrow$ $p_m - open$ $\Rightarrow$ $b_m - open$ $\Rightarrow$ $\beta_m - open$

II. M- HYPERCONNECTED SPACES

In this section we introduce the notion of m - *Hyperconnectedness*. Some properties and characterizations are investigated.

Definition 2.1. A subset *A* of a *m*- space (X, M) is said to be

(1) $m - dense$ if cl^m(A)= X.

(2) $m - nowhere$ dense if $\text{int}^m(cl^m$(A))= ϕ .

Definition 2.2. An *m*- space (X, M) is said to be

(i) *Hyperconnected* (equivalently X is m- hyperconnected) if A is $m - dense$ for every subset A$\neq$ ϕ of X.
(ii) *Connected* (equivalently X is m- connected) if X cannot be written as a union of nonempty and disjoint *m*-open sets.

Remark 2.3. From Definitions 2.2 we have the following relation.
m- hyperconnectedness $\Rightarrow$ m - connectedness

The converse of this implication is not true in general as shown by the following example,

Example 2.4. X= {a, b, c}, M = {X,ϕ , {c}, {b}, {c, b}}. Then (X, M) is m- connected but not m-hyperconnected.

Notation 2.5. We say that (X, M) satisfies property (υ) if M is closed under arbitrary union.

Theorem 2.6. Let (X, M) be an m - space with property (υ) and A be a subset of X. Then the following are equivalent.

(1) (X, M) is hyperconnected.
(2) A is $m - dense$ or $m - nowhere$ dense, for every subset A of X.
(3) A$\cap$ B$\neq$ ϕ , for every nonempty m - open subset A and B of X.

Proof. (1)$\Rightarrow$ (2): Let (X, M) be a hyperconnected and A be a subset of X. Suppose that A is not $m-nowhere$ dense. Then cl^m (X\ cl^m (A)) = X\ int^m(cl^m (A))$\neq$ X. Since $\text{int}^m(cl^m$ (A))$\neq \phi$, so by (1), $cl^m(\text{int}^m(cl^m$ (A))) = X. Since $cl^m(\text{int}^m(cl^m$ (A))) = X$\subset$ cl^m (A). Then cl^m (A) = X. Hence A is $m-dense$.

(2) $\Rightarrow$ (3): Suppose that A$\cap$ B=ϕ for some nonempty $m-open$ subsets A and B of X.

Then cl^m (A) $\cap$ B=ϕ and A is not $m-dense$. Since A is m - open, so $\phi\neq$ A $\subset$ int^m (cl^m (A)). Hence A is not $m-nowhere$ dense. This is a contradiction.

(3) $\Rightarrow$ (1): Let A$\cap$ B$\neq\phi$ for every nonempty $m-open$ subsets A and B of X. Suppose that (X,M) is not hyperconnected. Then there is a nonempty $m-open$ subset U of X such that U is not $m-dense$ in X, thus cl^m (U) $\neq$ X. Hence X \cl^m (U) and U are disjoint nonempty $m-open$ subsets of X. This is a contradiction. Hence (X, M) is hyperconnect.

Definition 2.7. The m - semi- closure (resp. m - pre- closure, m -β -closure) of a subset A of a minimal space (X, M), denoted by c_m^s(A) (resp. $c_m{}^p$(A), c_m^β (A)) is the intersection of all $s_m-closed$ (resp. $p_m-closed$,$\beta_m-closed$) sets of X containing A.

The following Lemma is easy to prove.

Theorem 2.8. If (X, M) is a minimal space with property (υ), then the following are equivalent.

(1) (X, M) is hyperconnected.

(2) A is $m-dense$ for every β_m-open subset $\phi\neq$ A$\subset$ X.

(3) A is $m-dense$ for every b_m-open subset $\phi\neq$ A$\subset$ X.

(4) A is $m-dense$ for every p_m-open subset $\phi\neq$ A$\subset$ X.

(5) c_m^s (A)= X for every p_m-open subset $\phi\neq$ A$\subset$ X.

(6) c_m^p (A)= X for every s_m-open subset $\phi\neq$ A$\subset$ X.

Proof. (1) $\Rightarrow$ (2): Suppose that A is a nonempty β_m-open subset of X. Hence $\text{int}_m(cl_m$ (A)))$\neq\phi$. Then cl_m (A) = $cl_m(\text{int}_m(cl_m$ (A)))= X.

(2) $\Rightarrow$ (3): Since every b_m-open is β_m-open , we have (3).

(3) $\Rightarrow$ (4): Since every p_m-open is b_m-open , we have (4).

(4) $\Rightarrow$ (5): Suppose that A$\neq\phi$ is a p_m-open set such that c_m^s (A) $\neq$ X. Then there is a nonempty s_m-open set U such that U $\cap$ A= ϕ . Hence int_m (U) $\cap$ A= ϕ . Then, by (4), ϕ = int_m (U) $\cap$ cl_m (A) = int_m (U) which is a contradiction.

(5) $\Rightarrow$ (6): Suppose that there is a nonempty s_m-open set A such that c_m^p (A) $\neq$ X. Hence there is a p_m-open set U$\neq\phi$ such that U$\cap$ A= ϕ . So U $\cap$ int_m(A)=ϕ . Hence, by (5), int_m(A)= c_m^s (U)$\cap$ int_m (A) $\subset$ cl_m U$\cap$ int_m (A)=ϕ . This is a contradiction.

(6) $\Rightarrow$ (1): Let A be a nonempty $m-open$ subset of X. So A is s_m-open . Hence by (6), c_m^p (A)= X. Since c_m^p (A) $\subset$ cl_m (A). Then cl_m (A)= X. So (X, M) is hyperconnected.

Corollary 2.9. Let (X,M) be an *m*- space with property (υ). Then the following are equivalent.

(1) (X, M) is hyperconnected.
(2) A ⋂ B$\neq \phi$ for every nonempty $s_m - open$ set A and nonempty $p_m - open$ set B.
(3) A ⋂ B$\neq \phi$ for every nonempty $s_m - open$ sets A and B.

Proof. (1) $\Rightarrow$ (2): Assume that X is hyperconnected. Let A be a nonempty $s_m - open$ set and B be a nonempty $p_m - open$ set such that A ⋂ B=ϕ . Then by Theorem 2.8(6), c_m^s (B)= X. But X= c_m^s (B) $\subset$ c_m^s (X- A) = X- A. Hence A= ϕ , this is a contradiction.

(2) $\Rightarrow$ (1): Suppose that A and B are any two nonempty $m - open$ subsets of X. Hence, by (2), A ⋂ B$\neq \phi$, by Theorem 2.6, X is hyperconnected.
Similarly we can prove that (1) is equivalent to (3).

III. PRESERVATION THEOREMS

In this **s**ection, we introduce some types of functions under which *m*- hyperconnectedness is preserved and some related properties are given.

Definition 3.1. The m - semi- interior of a subset A of an *m*- space X denoted by i_m^s (A) is the union of all $s_m - open$ sets of X contained in A.

Definition 3.2. Let (X, M_1) and (Y, M_2) be two *m*- spaces. A function $f:(X,M_1)\rightarrow(Y,M_2)$ is called rs -continuous if for each nonempty $r_{m_2} - open$ set V of Y, $f^{-1}(V)\neq\phi$ implies $i_{m_1}^s(f^{-1}(V))\neq\phi$.

Definition 3.3. A function $f:(X,M_1)\rightarrow(Y,M_2)$ is called m -semi-continuous function if $f^{-1}(V)$ is $s_{m_1} - open$ in X for each m_2 - open set V of Y.

Theorem 3.4. Let $f:(X,M_1)\rightarrow(Y,M_2)$ be an m - semi-continuous function and M_2 satisfies property (υ), then f is rs -continuous function.

Proof. Suppose that V is an $r_{m_2} - open$ subset of Y such that $f^{-1}(V)\neq\phi$. Since every $r_{m_2} - open$ set is m_2 -open and f is m - semi-continuous continuous. So $f^{-1}(V)$ is a nonempty $s_{m_1} - open$ in X. Hence $f^{-1}(V)=i_{m_1}^s(f^{-1}(V))$ and $i_{m_1}^s(f^{-1}(V))\neq\phi$ .Thus $f:(X,M_1)\rightarrow(Y,M_2)$ is rs -continuous.

Theorem 3.5. If $f:(X,M_1)\rightarrow(Y,M_2)$ is an rs-continuous surjection and (X, M_1) is hyperconnected with property (υ), then (Y, M_2) is hyperconnected.

Proof. Assume that (Y, M_2) is not hyperconnected. Then there are disjoint m_2 -open sets A and B. Put $U=\text{int}_{m_2}(cl_{m_2}(A))$ and $V=\text{int}_{m_2}(cl_{m_2}(B))$. Hence $U=\text{int}_{m_2}(cl_{m_2}(U))$ and $V=\text{int}_{m_2}(cl_{m_2}(V))$ .Thus U and V are disjoint nonempty $r_{m_2} - open$ sets. Hence $i_{m_1}^s(f^{-1}(U)) \cap i_{m_1}^s(f^{-1}(V)) \subset f^{-1}(U)\cap f^{-1}(V)=\phi$. Since $f:(X,M_1)\rightarrow(Y,M_2)$ is an rs -continuous surjection, then $i_{m_1}^s(f^{-1}(U))\neq\phi$ and $i_{m_1}(f^{-1}(V))\neq\phi$. Hence, by Corollary 2.9, thus (X, M_1) is not a hyperconnected. This is a contradiction.

Combining Theorem 3.4 and Theorem 3.5, we get the following result.

Corollary 3.6. If $f:(X,M_1)\rightarrow(Y,M_2)$ is m - semi-continuous surjection and (X, M_1) is hyperconnected with property (υ), then (Y, M_2) is hyperconnected.

Definition 3.7. A function $f:(X,M_1)\to(Y,M_2)$ is called m -continuous function if $f^{-1}(V)\in M_1$ for every V$\in M_2$.

Remark 3.8. It is clear that every m - continuous function is m - semi-continuous but not conversely as shown by the following example.

Example 3.9. Let X=Y= {a,b,c,d} and $M_1 = M_2$ = {X, ϕ ,{b},{d},{b, d}}. Define $f:(X,M_1)\to(Y,M_2)$ as follows, f(a)= a, f(b)= d, f(c)= d, f(d)= b. Then f is m - semi-continuous but not m -continuous.

From Theorem 3.4 and Remark 3.8 we have the following implications.

Remark 3.10. m -continuous$\Rightarrow$ m - semi-continuous$\Rightarrow$ rs-continuous**.**

From Theorem 3.5 and Remark 3.10 we obtained the following result.

Corollary 3.11. If $f:(X,M_1)\to(Y,M_2)$ is m − continuous surjection and (X, M_1) is hyperconnected with property (υ), then (Y, M_2) is hyperconnected.

V. CONCLUSIONS

Hyperconnectedness is an important property depends on the classes of open and dense sets. In this work we study hyperconnectedness in minimal topological spaces. Several characterizations of minimal hyperconnectedness are provided, preservation theorems are given. This work help researchers to study hypercnenectedness in many types of topological spaces.

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