Unification of Ramanujan Integrals with Some Infinite Integrals and Multivariable Gimel-Function

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ABSTRACT
In the present paper some Ramanujan integrals are unified with some infinite integrals and multivariable Gimel-function. The importance of our main results lies in the fact that they involve special functions and multivariable Gimel-function which are sufficiently general in nature and capable of yielding a large number or simpler and useful results merely by specializing the parameters therein.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, Ramanujan integrals, generalized hypergeometric function.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$$

$$\begin{array}{c}[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},\\ [\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};\end{array}$$

$$\begin{array}{l}[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],\\ [\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\qquad\qquad \ldots\end{array}$$

$$\begin{array}{l}[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]\\ [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]\end{array}$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

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$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \quad \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right)$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([4] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}) , max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}) , min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] \text{ and } \beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [2].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1and\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [8,9].

In your investigation, we shall use the following notations.

$\mathbb{A} = [(\mathrm{a}_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha^{(1)}_{(r-1)j},\cdots,\alpha^{(r-1)}_{(r-1)j};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha^{(1)}_{(r-1)ji_{r-1}},\cdots,\alpha^{(r-1)}_{(r-1)ji_{r-1}};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}];\cdots;$$

$$[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta^{(1)}_{(r-1)ji_{r-1}},\cdots,\beta^{(r-1)}_{(r-1)ji_{r-1}};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}^{(1)}_j,\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}];\cdots;$$

$$[(\mathrm{d}^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{m^{(r)}+1,q^{(r)}_i}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

In excellent five parts [3] Berndt has examined 3254 results from the note book pf S.Ramanujan. Also, Agarwal has made a comprehensive study of Ramanujan's work in his remarkable three volume [1]. These works also contain complete references of the contributions made by other eminent mathematicians on Ramanujan's Mathematics. Undoubtedly, some of Ramanujan's work has embedded in it unparalleled motivation, wisdom, depth and imagination and enough scope to pursue further research. Thus motivated by the following result obtained by Ramanujan ([1], p.191, Eq(20)).

Lemma 1.

$$\int_0^\infty x^{p-1}\left[\frac{2}{1+\sqrt{1+4x}}\right]^n \mathrm{d}x = \frac{n\Gamma(p)\Gamma(n-2p)}{\Gamma(n-p+1)} \tag{2.1}$$

provided $n > 0, 0 < p < \frac{n}{2}$

Lemma 2.

$$\int_0^\infty x^{p-1}\left[\frac{1}{1+\sqrt{1+x^2}}\right]^n \mathrm{d}x = \frac{n\Gamma(p)\Gamma\left(\frac{n-p}{2}\right)}{2^{p+1}\Gamma\left(\frac{n+p+2}{2}\right)} \tag{2.2}$$

provided $0 < p < n$

Lemma 3.

$$\int_0^\infty x^{p-1}\left[1+\left(1-\frac{a}{c}\right)x\right](1+x)^{-a-1}\mathrm{d}x=\frac{(c-p)\Gamma(p)\Gamma(a-p)}{c\Gamma(a)} \tag{2.3}$$

provided $c \neq p, Re(p)>0, Re(a)>Re(a-p)>0$

Lemma 4.

$$\int_0^\infty x^{p-1}(1+x)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1+x}}\right]^{2a}\mathrm{d}x=\frac{4^p\Gamma(p)\Gamma(2a+1-2p)}{\Gamma(2a-p+1)} \tag{2.4}$$

provided $Re(p)>0, Re(2a+1-2p)>0$

Lemma 5.

$$\int_0^\infty {}_PF_Q\left[\begin{array}{c|c}\alpha_p & \\ . & \mathrm{c}y^h \\ \beta_q & \end{array}\right]{}_UF_V\left[\begin{array}{c|c}\gamma_u & \\ . & \mathrm{d}y^k \\ \delta_v & \end{array}\right]\mathrm{d}y=\frac{1}{m!}\sum_{s,t=0}^\infty\frac{(\alpha_p)_s(\gamma_u)_t c^s d^t\Gamma(1+\alpha-p+n-hs-kt)\Gamma(p+hs+kt)}{(\beta_q)_s s!'\delta_v)_t t!\Gamma(1+\alpha-hs-kt)} \tag{2.5}$$

provided $U<V, P<Q, |d|<1, |c|<1.$

3. Main integrals.

In this section, we shall give five double infinite integrals.

Theorem 1

$$\int_0^\infty\int_0^\infty x^{p-1}\left[\frac{1}{1+\sqrt{1+4x}}\right]^n y^\rho e^{-y}L_m^\alpha(y)\ {}_PF_Q\left[\begin{array}{c|c}\alpha_p & \\ . & \mathrm{c}y^h \\ \beta_q & \end{array}\right]{}_UF_V\left[\begin{array}{c|c}\gamma_u & \\ . & \mathrm{d}y^k \\ \delta_v & \end{array}\right]$$

$$\daleth\left(z_1x^{\sigma_1}y^{\eta_1}\left\{x+\sqrt{1+x^2}\right\}^{-\zeta_1},\cdots,z_rx^{\sigma_r}y^{\eta_r}\left\{x+\sqrt{1+x^2}\right\}^{-\zeta_r}\right)\mathrm{d}x\mathrm{d}y=$$

$$\frac{1}{m!}\sum_{s,t=0}^\infty\frac{(\alpha_p)_s(\gamma_u)_t c^s d^t\Gamma(1+\alpha-p+n-hs-kt)\Gamma(p+hs+kt)}{(\beta_q)_s s!'\delta_v)_t t!\Gamma(1+\alpha-hs-kt)}\ \daleth^{U;0,n_r+4:V}_{X;p_{i_r}+5,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\frac{z_1}{2^{\sigma_1}} & \\ \cdot & \\ \cdot & \\ \frac{z_r}{2^{\sigma_r}} & \end{array}\right.$$

$$\mathbb{A};\ (-n;\zeta_1,\cdots,\zeta_r;1),(1-p;\sigma_1,\cdots,\sigma_r;1),\left(\tfrac{2-n+p}{2};\tfrac{\zeta_1-\sigma_1}{2},\cdots,\tfrac{\zeta_r-\sigma_r}{2};1\right),(1-\rho-hs-kt;\eta_1,\cdots,\eta_r;1),$$

$$\mathbb{B};\ \mathbf{B},(1-n;\zeta_1,\cdots,\zeta_r:1),\left(\tfrac{-n-p}{2};\tfrac{\zeta_1+\sigma_1}{2},\cdots,\tfrac{\zeta_r+\sigma_r}{2};1\right),(1+m+\alpha-\rho-hs-kt;\eta_1,\cdots,\eta_r;1):B$$

$$\left.(1+\alpha-\rho-hs-kt;\eta_1,\cdots,\eta_r;1),\mathbf{A}:A \atop \cdots\cdots\right) \tag{3.1}$$

Provided

$0<p<\dfrac{n}{2};\zeta_i,\eta_i,\sigma_i,\zeta_i-\sigma_i>0(i=1,\cdots,r).\ Re\,(p+hs)+\sum_{i=1}^r\sigma_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\dfrac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$

$$Re\,(\rho+kt)+\sum_{i=1}^{r}\eta_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,\; Re\,(n)+\sum_{i=1}^{r}\zeta_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$\left|\arg(z_i x^{\sigma_i} y^{\eta_i}\left\{x+\sqrt{1+x^2}\right\}^{-\zeta_i}\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof
To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations which is justified under the conditions mentioned above, evaluating the inner x-integral with the help of the lemma 2 and evaluating the inner y-integral with the help of the lemma 4. and interpreting the resulting multiple integrals contour with the help of (1.1) about the gimel-function of r-variables, we obtain the desired theorem 1.

Theorem 2.

$$\int_0^\infty \int_0^\infty x^{p-1}\left[\frac{1}{1+\sqrt{1+4x}}\right]^n y^{\alpha-1}\left[1+\left(1-\frac{a}{c}\right)y\right](1+y)^{-\alpha-1}$$

$$\gimel\left(z_1 x^{\sigma_1} y^{\eta_1}\left\{x+\sqrt{1+4x}\right\}^{-\zeta_1},\cdots,z_r x^{\sigma_r} y^{\eta_r}\left\{x+\sqrt{1+4x}\right\}^{-\zeta_r}\right)\mathrm{d}x\mathrm{d}y=\frac{1}{c\Gamma(\alpha)}$$

$$\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+5,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\,(-n;\zeta_1,\cdots,\zeta_r;1)\,,(1-p;\sigma_1,\cdots,\sigma_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\,\mathbf{B},(\text{1-n};\zeta_1,\cdots,\zeta_r;1)\,(1+c-\alpha;\eta_1,\cdots,\eta_r:1)\,, \end{array}\right.$$

$$\left.\begin{array}{c} (\text{1-n+2p};\,\zeta_1-2\sigma_1,\cdots,\zeta_r-2\sigma_r;1),(1-\alpha;\eta_1,\cdots,\eta_r;1),(c-\alpha;\eta_1,\cdots,\eta_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (\text{p-n };\zeta-\sigma_1,\cdots,\sigma_r-\sigma_r;1),(a-\alpha;\eta_1,\cdots,\eta_r;1);B \end{array}\right) \tag{3.2}$$

Provided

$$Re(a)>0,\zeta_i,\eta_i,\sigma_i;\zeta_i-2\sigma_i>0(i=1,\cdots,r),\; Re\,(a-\alpha)-\sum_{i=1}^{r}\eta_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]<0$$

$$Re\,(n-2p)+\sum_{i=1}^{r}(\zeta_i-2\sigma_i) \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,\; Re\,(\alpha)+\sum_{i=1}^{r}\eta_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$Re\,(n)+\sum_{i=1}^{r}\zeta_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$\left|\arg(z_i x^{\sigma_i} y^{\eta_i}\left\{x+\sqrt{1+4x}\right\}^{-\zeta_i}\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof
To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations which is justified under the conditions mentioned above, evaluating the inner x-integral with the help of the lemma 1 and evaluating the inner y-integral with the help of the lemma 3. and interpreting the resulting multiple integrals contour with the help of (1.1) about the gimel-function of r-variables, we obtain the desired theorem 2.

Theorem 3.

$$\int_0^\infty \int_0^\infty x^{p-1}\left[\frac{1}{1+\sqrt{1+x^2}}\right]^n y^{\alpha-1}\left[1+\left(1-\frac{a}{c}\right)y\right](1+y)^{-\alpha-1}$$

$$\gimel\left(z_1x^{\sigma_1}y^{\eta_1}\left\{x+\sqrt{1+x^2}\right\}^{-\zeta_1},\cdots,z_rx^{\sigma_r}y^{\eta_r}\left\{x+\sqrt{1+x^2}\right\}^{-\zeta_r}\right)\mathrm{d}x\mathrm{d}y=\frac{1}{2^{p+1}c\Gamma(\alpha)}$$

$$\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+5,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 2^{-\sigma_1} & \mathbb{A};\ (-n;\zeta_1,\cdots,\zeta_r;1),(1-p;2\sigma_1,\cdots,2\sigma_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r 2^{-\sigma_r} & \mathbb{B};\ \mathbf{B},(1\text{-n};\zeta_1,\cdots,\zeta_r;1)\left(\frac{-p-n}{2};\frac{\eta_1+\sigma_1}{2},\cdots,\frac{\eta_r+\sigma_r}{2}:1\right), \end{array}\right.$$

$$\left.\begin{array}{c} \left(\frac{2+p-n}{2};\frac{\eta_1-\sigma_1}{2},\cdots,\frac{\eta_r-\sigma_r}{2};1\right),(1-\alpha;\eta_1,\cdots,\eta_r;1),(c-\alpha;\eta_1,\cdots,\eta_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (1\text{+c-}\alpha;\eta_1,\cdots,\eta_r;1),(a-\alpha;\eta_1,\cdots,\eta_r;1);B \end{array}\right) \tag{3.3}$$

Provided

$$Re(a)>0,\zeta_i,\eta_i,\sigma_i;\eta_i-\sigma_i>0(i=1,\cdots,r),\ Re\,(\alpha)+\sum_{i=1}^r\eta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]<a$$

$$Re\,(n-p)+\sum_{i=1}^r(\eta_i-\sigma_i)\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,\ Re\,(n)+\sum_{i=1}^r\zeta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$\left|\arg(z_ix^{\sigma_i}y^{\eta_i}\left\{x+\sqrt{1+x^2}\right\}^{-\zeta_i}\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations which is justified under the conditions mentioned above, evaluating the inner x-integral with the help of the lemma 2 and evaluating the inner y-integral with the help of the lemma 3. and interpreting the resulting multiple integrals contour with the help of (1.1) about the gimel-function of r-variables, we obtain the desired theorem 3.

Theorem 4.

$$\int_0^\infty \int_0^\infty x^{p-1}\left[\frac{1}{1+\sqrt{1+4x}}\right]^n y^{\alpha-1}(1+y)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1+y}}\right]^{2a}$$

$$\gimel\left(z_1x^{\sigma_1}y^{\eta_1}\left[\frac{2}{1+\sqrt{1+4x}}\right]^{\zeta_1}\left[\frac{2}{1+\sqrt{1+y}}\right]^{\lambda_1},\cdots,z_rx^{\sigma_r}y^{\eta_r}\left[\frac{2}{1+\sqrt{1+4x}}\right]^{\zeta_r}\left[\frac{2}{1+\sqrt{1+y}}\right]^{\lambda_r}\right)\mathrm{d}x\mathrm{d}y=4^a$$

$$\gimel^{U;0,n_r+5:V}_{X;p_{i_r}+5,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 a^{\delta_1} & \mathbb{A};\ (-n;\zeta_1,\cdots,\zeta_r;1),(1-p;\sigma_1,\cdots,\sigma_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r 4^{\delta_r} & \mathbb{B};\ \mathbf{B},(1\text{-n};\zeta_1,\cdots,\zeta_r;1),\left(\frac{-p-n}{2};\frac{\zeta_1+\sigma_1}{2},\cdots,\frac{\zeta_r+\sigma_r}{2};1\right), \end{array}\right.$$

$$\left(\begin{array}{c}\frac{2+p-n}{2};\frac{\zeta_1-\sigma_1}{2}\cdots,\frac{\zeta_r-\sigma_r}{2};1),(1-\alpha;\eta_1,\cdots,\eta_r;1),(2\alpha-2a;2\delta_1-2\eta_1,\cdots,2\delta_r-2\eta_r;1),\mathbf{A}:A\\ \cdot\\ \cdot\\ (\alpha-2a;2\lambda_1-\eta_1,\cdots,2\lambda_r-\eta_r;1);B\end{array}\right) \quad (3.5)$$

Provided

$\zeta_i,\eta_i,\sigma_i,\lambda_i;\zeta_i-2\sigma_i>0,2\lambda_i-\zeta_i>0(i=1,\cdots,r)$

$$Re\,(2a-\alpha)+\sum_{i=1}^{r}(2\lambda_i-\zeta_i)\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1,\,Re\,(\alpha)+\sum_{i=1}^{r}\eta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$Re\,(n-2p)+\sum_{i=1}^{r}(\zeta_i-2\sigma_i)\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,\,Re\,(n)+\sum_{i=1}^{r}\zeta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$\left|arg\left(z_ix^{\sigma_i}y^{\eta_i}\left[\frac{2}{1+\sqrt{1+4x}}\right]^{\zeta_i}\left[\frac{2}{1+\sqrt{1+y}}\right]^{\lambda_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi \text{ where } A_i^{(k)} \text{ is defined by } (1.4)$$

Proof

To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations which is justified under the conditions mentioned above, evaluating the inner x-integral with the help of the lemma 1 and evaluating the inner y-integral with the help of the lemma 4. and interpreting the resulting multiple integrals contour with the help of (1.1) about the gimel-function of r-variables, we obtain the desired theorem 4.

Theorem 5.

$$\int_0^\infty\int_0^\infty x^{p-1}\left[\frac{1}{1+\sqrt{1+x^2}}\right]^n y^{\alpha-1}(1+y)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1+y}}\right]^{2a}$$

$$\beth\left(z_1x^{\sigma_1}y^{\eta_1}\left[\frac{2}{1+\sqrt{1+x^2}}\right]^{\zeta_1}\left[\frac{2}{1+\sqrt{1+y}}\right]^{\lambda_1},\cdots,z_rx^{\sigma_r}y^{\eta_r}\left[\frac{2}{1+\sqrt{1+x^2}}\right]^{\zeta_r}\left[\frac{2}{1+\sqrt{1+y}}\right]^{\lambda_r}\right)\mathrm{d}x\mathrm{d}y=4^a$$

$$\beth^{U;0,n_r+5:V}_{X;p_{i_r}+5,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c}\mathrm{z}_1a^{2\delta_1-\sigma_1}\\ \cdot\\ \cdot\\ \mathrm{z}_r4^{2\delta_r-\sigma_r}\end{array}\left|\begin{array}{c}\mathbb{A};\,(-n;\zeta_1,\cdots,\zeta_r;1),(1-p;\sigma_1,\cdots,\sigma_r;1),\\ \cdot\\ \cdot\\ \mathbb{B};\,\mathbf{B},(1\text{-n};\zeta_1,\cdots,\zeta_r;1),(p-n;\zeta_1-\sigma_1,\cdots,\zeta_r-\sigma_r;1),\end{array}\right.\right.$$

$$\left.\begin{array}{c}1-n+2p;\zeta_1-2\sigma_1,\cdots,\zeta_r-2\sigma_r;1),(1-\alpha;\eta_1,\cdots,\eta_r;1),(2\alpha-2a;2\delta_1-2\eta_1,\cdots,2\delta_r-2\eta_r;1),\mathbf{A}:A\\ \cdot\\ \cdot\\ (\alpha-2a;2\lambda_1-\eta_1,\cdots,2\lambda_r-\eta_r;1);B\end{array}\right) \quad (3.4)$$

Provided

$\zeta_i,\eta_i,\sigma_i,\lambda_i;2\zeta_i-\sigma_i>0,2\lambda_i-\zeta_i>0(i=1,\cdots,r),$

$$Re\,(2a-\alpha)+\sum_{i=1}^{r}(2\lambda_i-\zeta_i)\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1,\,Re\,(\alpha)+\sum_{i=1}^{r}\eta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$Re\,(n-2p)+\sum_{i=1}^{r}(2\zeta_i-\sigma_i)\min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,\; Re\,(n)+\sum_{i=1}^{r}\zeta_i\min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$\left|arg\left(z_i x^{\sigma_i}y^{\eta_i}\left[\frac{2}{1+\sqrt{1+x^2}}\right]^{\zeta_i}\left[\frac{2}{1+\sqrt{1+y}}\right]^{\lambda_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi \text{ where } A_i^{(k)} \text{ is defined by } (1.4).$$

Proof

To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations which is justified under the conditions mentioned above, evaluating the inner x-integral with the help of the lemma 2 and evaluating the inner y-integral with the help of the lemma 4. and interpreting the resulting multiple integrals contour with the help of (1.1) about the gimel-function of r-variables, we obtain the desired theorem 4.

Remark :

We obtain the same double finite integrals with the functions defined in section I. Kumar and Nagar [5] have obtained the same relations about the multivariable H-function.

4. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these double integrals, we can obtain a large simpler double or single finite integrals, Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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