Largest Possible Variable Delay In Control Problem

Amit Ghosh[1], Tanmoy Hazra[2], and Towhid E Aman[3]

[1]Department of Mathematics, Raiganj University, West Bengal, India
[2]Department of Mathematics,Gangarampur College, West Bengal India
[3]Department of Mathematics,Gangarampur College, West Bengal India

Abstract

In this paper we consider optimal control problem involbing largest possible delay. It considers variable delay in connection with optimal control problem.Here the necessary conditions of optimality are obtained and it is shown that the conditions agree, in particular case with the corresponding known conditions when there is no delay, or small delay where second degree terms are not considered.

Key words and phrases: Pontryagin's maximum principle, Optimal control problem, Delay differential equation, Dynamical system

1   INTRODUCTION

So far as real world continuous phenomena are concerned, delay differential equation is more be fitting than ordinary differential equations. Now control problems associated with ordinary differential equations representing dynamical systems are well studied in the literature. Also control problems modeled by differential equation involving small time delay[1] and comparatively large delay [2], ignoring higher degree terms are well studied. Also the problems without ignoring any higher degree terms are discussed for constant time delay[5]. But similar problems without ignoring higher terms for variable delay are not well studied. Hence it is essential to consider control problems with large variable time delay, without ignoring any higher degree terms. With this in view, the present paper suitably generalizes the well known Pontryagin's maximum principle, which is an essential tool for solving an optimal control problem, without ignoring any higher degree terms for largest possible variable delay.

1. GENERAL FORM OF THE CONTROL PROBLEM INVOLVING VARIABLE DELAY

Problem I

Let the dynamical system be given by
$\dot{x} = f(t, x(t), x_{\tau(t)}(t), u(t), u_{\tau_1(t)}(t))........$ (1)

where $t \in [0,1] \subset R$, $x(t) \in R^n$, $u(t) \in R^m$,$x_{\tau(t)}(t) = x(t - \tau(t)) \in R^n$,$u_{\tau_1(t)}(t) = u(t - \tau_1(t)) \in$

R^m and $f\colon R \times R^n \times R^n \times R^m \times R^m \longrightarrow R^n$ is a continuously differentiable mapping and $\tau\colon [0, t_1] \longrightarrow (-\delta, \delta) \subset R$ and $\tau_1\colon [0, t_1] \longrightarrow (-\delta_1, \delta_1) \subset R$ are also differentiable . Here the delay are not small, hence higher order τ and τ' are not negligible.

Let the objective criteria be given by

Maximize $J(u) = \phi(t_1, x(t_1)) + \int_0^{t_1} F(t, x(t), x_{\tau(t)}(t), u(t), u_{\tau_1}(t)) dt$..... (2)
where $\phi\colon R \times R^n \longrightarrow R$, $F\colon R \times R^n \times R^n \times R^m \times R^m \longrightarrow R$ are also differentiable mappings.

Here we would like to develop Pontryagin's Maximum principle without ignoring any higher order terms of $\tau(t)$ and $\tau_1(t)$. Before going to prove the actual Theorem let us prove the following lemma.

1.1 Lemma 1

The series of the form
$\tau(t)\ddot{x}(t) - \frac{\tau^2(t)}{2!}\dddot{x}(t) + ... + (-1)^n \frac{\tau^n(t)}{n!} x^{n+1}(t) + ...$
is uniformly concergent if $|\tau^n(t)x^{n+1}(t)| < 1$ and x(t) is continuously differentiable .
Proof of the lemma:
The given series is a series of real valued functions. By the given condition $|\tau^n(t)x^{n+1}(t)| < 1$ and $\forall n$. If we take $M_n = \frac{1}{n!}$ then every term $u_n(x)$ of the series satisfies $|u_n| \le M_n \ \forall n$. And since the series $\Sigma |M_n|$ is convergent. Then by the Weierstrass's M-test the given series is uniformly convergent . this proves the lemma
By the above lemma we can denote
$\psi_1(\ddot{x}(t), \dddot{x}(t), ...) = \tau(t)\ddot{x}(t) - \frac{\tau^2(t)}{2!}\dddot{x}(t) + ... + (-1)^n \frac{\tau^n(t)}{n!} x^{n+1}(t) + ...$
if $|\tau^n(t)x^{n+1}(t)| < 1$ and x(t) is continuously differentiable function $\forall n$.
similarly we can write
$\psi_2(\ddot{u}(t), \dddot{u}(t), ...) = \tau_1(t)\ddot{u}(t) - \frac{\tau_1^2(t)}{2!}\dddot{u}(t) + ... + (-1)^n \frac{\tau_1^n(t)}{n!} u^{n+1}(t) + ...$
for $|\tau_1^n(t)u^{n+1}(t)| < 1$, and continuously differentiable function u(t). $\forall n$.

1.2 Theorem 1

A necessary condition that (x^*, u^*) is a solution of the control problem I with $|\tau^n(t)x^{n+1}(t)| < 1$, $|\tau_1^n(t)u^{n+1}(t)| < 1$, there exist a co-state vector $p(t) \in R^n$ such that $P^T(t_1) = (\frac{\partial\phi}{\partial x})_{t=t_1}$, $\dot{P}^T + \frac{\partial H}{\partial x} + \frac{\partial H}{\partial x_{\tau(t)}} = 0$ and

$(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_{\tau_1(t)}})\dot{u} = \frac{\partial H}{\partial x_{\tau(t)}}\{(1-\dot{\tau}(t))\psi_1 + \dot{x}(t)\dot{\tau}(t)\} + \frac{\partial H}{\partial u_{\tau_1(t)}}\{(1-\tau_1\dot{(t)}\psi_2 + \dot{u}\tau_1\dot{(t)}\}$ at $u = u^*$. where

$H(t, x, x_{\tau(t)}, u, u_{\tau_1(t)}, p) = F(t, x, x_{\tau(t)}, u, u_{\tau_1(t)}) + p^T f(t, x, x_{\tau(t)}, u, u_{\tau_1(t)});$

Proof of the Theorem:

We first introduce the co-state vector $p(t)$ to construct the augmented functional $J_a(u)$,

as given by

$$J_a(u) = \phi(\ t_1,\ x(t_1)) + \int_0^{t_1}[F(t,x,\ x_{\tau(t)},u,u_{\tau_1(t)}) + p^T(f(t,x,\ x_{\tau(t)},u,u_{\tau_1(t)})) - \dot{x}]dt$$

Applying the rule of integration by parts we get

$$J_a(u) = \phi(\ t_1,\ x(t_1)) - [p^T x]_0^{t_1} + \int_0^{t_1}[H(t,x,\ x_{\tau(t)},u,u_{\tau_1(t)},p) + \dot{p}^T x]dt$$

Where the Hamiltonian function $H(t,x,\ x_{\tau(t)},u,u_{\tau_1(t)},p)$ is defined by

$$H(t,x,\ x_{\tau(t)},u,u_{\tau_1(t)},p) = F(t,x,\ x_{\tau(t)},u,u_{\tau_1(t)}) + p^T f(t,x,\ x_{\tau(t)},u,u_{\tau_1(t)})$$

Now the variation δJ_a of $J_a(u)$ is given by
$$\delta J_a = (\frac{\partial \phi}{\partial x} - p^T)\delta x|_{t=t_1} + \int_0^{t_1}[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial x_{\tau(t)}}\delta x_{\tau(t)} + \frac{\partial H}{\partial u}\delta u + \frac{\partial H}{\partial u_{\tau_1(t)}}\delta u_{\tau_1(t)} + \dot{p}^T\delta x]dt$$

Now applying Taylor's theorem for a fixed $t \in [0,t_1]$ we get

$$\mathrm{x}_{\tau(t)}(t) = x(t - \tau(t)) = x(t) - \tau(t)\dot{x}(t) + \frac{\tau^2(t)}{2!}\ddot{x}(t) - \frac{\tau^3(t)}{3!}\dddot{x}(t) + ... + (-1)^{n-1}\frac{\tau^{n}(t)}{n!}x^n(t) + ...$$

Thus

$$\mathrm{x}_{\tau(t)}(t) = x(t - \tau(t)) = x(t) - \tau(t)\dot{x}(t) + \frac{\tau^2(t)}{2!}\ddot{x}(t) - \frac{\tau^3(t)}{3!}\dddot{x}(t) + ... + (-1)^{n-1}\frac{\tau^{n}(t)}{n!}x^n(t) + ...$$
for every $t \in [0,t_1]$

This implies

$$\delta x_{\tau(t)} = \delta x - \tau(t)\ddot{x}(t)\delta t - \dot{x}(t)\dot{\tau}(t)\delta t + \frac{\tau^2(t)}{2!}\dddot{x}\delta t + \tau(t)\ddot{x}(t)\dot{\tau}(t)\delta t - \frac{\tau^3(t)}{3!}x^{iv}(t)\delta t - \frac{\tau^2(t)}{2!}\dddot{x}(t)\dot{\tau}(t)\delta t +$$
$$..... + (-1)^{n-1}\frac{\tau^n(t)}{n!}x^{n+1}(t)\delta t + (-1)^n\frac{\tau^{n-1}}{(n-1)!}x^n(t)\delta t + ...$$

$$= \delta\ \mathrm{x} - (\ \tau(t)\ddot{x}(t) - \frac{\tau^2(t)}{2!}\dddot{x}(t) + ... + (-1)^n\frac{\tau^n(t)}{n!}x^{n+1}(t) + ...)\delta t - (\mathrm{t})(\mathrm{t})\delta t + (\ \tau(t)\ddot{x}(t) - \frac{\tau^2(t)}{2!}\dddot{x}(t) +$$
$$...... + (-1)^n\frac{\tau^n(t)}{n!}x^{n+1}(t) + ...)\dot{\tau}(\mathrm{t})\delta t$$

Now by lemma 1 and by the given condition of the theorem the series

$$\tau(t)\ddot{x} - \frac{\tau^2(t)}{2!}\dddot{x} + ...(-1)^{n-1}\frac{\tau^n(t)}{n!}x^{n+1}(t) + ... \text{ is convergent.}$$

And thus

$$\delta x_{\tau(t)} = \delta x - \psi_1(\ddot{x}(t),\dddot{x}(t),...)\delta t - \dot{x}(t)\dot{\tau}(t)\delta t + \psi_1(\ddot{x}(t),\dddot{x}(t),...)\dot{\tau}(t)\delta t$$

$$= \delta x - (1 - \dot{\tau}(t))\psi_1\delta t - \dot{x}(t)\dot{\tau}(t)\delta t$$

Where $\psi_1(\ddot{x}(t),\dddot{x}(t),...) = \tau(t)\ddot{x}(t) - \frac{\tau^2(t)}{2!}\dddot{x}(t) + ... + (-1)^n\frac{\tau^n(t)}{n!}x^{n+1}(t) + ...$

Similarly we can write

$$\delta u_{\tau_1(t)} = \delta u - (1 - \dot{\tau_1}(t)(t))\psi_2\delta t - \dot{u}(t)\dot{\tau_1}(t)\delta t$$

Where $\psi_2(\ddot{u}(t), \dddot{u}(t), ...) = \tau_1(t)\ddot{u}(t) - \frac{\tau_1^2(t)}{2!}\dddot{u}(t) + ... + (-1)^n \frac{\tau_1^n(t)}{n!} u^{n+1}(t) + ...$

Therefore finally we get

$\delta J_a = (\frac{\partial \phi}{\partial x} - p^T)\delta x|_{t=t_1} + \int_0^{t_1} [\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial x_{\tau(t)}}\{\delta x - (1-\dot{\tau}(t))\psi_1 \delta t - \dot{x}(t)\dot{\tau}(t)\delta t\} + \frac{\partial H}{\partial u}\delta u + \frac{\partial H}{\partial u_{\tau_1(t)}}\{\delta u - (1 - \dot{\tau}_1(t))\psi_2 \delta t - \dot{u}(t)\dot{\tau}_1(t)\delta(t)\} + \dot{p}^T \delta x]dt$

$= (\frac{\partial \phi}{\partial x} - p^T)\delta x|_{t=t_1} + \int_0^{t_1} [(\frac{\partial H}{\partial x} + \frac{\partial H}{\partial x_{\tau(t)}} + \dot{p}^T)\delta x + (\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_{\tau_1(t)}})\delta u - \frac{\partial H}{\partial x_{\tau(t)}}\{(1-\dot{\tau}(t))\psi_1 + \dot{x}(t)\dot{\tau}(t)\}\delta t - \frac{\partial H}{\partial u_{\tau_1(t)}}\{(1 - \dot{\tau_1}(t)\psi_2 + \dot{u}\dot{\tau_1}(t)\}\delta t]dt$

We now choose $p(t)$ so that

$p^T = \frac{\partial \phi}{\partial x}$ at $t = t_1$ i.e. $p^T(t_1) = \frac{\partial \phi}{\partial x}|_{t=t_1}$ and $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial x_\tau} + \dot{p}^T = 0$

Then under this choice of $p(t), \delta J_a$ becomes

$\delta J_a = \int_0^{t_1} [(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_{\tau_1(t)}})\delta u - \frac{\partial H}{\partial x_{\tau(t)}}\{(1-\dot{\tau}(t))\psi_1 + \dot{x}(t)\dot{\tau}(t)\}\delta t - \frac{\partial H}{\partial u_{\tau_1(t)}}\{(1 - \dot{\tau_1}(t)\psi_2 + \dot{u}\dot{\tau_1}(t)\}\delta t]dt$

Assuming that J_a is maximum corresponding to $u = u^*$, we have $\delta J_a = 0$ at $u = u^*$ this gives

$(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_{\tau_1(t)}})\delta u - \frac{\partial H}{\partial x_{\tau(t)}}\{(1-\dot{\tau}(t))\psi_1 + \dot{x}(t)\dot{\tau}(t)\}\delta t - \frac{\partial H}{\partial u_{\tau_1(t)}}\{(1 - \dot{\tau_1}(t)\psi_2 + \dot{u}\dot{\tau_1}(t)\}\delta t = 0$ at u=u*

$i.e (\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_{\tau_1(t)}})\delta u = \frac{\partial H}{\partial x_{\tau(t)}}\{(1-\dot{\tau}(t))\psi_1 + \dot{x}(t)\dot{\tau}(t)\}\delta t + \frac{\partial H}{\partial u_{\tau_1(t)}}\{(1 - \dot{\tau_1}(t)\psi_2 + \dot{u}\dot{\tau_1}(t)\}\delta t$ at u=u*

$or, (\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_{\tau_1(t)}})\frac{\delta u}{\delta t} = \frac{\partial H}{\partial x_{\tau(t)}}\{(1-\dot{\tau}(t))\psi_1 + \dot{x}(t)\dot{\tau}(t)\} + \frac{\partial H}{\partial u_{\tau_1(t)}}\{(1 - \dot{\tau_1}(t)\psi_2 + \dot{u}\dot{\tau_1}(t)\}$ at u=u*
Taking the limit as $\delta t \to 0$ we get

$(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_{\tau_1(t)}})\dot{u} = \frac{\partial H}{\partial x_{\tau(t)}}\{(1-\dot{\tau}(t))\psi_1 + \dot{x}(t)\dot{\tau}(t)\} + \frac{\partial H}{\partial u_{\tau_1(t)}}\{(1 - \dot{\tau_1}(t)\psi_2 + \dot{u}\dot{\tau_1}(t)\}$ at $u = u^*$

This proves the theorem.

1.3 OBSERVATION:

If $|\tau(t)| < 1$ and $|x^{(n+1)}(t)| < 1$, then $|\tau^n(t)x^{(n+1)}(t)| < 1$ and in this case the necessary conditions remain unchange. In this paper we try to find the largest possible delay. If $|\tau(t)| > 1$ then from the condition $|\tau^n(t)x^{(n+1)}(t)| < 1$ we can say that the delay $\tau(t)$ is largest possible depending on $|x^{(n+1)}(t)| < \frac{1}{\tau^n(t)}$.
similarly the delay $\tau_1(t)$ is largest possible depending on $|u^{(n+1)}(t)| < \frac{1}{\tau_1^n(t)}$.

1.4   SOME SPECIAL CASES:

(1) If the delay occurs only in the state vector x(t), then the above necessary conditions of optimality becomes;
$p^T(t_1) = (\frac{\partial \phi}{\partial x})_{t=t_1}$, $\dot{p}^T + \frac{\partial H}{\partial x} + \frac{\partial H}{\partial x_{\tau(t)}} = 0$, and $\frac{\partial H}{\partial u}\dot{u} = \{(1 - \dot{\tau}(t))\psi_1 + \dot{x}(t)\dot{\tau}(t)\}\frac{\partial H}{\partial x_{\tau(t)}}$

(2) If the delay occurs only in the control vector u(t), then the above conditions becoms:
$p^T(t_1) = (\frac{\partial \phi}{\partial x})_{t=t_1}$, $\dot{p}^T + \frac{\partial H}{\partial x} = 0$, and $(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_{\tau'}})\dot{u} = \frac{\partial H}{\partial u_{\tau_1(t)}}\{(1 - \dot{\tau_1}(t)\psi_2 + \dot{u}\dot{\tau_1}(t)\}$

(3) If there is no delay, then the above results agrees with the condition given by Pontryagin's maximum principle.

(4) If both the delays are very small then as like as [1] then the result agrees with the corresponding result in [1].

(5) If both the delays are comparatively large such as [2], then the result agrees with the corresponding result in [2].

1.5   REFERENCES:

[1]. T.E.Aman and D.K.Bhattacharya: Control theoretic Optimization in general delay-differential equations with small delay. Rev. Acad. Canar. Ciens., XIX (nums1-2), 37-47 (2007).
[2]. T.E.Aman, A.Ghosh, S.K.Sur and D.K.Bhattacharya: Optimal Control Problem With Comparatively Large Delay.Rev. Acad. Canar. Ciens., XXII (nums1-2), 23-29 (2010).
[3]. Lee, E.B. and L.Markus: Foundations of optimal control theory. John Willey and sons, Inc. New York (1968).
[4]. Pontryagin, L.S., V.S.Boltyanski, R. V. Gamkrelidze, and E.F. Ischenko: The mathematical theory of optimal process. Wiley Interscience, New York (1962).
[5] A.Ghosh, T.Hazra , T.E.Aman : 'Largest Possible Delay in Control Problem' . International Journal Of Mathematics Trends and Technology 60.2 (2018)106-109 .