Double Integrals Involving Multivariable Gimel-Function, General Class of Polynomials and Biorthogonal Polynomial

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ABSTRACT
In the present paper, we evaluate a finite double integrals involving product of multivariable Gimel-function, biorthogonal polynomial and general class of polynomials. The integral evaluated is quite general in nature and yields a number of new integrals and its special cases.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, finite double integral, general class of polynomials, biorthogonal polynomials.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1, \cdots, z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2}:R_2;p_{i_3},q_{i_3},\tau_{i_3}:R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left. \begin{array}{l} ;\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \quad \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right)$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(\, |z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}\,), max(\, |z_1|, \cdots, |z_r|\,) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(\, |z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}\,), min(\, |z_1|, \cdots, |z_r|\,) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [6].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1and\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [5].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [8,9].

In your investigation, we shall use the following notations.

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

The biorthogonal pair of polynomial sets occurring in this paper are defined in the following manner [3] :

$$J_n^{(\alpha,\beta)}(x;k) = \frac{(\alpha+1)_{kn}}{n!}\sum_{j=0}^{n}(-)^j\binom{n}{j}\frac{(\alpha+\beta+n+1)_{kj}}{(\alpha+1)_{kj}}\left(\frac{1-x}{2}\right)^{kj} \tag{1.13}$$

and

$$K_n^{(\alpha,\beta)}(x;k) = \frac{1}{n!}\sum_{j=1}^{n}(-)^j\binom{\beta+n}{j}\left(\frac{x-1}{2}\right)^j\left(\frac{x+1}{2}\right)^{n-j}\sum_{w=0}^{j}(-)^w\binom{j}{w}\left(\frac{\alpha+w+1}{k}\right)_n \tag{1.14}$$

provided $Re(\alpha),Re(\beta) > -1$.

When $k=1$ the above polynomials set reduce to the Jacobi polynomials.

Srivastava ([7], p. 1, Eq. 1). have introduced the general class of polynomials :

$$S_N^M(x) = \sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}x^K \tag{1.15}$$

where M is an arbitrary positive integer and the coeficients $A_{N,K}$ are arbitrary constants real or complex. On specializing these coefficients $A_{N,K}$, $S_N^M[.]$ yields a number of known polynomials as special cases. These include, among others, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, bessel polynomials and several others, see ([10], p. 158-161).

2. Required results.

Lemma 1.

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}J_n^{(\alpha,\beta)}(1-2x,k)\mathrm{d}x = \frac{(\alpha+1)_{nk}}{n!}\sum_{m=0}^{n}\frac{(-n)_m}{m!}\frac{(\alpha+\beta+n+1)_{mk}}{(\alpha+1)_{mk}}\frac{\Gamma(\mu)\Gamma(\lambda+mk)}{\Gamma(\lambda+\mu+mk)} \tag{2.1}$$

provided $Re(\lambda)>0, Re(\mu)>0, Re(\alpha)>-1$ and $Re(\beta)>-1$.

Lemma 2.

$$\int_0^1 y^{\alpha-1}(1-y^2)^{\frac{\beta}{2}-1}(\sqrt{1-y^2}+iy)^{\alpha+\beta}\mathrm{d}y = \int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}\mathrm{d}\theta = e^{\omega\pi\frac{\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.2}$$

provided that $Re(\alpha), Re(\beta)>0$

3. Main integral.

In this section, we evaluate a general double finite integral.

Theorem.

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}\left(\sqrt{1-y^2}+iy\right)^{\rho+2\sigma}(1-x)^{\mu-1}(1-y^2)^{\sigma-1}J_n^{(\alpha,\beta)}(1-2x;k)S_N^M\left[ay^u(1-y^2)^v\left(\sqrt{1-y^2}+iy\right)^{u+2v}\right]$$

$$\gimel\left(z_1x^{\gamma_1}(1-x)^{\eta_1}y^{v_1}(1-y^2)^{\delta_1}\left(\sqrt{1-y^2}+iy\right)^{v_1+2\delta_1},\cdots,z_rx^{\gamma_r}(1-x)^{\eta_r}y^{v_r}(1-y^2)^{\delta_r}\left(\sqrt{1-y^2}+iy\right)^{v_r+2\delta_r}\right)$$

$$\mathrm{d}x\mathrm{d}y = \frac{(\alpha+1)_{kn}}{n!}exp\left(\frac{\omega\pi\rho}{2}\right)\sum_{j=0}^{n}(-)^j\sum_{K=0}^{[M/N]}\frac{(-N)_{MK}}{K!}A_{N,K}\binom{n}{j}\frac{(\alpha+\beta+n+1)_{kj}}{(\alpha+1)_{kj}}\left(\frac{1-x}{2}\right)^{kj}a^K$$

$$\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c} \mathrm{z}_1e^{\frac{\omega\pi v_1}{2}} \\ \cdot \\ \cdot \\ \mathrm{z}_re^{\frac{\omega\pi v_r}{2}} \end{array}\middle| \begin{array}{c} \mathbb{A};\ (1-\mu;\eta_1,\cdots,\eta_r;1),(1-\lambda-mk;\gamma_1,\cdots,\gamma_r;1), \\ \cdot \\ \cdot \\ \mathbb{B};\ \mathbf{B},(1\text{-}\lambda-mk-\mu;\gamma_1+\eta_1,\cdots,\gamma_r+\eta_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\rho-ut;v_1,\cdots,v_r;1),(1-2\sigma-2vt:2\delta_1,\cdots,2\delta_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (1\text{-}\rho-ut-2-2vt;\gamma_1+2\delta_r,\cdots,\gamma_r+2\delta_r;1):B \end{array}\right) \tag{3.1}$$

provided

$$u,v,\eta_i,\gamma_i,v_i,\delta_i>0(i=1,\cdots,r),\quad Re(\lambda)+\sum_{i=1}^{r}\gamma_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.\ \ Re(\mu)>0.$$

$$Re(\alpha) > -1. \ Re(\beta) > -1, \ Re(\rho) + \sum_{i=1}^{r} v_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0.$$

$$Re(\sigma) + \sum_{i=1}^{r} \delta_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0. \ Re(\mu) + \sum_{i=1}^{r} \eta_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0.$$

$$\left|\left(z_i x^{\gamma_i} y^{u_i}(1-y^2)^{\delta_i}\left(\sqrt{1-y^2}+iy\right)^{v_i+2\delta_i}\right| < \frac{1}{2}A_i^{(k)}\pi \quad , \quad \text{where } A_i^{(k)} \text{ is defined by (1.4).}$$

Proof
To prove (3.1), using the series representation of $S_N^M[.]$ with the help of (1.15), and expressing the multivariable Gimel-function as Mellin-Barnes multiple integrals contour with the the help of (1.1), interchanging the order of summations and integrations which is justified under the conditions mentioned above, we get (say I)

$$I = \sum_{K=0}^{[M/N]} \frac{(-N)_{MK}}{K!} A_{N,K} a^K \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\cdots,s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k}$$

$$\left[\int_0^1 x^{\lambda+\sum_{i=1}^r \gamma_i s_i}(1-x)^{u+\sum_{i=1}^r \eta_i s_i} J_n^{(\alpha,\beta)}(1-2x;k)\mathrm{d}x\right]$$

$$\left[\int_0^1 y^{\rho+ut\sum_{i=1}^r v_i s_i}(1-y^2)^{\sigma+vt+\sum_{i=1}^r v_i s_i}\left(\sqrt{1-y^2}+iy\right)^{\rho+2\sigma+ut+2vt+\sum_{i=1}^r (v_i+2\delta_i)s_i}\mathrm{d}y\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.2}$$

Now using the lemmae 1 and 2 respectively and finally interpreting the Mellin-Barnes multiple integrals contour in terms of the multivariable Gimel-function, we get the desired result (3.1).

4. Particular cases.

When $k=1$ the polynomials $J_n^{(\alpha,\beta)}(1-2x;1)$ reduce to the Jacobi polynomials. Taking $m=1$ and

$A_{N,K} = \dfrac{(a'+1)_N(\alpha'+\beta'+N+1)_K}{N!(\beta'+1)_K}$ in (2.1), taking $\eta_1,\cdots,\eta_r \to 1$and $\mu=\beta+1$, we obtain

Corollary.

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}\left(\sqrt{1-y^2}+iy\right)^{\rho+2\sigma}(1-x)^{\beta}(1-y^2)^{\sigma-1}P_n^{(\alpha,\beta)}(x)$$

$$P_N^{(\alpha',\beta')}\left[1-ay^u(1-y^2)^v\left(\sqrt{1-y^2}+iy\right)^{u+2v}\right]$$

$$\gimel\left(z_1x^{\gamma_1}y^{v_1}(1-y^2)^{\delta_1}\left(\sqrt{1-y^2}+iy\right)^{v_1+2\delta_1},\cdots,z_rx^{\gamma_r}y^{v_r}(1-y^2)^{\delta_r}\left(\sqrt{1-y^2}+iy\right)^{v_r+2\delta_r}\right)\mathrm{d}x\mathrm{d}y =$$

$$\frac{(-)^n\Gamma(1+\beta+n)(\alpha'+1)_N}{n!N!}\sum_{t=0}^{N}\frac{(-N)_t}{t!}\frac{(\alpha'+\beta'+n+1)_t}{(\alpha'+1)_t}a^t exp\left(\frac{\omega\pi(e+ut)}{2}\right)$$

$$\sum_{j=0}^{n}(-)^j\binom{n}{j}\frac{(\alpha+\beta+n+1)_j}{(\alpha+1)_j}\left(\frac{1-x}{2}\right)^j$$

$$\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 e^{\frac{\omega\pi v_1}{2}} & \mathbb{A};\ (1\text{-}\lambda;\gamma_1,\cdots,\gamma_r;1),(1-\alpha-\lambda;\gamma_1,\cdots,\gamma_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r e^{\frac{\omega\pi v_r}{2}} & \mathbb{B};\ \mathbf{B},(1\text{-}\lambda-mk-\mu;\gamma_1+\eta_1,\cdots,\gamma_r+\eta_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-e-ut;v}_1,\cdots,v_r;1),(1-2\sigma-2vt:2\delta_1,\cdots,2\delta_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (1\text{-}\rho-ut-2-2vt;\gamma_1+2\delta_r,\cdots,\gamma_r+2\delta_r;1):B \end{array}\right) \tag{4.1}$$

under the same existence conditions that (3.1).

Remarks :

We obtain the same double finite integrals with the functions defined in section I.

We can also evaluate three more integrals by replacing $J_n^{(\alpha,\beta)}(x;k)$ by $K_n^{(\alpha,\beta)}(x;k)$.

Khan have obtained the same relations with multivariable I-function defined by prasad [4].

4. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these double integrals, we can obtain a large simpler double or single finite integrals. Secondly by specialising the various parameters as well as variables in the multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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