Linear Rational Interpolant

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Abstract

An explicit representation of a C^0 piecewise rational function is developed which can be used to solve the problems of shape preserving interpolation. It is shown that the interpolation method can be applied to convex and/or monotonic set of data.

Key Words: *Rational, Shape Preserving, Monotonic, Convex Interpolation.*

I. INTRODUCTION

Many authors have been studied the problem of shape preserving interpolation .Fritsch and Carlson [4] and Fritsch and Butland [5] have discussed the piecewise cubic interpolation of monotonic data. Also Mcallister, Passow and Roulier [8] , [9] and [10] consider the piecewise polynomial interpolation of monotonic and convex data.(see [1],[2],[3],[6],[7] also)

In this paper we discuss a piecewise linear function (with linear numerator and linear denominator) which can be used to solve the problem of shape preserving interpolation.

The paper begins with the definition of Rational Linear Interpolant. The application of the interpolant to monotonic and/or convex sets of data is then discussed in Section 3. Section 4 contains the convergence properties of the interpolant.

II. THE RATIONAL LINEAR INTERPOLANT

Let (x_i, f_i),i =1, ..., n be a given set of data points, where $x_1 < x_2 < \ldots < x_n$.

Let $h_i = x_{i+1} - x_i$

$\Delta_i = (f_{i+1} - f_i) / h_i$

(2.1)

A piecewise rational polynomial function s $\in$ $C^o[x_1, x_n]$ defined as follows. For xε $C^o[x_1, x_n]$, let

$\theta = (x-x_i) / h_i$, i = 1,2...,n-1.

(2.2)

Then

$s(x) = P_i(\square) / Q(\square)$

(2.3)

where

$P_i(\square) = f_{i+1}\square + 2f_i(1-\square)$

(2.4)

$Q_i(\square) = 2-\square$

(2.5)

Let $s(x) = R(\square)\frac{Pi\ (\theta)}{Qi\ (\theta)}$

so that

$$s(x) = s_i(x) = \frac{f_{i+1}\theta + 2f_i(1-\theta)}{2-\theta}$$

(2.6)

The rational linear has the following interpolatory property

$s_i(x_i) = f_i$, $\qquad s_{i+1}(x_{i+1}) = f_{i+1}$

(2.7)

III. SHAPE PRESERVING INTERPOLATION

3.1 Monotonic Data: For simplicity of presentation, we assume a monotonic increasing set of data so that

$$f_1 \le f_2 \le \ldots\ldots \le f_n$$

(3.1)

or equivalently

$\Delta_i \ge 0$, i = 1, ..., n-1.

(3.2)

(In case of a monotonic decreasing set of data can be stated in a similar manner)

s(x) is monotonic increasing iff $s'(x) \ge 0$

(3.3)

We find that

$$s'(x) = = \frac{2\Delta_i}{(2-\theta)^2}$$

(3.4)

Now since $\Delta_i \ge 0 \quad \forall$ i, we find that

$s'(x) \ge 0$, x □ $[x_1, x_n]$

Thus s'(x) is always ≥0, so s is monotone.

Thus piecewise rational linear spline interpolant is a monotonicity preserving interpolant.

3.2 Convex Data: We assume a strictly convex set of data

$\Delta_1 <, \ldots\ldots, < \Delta_{n-1}$

(3.5)

(The case of concave data, where the inequalities are reversed, can be treated in a similar way).

Now s(x) is convex iff

$$s''(x) \geq 0$$

(3.6)

We find that

$$s''(x) = \frac{4\Delta_i}{h_i(2-\theta)^3}$$

θ lies between 0 and 1, (2- θ) is positive. Hence denominator is always positive.

∴ $s''(x)$ is positive in each subinterval.

S is convex in each subinterval.

(A)

Continuity of 1st derivative

$$s'(x_i^+) = \frac{2\Delta_i}{(2-\theta)^2} = \frac{2 f_{i+1} - f_i}{h_i(2 - \frac{x - x_i}{h_i})^2}$$

$$s'(x_i^-) = \frac{2 f_i - f_{i-1}}{h_{i-1}(2 - \frac{x - x_{i-1}}{h_{i-1}})^2} = \frac{2\Delta_{i-1}}{(2-\theta)^2}$$

Equating $s'(x_i^+) = s'(x_i^-)$ we get

$$\frac{2\Delta_i}{(2-\theta)^2} = \frac{2\Delta_{i-1}}{(2-\theta)^2}$$

Thus continuity of 1st derivative is satisfied if $\Delta_i = \Delta_{i-1}$

(3.7)

Suppose s□ C1 $[x_1, x_n]$. Now we examine $s''(x_i)$

We see

$$s''(x) = \frac{4\Delta_i}{h_i(2-\theta)^3}, \quad x \,\square\, [x_i, x_{i+1}].$$

Clearly at $\theta = 0$

$$s''(x) = \frac{\Delta_i}{2h_i}$$ which is positive.

Similarly we can see for $\theta = 1$

$s''(x)$ is positive provided

$\Delta_i \geq 0 \quad \forall \quad i$

(3.8)

Hence $s''(x)$ is positive $\forall$ x □ [x_1, x_n], and so s is convex in [x_1, x_n] if (3.7) holds.

Therefore, $s''(x)$ is positive in each subinterval provided (3.8) holds.

So s is convex at each mesh point (B)

By (A) and (B) s is convex if

$\Delta_i = \Delta_{i-1} = \Delta_{i+1} = \ldots\ldots$

IV. ERROR-ANALYSIS

Now we consider the convergence properties of the rational linear spline interpolant studied above. We prove the following:

***THEOREM 4.1**: Let f∊ C^1 [x_1, x_n] and let s be its piecewise rational (linear/linear) spline interpolant matching data values at knots. Then*

$$\left| f(x) - s(x) \right| \leq \omega(f,h) \quad x \; \square \; \square \; [x_i, x_{i+1}],\ i=1,2,\ldots n-1;$$

and $||e|| \leq \omega(f,h)$

where ||.|| *denotes the uniform norm on [x_i, x_{i+1}] and* $\omega(f,h)$ *denotes the modulus of continuity.*

Proof: We find that

$$f(x) - s(x) = \frac{f(x)(2-\theta) - f_{i+1}\theta - 2f_i(1-\theta)}{2-\theta}$$

$$= \frac{(f(x) - f_{i+1})\theta + 2(f(x) - f_i)(1-\theta)}{2-\theta}$$

Hence

$$\left| f(x) - s(x) \right| \leq \omega(f,h)$$

Therefore

$$||e|| = \max_{1 \leq i \leq n} (Sup\ |f(x)-s(x)|) \leq \omega\ (f,h)$$

Clearly as h→ 0 then ||e||→ 0.

Thus we find that the piecewise rational linear spline interpolant obtained in (2.6) preserves the monotonicity and convexity of the data and at the same time converges efficiently to the function to be interpolated.

V. CONCLUSION

We establish a piecewise rational (linear/linear) spline matching function values at the knots. We observe that the rational spline is monotonicity preserving. Moreover under the simple constraints given by (3.7), the spline preserves the convexity of the given data.

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