

# New Type of Quadratic Functional Equation and Its Stability

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## Abstract

We prove the generalized Hyers-Ulam Stability of the Quadratic functional equation

$f(2x_1 \pm x_2 \pm x_3 \pm x_4) = f(x_1 \pm x_3 \pm x_4) + f(x_1 \pm x_2 \pm x_3) + f(x_1 \pm x_2 \pm x_4) + f(\pm x_1) - f(\pm x_2) - f(\pm x_3) - f(\pm x_4)$   
in non-Archimedean Banach Spaces using Direct Method and Fixed Point Method.

## Keywords

Fixed point method, Hyers-Ulam stability, Non-Archimedean Banach space, Quadratic functional equation.

MSC: 39B22, 39B82, 46S10.

## I. INTRODUCTION

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (1.1)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be quadratic mapping. In 1996, Isac and Rassias [8] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [1],[4],[5],[11] - [16]). Recently, Brzdek et al.,[2] and Dong Yun Shin et al.,[7] gave a survey on the fixed point method and the direct method to prove the Hyers-Ulam stability of functional equations and functional inequalities. In this paper, we introduce the following quadratic functional equation

$$f(2x_1 + x_2 + x_3 + x_4) = f(x_1 + x_3 + x_4) + f(x_1 + x_2 + x_3) + f(x_1 + x_2 + x_4) + f(x_1) - f(x_2) - f(x_3) - f(x_4) \quad (1.2)$$

$$f(2x_1 - x_2 - x_3 - x_4) = f(x_1 - x_3 - x_4) + f(x_1 - x_2 - x_3) + f(x_1 - x_2 - x_4) + f(-x_1) - f(-x_2) - f(-x_3) - f(-x_4) \quad (1.3)$$

$$f(2x_1 \pm x_2 \pm x_3 \pm x_4) = f(x_1 \pm x_3 \pm x_4) + f(x_1 \pm x_2 \pm x_3) + f(x_1 \pm x_2 \pm x_4) + f(\pm x_1) - f(\pm x_2) - f(\pm x_3) - f(\pm x_4) \quad (1.4)$$

in non-Archimedean Banach Spaces using Direct Method and Fixed Point Method.

All over this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

**Definition 1.1[9]** Let  $X$  be a vector space over a field  $K$  with a non-Archimedean valuation  $|\cdot|$ . A function  $\|\cdot\|: X \rightarrow [0, \infty)$  is said to be a non-Archimedean norm (valuation) if it satisfies the following conditions:

- i)  $\|x\| = 0$  if and only if  $x = 0$ ;
- ii)  $\|rx\| = |r|\|x\|$  for all  $r \in K, x \in X$ ;
- iii) The strong triangle inequality

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}$$

for all  $x, y \in X$ . Then  $(X, \|\cdot\|)$  is called a non-Archimedean normed space.

We recall a fundamental result in fixed point theory.

**Theorem 1.2 ([3],[6])** Let  $(X, d)$  be a complete generalized metric space and let  $J: X \rightarrow X$  be a strictly contractive mapping with Lipschitz constant  $L > 1$ . Then for each given element  $x \in X$ , either  $d(J^n x, J^{n+1} x) = \infty$  for all non negative integers  $n$  or there exists a positive integer  $n_0$  such that

- i)  $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0$ ;

- ii) The sequence  $\{J^n x\}$  converges to a fixed point  $y^*$  of  $J$ ;
- iii)  $y^*$  is the unique fixed point of  $J$  in the set  $Y = \left\{y \in X \mid d\left(J^n x, y\right) < \infty\right\}$ ;
- iv)  $d\left(y, y^*\right) \leq \frac{1}{1-L} d\left(y, Jy\right)$  for all  $y \in Y$ .

Throughout this paper, assume that  $X$  is a non-Archimedean normed space and that  $Y$  is a non-Archimedean Banach Space.

## II. GENERAL SOLUTION OF THE QUADRATIC FUNCTIONAL EQUATION

In this section, we find out the general solution of the Quadratic functional equation .

**Theorem 2.1** If a mapping  $f : X \rightarrow Y$  satisfies the functional equation (1.1) for all  $x, y \in X$ , iff the function  $f : X \rightarrow Y$  satisfies the functional equation (1.2) for all  $x, y \in X$ .

**Proof.** Setting  $(x, y)$  by  $(0, 0)$  in (1.1), we get  $f(0) = 0$ . Again replacing  $(x, y)$  by  $(0, x)$  in (1.1), we have  $f(-x) = f(x)$ , for all  $x \in X$ . Replacing  $(x, y)$  by  $(x, x)$  and  $(2x, x)$  in (1.1), we obtain  $f(2x) = 4x$  and  $f(3x) = 9f(x)$ , for all  $x \in X$ . respectively. In general for any positive integer  $a$ , we receive  $f(ax) = a^2 f(x)$ , for all  $x \in X$ . Replacing  $x = x_3$  and  $y = x_4$  in (1.1), we get

$$f\left(x_3 + x_4\right) + f\left(x_3 - x_4\right) = 2f\left(x_3\right) + 2f\left(x_4\right) \tag{2.1}$$

for all  $x_3, x_4 \in X$ . It follows from (2.1), we have

$$f\left(x_3 + x_4\right) + f\left(x_1 + x_2 + x_3 - \left(x_1 + x_2 + x_4\right)\right) = 2f\left(x_3\right) + 2f\left(x_4\right) \tag{2.2}$$

for all  $x_1, x_2, x_3, x_4 \in X$ . Adding  $f\left(x_1 + x_2 + x_3 + \left(x_1 + x_2 + x_4\right)\right)$  on both sides in (2.2), we arrive

$$\begin{aligned} f\left(x_3 + x_4\right) + f\left(x_1 + x_2 + x_3 - \left(x_1 + x_2 + x_4\right)\right) + f\left(x_1 + x_2 + x_3 + \left(x_1 + x_2 + x_4\right)\right) \\ = 2f\left(x_3\right) + 2f\left(x_4\right) + f\left(x_1 + x_2 + x_3 + \left(x_1 + x_2 + x_4\right)\right) \end{aligned} \tag{2.3}$$

for all  $x_1, x_2, x_3, x_4 \in X$ . Using (1.1) in (2.3), we get

$$f\left(x_3 + x_4\right) + 2f\left(x_1 + x_2 + x_3\right) + 2f\left(x_1 + x_2 + x_4\right) = 2f\left(x_3\right) + 2f\left(x_4\right) + f\left(2x_1 + x_2 + x_3 + x_4 + x_2\right) \tag{2.4}$$

for all  $x_1, x_2, x_3, x_4 \in X$ . It follows from (2.4), we have

$$f\left(x_1 + x_3 + x_4 - x_1\right) + 2f\left(x_1 + x_2 + x_3\right) + 2f\left(x_1 + x_2 + x_4\right) = 2f\left(x_3\right) + 2f\left(x_4\right) + f\left(2x_1 + x_2 + x_3 + x_4 + x_2\right) \tag{2.5}$$

for all  $x_1, x_2, x_3, x_4 \in X$ . Adding  $f\left(x_1 + x_3 + x_4 + x_1\right)$  on both sides in (2.5), we arrive

$$\begin{aligned} f\left(x_1 + x_3 + x_4 - x_1\right) + f\left(x_1 + x_3 + x_4 + x_1\right) + 2f\left(x_1 + x_2 + x_3\right) + 2f\left(x_1 + x_2 + x_4\right) \\ = 2f\left(x_3\right) + 2f\left(x_4\right) + f\left(2x_1 + x_2 + x_3 + x_4 + x_2\right) + f\left(x_1 + x_3 + x_4 + x_1\right) \end{aligned} \tag{2.6}$$

for all  $x_1, x_2, x_3, x_4 \in X$ . Using (1.1) in (2.6), we obtain

$$\begin{aligned} 2f\left(x_1 + x_3 + x_4\right) + 2f\left(x_1\right) + 2f\left(x_1 + x_2 + x_3\right) + 2f\left(x_1 + x_2 + x_4\right) \\ = 2f\left(x_3\right) + 2f\left(x_4\right) + f\left(2x_1 + x_2 + x_3 + x_4 + x_2\right) + f\left(2x_1 + x_3 + x_4\right) \end{aligned} \tag{2.7}$$

for all  $x_1, x_2, x_3, x_4 \in X$ . It follows from (2.7), we have

$$\begin{aligned} 2f\left(x_1 + x_3 + x_4\right) + 2f\left(x_1\right) + 2f\left(x_1 + x_2 + x_3\right) + 2f\left(x_1 + x_2 + x_4\right) \\ = 2f\left(x_3\right) + 2f\left(x_4\right) + f\left(2x_1 + x_2 + x_3 + x_4 + x_2\right) + f\left(2x_1 + x_2 + x_3 + x_4 - x_2\right) \end{aligned} \tag{2.8}$$

for all  $x_1, x_2, x_3, x_4 \in X$ . Using (1.1) in (2.8), we get

$$2f(2x_1 + x_2 + x_3 + x_4) = 2f(x_1 + x_3 + x_4) + 2f(x_1) + 2f(x_1 + x_2 + x_3) + 2f(x_1 + x_2 + x_4) - 2f(x_2) - 2f(x_3) - 2f(x_4) \quad (2.9)$$

for all  $x_1, x_2, x_3, x_4 \in X$ . Dividing 2 on both sides of (2.9), we receive (1.2)

Conversely, let  $f : X \rightarrow Y$  satisfies the functional equation (1.2). Setting  $(x_1, x_2, x_3, x_4)$  by  $(0,0,0,0)$ , we get  $f(0) = 0$ . Replacing  $(x_1, x_2, x_3, x_4)$  by  $(x, -x, x, -x)$  in (1.2) we obtain  $f(-x)=f(x)$ . Hence  $f$  is even. Again setting  $(x_1, x_2, x_3, x_4)$  by  $(x,0,0,0)$  in (1.2), we get  $f(2x) = 4f(x)$  for all  $x \in X$ . Replace  $(x_1, x_2, x_3, x_4)$  by  $(x, x, 0, 0)$ ,  $(x, x, x, 0)$  and  $(x, x, x, x)$  in (1.2), we arrive  $f(3x) = 9f(x)$ ,  $f(4x) = 16f(x)$  and  $f(5x) = 25f(x)$ , for all  $x \in X$ , respectively. In general for any positive integer “a”, we have  $f(ax) = a^2 f(x)$ . Put  $x_1 = 0$  in (1.2) we get,

$$f(x_2 + x_3 + x_4) = f(x_3 + x_4) + f(x_2 + x_3) + f(x_2 + x_4) - f(x_2) - f(x_3) - f(x_4) \quad (2.10)$$

for all  $x_2, x_3, x_4 \in X$ . Adding  $f(x_2 + x_3 - x_4)$  on both sides in (2.10), we have

$$2f(x_2 + x_3) + 2f(x_4) = f(x_3 + x_4) + f(x_2 + x_3) + f(x_2 + x_4) - f(x_2) - f(x_3) - f(x_4) + f(x_2 + x_3 - x_4) \quad (2.11)$$

for all  $x_2, x_3, x_4 \in X$ . Put  $x_2 = x$ ,  $x_3 = 0$  and  $x_4 = y$  in (2.11), we get our desired result (1.1).

**Theorem 2.2** If a mapping  $f : X \rightarrow Y$  satisfies the functional equation (1.1) for all  $x, y \in X$ , iff the function  $f : X \rightarrow Y$  satisfies the functional equation (1.3) for all  $x, y \in X$ .

**Theorem 2.3** If a mapping  $f : X \rightarrow Y$  satisfies the functional equation (1.1) for all  $x, y \in X$ , iff the function  $f : X \rightarrow Y$  satisfies the functional equation (1.4) for all  $x, y \in X$ .

### III. STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION (1.4) –DIRECT METHOD

In this section, we investigate the stability of the Quadratic functional equation (1.4) in Non-Archimedean Banach Space using Direct Method.

**Theorem 3.1** Let  $G$  is an quadratic semi group and  $X$  is a complete non-Archimedean space. Assume that  $\varphi : G^4 \rightarrow [0, +\infty)$  be a function such that

$$\lim_{n \rightarrow \infty} \frac{\varphi(5^n x_1, 5^n x_2, 5^n x_3, 5^n x_4)}{|5|^{2n}} = 0 \quad (3.1)$$

for all  $x_1, x_2, x_3, x_4 \in G$ . Let for all  $x \in G$

$$\Phi(x) = \sup_{k \geq 0} \left\{ \frac{\varphi(5^k x, 5^k x, 5^k x, 5^k x)}{|5|^{2k}}; k \in \mathbb{N} \cup \{0\} \right\} \quad (3.2)$$

exists. Suppose that  $f : G \rightarrow X$  be a mapping satisfying the inequality

$$\begin{aligned} \|f(2x_1 \pm x_2 \pm x_3 \pm x_4) - f(x_1 \pm x_3 \pm x_4) - f(x_1 \pm x_2 \pm x_3) - f(x_1 \pm x_2 \pm x_4) + f(\pm x_1) - f(\pm x_2) - f(\pm x_3) - f(\pm x_4)\| \\ \leq \varphi(x_1, x_2, x_3, x_4) \end{aligned} \quad (3.3)$$

for all  $x_1, x_2, x_3, x_4 \in G$ . Then the limit

$$Q(x) := \lim_{n \rightarrow \infty} \frac{f(5^n x)}{5^{2n}}$$

exist for all  $x \in G$  and  $Q : G \rightarrow X$  is an quadratic mapping satisfying

$$\|f(x) - Q(x)\| \leq \frac{1}{|25|} \Phi(x) \quad (3.4)$$

for all  $x \in G$ . More over, if

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \max \left\{ \frac{\varphi(5^k x, 5^k x, 5^k x, 5^k x)}{|5|^{2k}}; j \leq k < n + j \right\} = 0$$

Then Q is the unique mapping satisfying (3.4).

**Proof.** Setting  $(x_1, x_2, x_3, x_4)$  by  $(x, x, x, x)$  in (3.3), we have

$$\left\| \frac{f(5x)}{5^2} - f(x) \right\|_X \leq \frac{1}{|25|} \varphi(x, x, x, x) \tag{3.5}$$

for all  $x \in G$ . Replacing  $x$  by  $5^n x$  in (3.5), we obtain

$$\left\| \frac{f(5^{n+1}x)}{5^{2(n+1)}} - \frac{f(5^n x)}{5^{2n}} \right\|_X \leq \frac{1}{|5|^{2(n+1)}} \varphi(5^n x, 5^n x, 5^n x, 5^n x) \tag{3.6}$$

It follows from (3.1) and (3.6) that the sequence  $\left\{ \frac{f(5^n x)}{5^{2n}} \right\}_{n=1}^{\infty}$  is a Cauchy sequence. Since X is complete, so

$\left\{ \frac{f(5^n x)}{5^{2n}} \right\}_{n=1}^{\infty}$  is convergent. Set

$$Q(x) := \lim_{n \rightarrow \infty} \frac{f(5^n x)}{5^{2n}}$$

Using induction, we see that

$$\left\| \frac{f(5^n x)}{5^{2n}} - f(x) \right\|_X \leq \frac{1}{|5|^2} \max \left\{ \frac{\varphi(5^k x, 5^k x, 5^k x, 5^k x)}{|5|^{2k}}; 0 \leq k < n \right\} \tag{3.7}$$

Indeed, (3.7) holds for  $n = 1$  by (3.5). Let (3.7) holds for  $n$ , so by (3.6), we have

$$\begin{aligned} \left\| \frac{f(5^{n+1}x)}{5^{2(n+1)}} - f(x) \right\|_X &= \left\| \frac{f(5^{n+1}x)}{5^{2(n+1)}} - \frac{f(5^n x)}{5^{2n}} + \frac{f(5^n x)}{5^{2n}} - f(x) \right\|_X \\ &\leq \max \left\{ \left\| \frac{f(5^{n+1}x)}{5^{2(n+1)}} - \frac{f(5^n x)}{5^{2n}} \right\|_X, \left\| \frac{f(5^n x)}{5^{2n}} - f(x) \right\|_X \right\} \\ &\leq \frac{1}{|5|^2} \max \left\{ \frac{\varphi(5^n x, 5^n x, 5^n x, 5^n x)}{|5|^{2n}}, \max \left\{ \frac{\varphi(5^k x, 5^k x, 5^k x, 5^k x)}{|5|^{2k}}; 0 \leq k < n \right\} \right\} \\ &= \frac{1}{|5|^2} \max \left\{ \frac{\varphi(5^k x, 5^k x, 5^k x, 5^k x)}{|5|^{2k}}; 0 \leq k < n + 1 \right\}. \end{aligned} \tag{3.8}$$

So for all  $n \in \mathbb{N}$  and all  $x \in G$ , (3.7) holds. By taking  $n$  to approach infinity in (3.8), one obtain (3.4). If R is another mapping satisfies (3.4), then for  $x \in G$ , we get

$$\left\| Q(x) - R(x) \right\|_X = \lim_{k \rightarrow \infty} \left\| \frac{Q(5^k x)}{5^{2k}} - \frac{R(5^k x)}{5^{2k}} \right\|_X$$

$$\begin{aligned}
 &= \lim_{k \rightarrow \infty} \left\| \frac{Q(5^k x)}{5^{2k}} \pm \frac{f(5^k x)}{5^{2k}} - \frac{R(5^k x)}{5^{2k}} \right\| \\
 &\leq \lim_{k \rightarrow \infty} \max \left\{ \left\| \frac{Q(5^k x) - f(5^k x)}{5^{2k}} \right\|_X, \left\| \frac{f(5^k x) - R(5^k x)}{5^{2k}} \right\|_X \right\} \\
 &\leq \lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \max \left\{ \frac{\varphi(5^k x, 5^k x, 5^k x, 5^k x)}{|5|^{2k}}; j \leq k < n + j \right\} \\
 &= 0
 \end{aligned}$$

Therefore  $Q = R$ . This completes the proof.

**Corollary 3.2** Let  $f : G \rightarrow X$  is a mapping satisfying the inequality

$$\begin{aligned}
 \|f(2x_1 \pm x_2 \pm x_3 \pm x_4) - f(x_1 \pm x_3 \pm x_4) - f(x_1 \pm x_2 \pm x_3) - f(x_1 \pm x_2 \pm x_4) - f(\pm x_1) - f(\pm x_2) - f(\pm x_3) - f(\pm x_4)\| \\
 \leq \delta (\eta (\|x_1\|) + \eta (\|x_2\|) + \eta (\|x_3\|) + \eta (\|x_4\|))
 \end{aligned} \tag{3.9}$$

for all  $x_1, x_2, x_3, x_4 \in G$ . Then the limit  $Q(x) = \lim_{n \rightarrow \infty} \frac{f(5^n x)}{5^{2n}}$  exists for all  $x \in G$  and  $Q : G \rightarrow X$  is a unique quadratic mapping such that

$$\|f(x) - Q(x)\| \leq \frac{25\delta\eta(\|x\|)}{6|5|^2}$$

for all  $x \in G$ .

#### IV. STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION (1.4) – FIXED POINT METHOD

In this section, we establish the stability of the Quadratic functional equation (1.4) in Non-Archimedean Banach Space using Fixed Point Method.

**Theorem 4.1** Let  $\varphi : G^4 \rightarrow [0, \infty)$  be a function such that there exists and  $L < 1$  with

$$\varphi\left(\frac{x_1}{5}, \frac{x_2}{5}, \frac{x_3}{5}, \frac{x_4}{5}\right) \leq \frac{L}{|25|} \varphi(x_1, x_2, x_3, x_4) \tag{4.1}$$

for all  $x_1, x_2, x_3, x_4 \in G$ . Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and (3.3) for all  $x_1, x_2, x_3, x_4 \in G$ . Then there exists a unique quadratic mapping  $Q : G \rightarrow X$  such that

$$\|f(x) - Q(x)\| \leq \frac{L}{|25|(1-L)} \varphi(x, x, x, x) \tag{4.2}$$

for all  $x \in G$ .

**Proof.** Setting  $(x_1, x_2, x_3, x_4)$  by  $(x, x, x, x)$  in (3.3), we have

$$\|f(5x) - 25f(x)\| \leq \varphi(x, x, x, x) \tag{4.3}$$

for all  $x \in G$ . Now, consider the set  $S := \{h : X \rightarrow Y, h(0) = 0\}$  and introduced the generalized metric on  $S$ :

$$d(g, h) = \inf \left\{ \mu \in \square_+ : \|g(x) - h(x)\| \leq \mu \varphi(x, x, x, x), \forall x \in G \right\},$$

where, as usual,  $\inf \phi = +\infty$ . It is easy to show that  $(S, d)$  is complete. Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := 25g\left(\frac{x}{5}\right)$$

for all  $x \in G$ . Let  $g, h \in S$  be given such that  $d(g, h) = \varepsilon$ . Then

$$\|g(x) - h(x)\| \leq \varepsilon \varphi(x, x, x, x)$$

for all  $x \in G$ . Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\| 25g\left(\frac{x}{5}\right) - 25h\left(\frac{x}{5}\right) \right\| \leq |25| \varepsilon \varphi\left(\frac{x}{5}, \frac{x}{5}, \frac{x}{5}, \frac{x}{5}\right) \\ &\leq |25| \varepsilon \frac{L}{|25|} \varphi(x, x, x, x) \leq L\varepsilon \varphi(x, x, x, x) \end{aligned}$$

for all  $x \in G$ . So  $d(g, h) = \varepsilon$  implies that  $d(Jg, Jh) \leq L\varepsilon$ . This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all  $g, h \in S$ . It follows from (4.3) that

$$\left\| f(x) - 25f\left(\frac{x}{5}\right) \right\| \leq \varphi\left(\frac{x}{5}, \frac{x}{5}, \frac{x}{5}, \frac{x}{5}\right) \leq \frac{L}{|25|} \varphi(x, x, x, x)$$

for all  $x \in G$ . So  $d(f, Jf) \leq \frac{L}{|25|}$ . By Theorem 1.2, there exists a mapping  $Q : G \rightarrow X$  satisfying the following:

(1)  $Q$  is a fixed point of  $J$ ,

$$\text{i.e., } Q(x) = 25Q\left(\frac{x}{5}\right) \tag{4.4}$$

for all  $x \in G$ . The mapping  $Q$  is a unique fixed point of  $J$  in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that  $Q$  is a unique mapping satisfying (4.4) such that there exists a  $\mu \in (0, \infty)$  satisfying

$$\|f(x) - Q(x)\| \leq \mu \varphi(x, x, x, x)$$

for all  $x \in G$ ;

(2)  $d(J^l f, Q) \rightarrow 0$  as  $l \rightarrow \infty$ . This implies the equality

$$\lim_{l \rightarrow \infty} 5^{2l} f\left(\frac{x}{5^l}\right) = Q(x)$$

for all  $x \in G$ ;

(3)  $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$ , which implies

$$\|f(x) - Q(x)\| \leq \frac{L}{|25|(1-L)} \varphi(x, x, x, x)$$

for all  $x \in G$ . It follows from (4.1) and (3.3) that

$$\begin{aligned} &\left\| Q(2x_1 \pm x_2 \pm x_3 \pm x_4) - Q(x_1 \pm x_3 \pm x_4) - Q(x_1 \pm x_2 \pm x_3) - Q(x_1 \pm x_2 \pm x_4) - Q(\pm x_1) + Q(\pm x_2) + Q(\pm x_3) + Q(\pm x_4) \right\| \\ &= \lim_{n \rightarrow \infty} |5|^{2n} \left\| f\left(\frac{2x_1 \pm x_2 \pm x_3 \pm x_4}{5^n}\right) - f\left(\frac{x_1 \pm x_3 \pm x_4}{5^n}\right) - f\left(\frac{x_1 \pm x_2 \pm x_3}{5^n}\right) - f\left(\frac{x_1 \pm x_2 \pm x_4}{5^n}\right) - f\left(\frac{\pm x_1}{5^n}\right) + f\left(\frac{\pm x_2}{5^n}\right) + f\left(\frac{\pm x_3}{5^n}\right) + f\left(\frac{\pm x_4}{5^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} |5|^{2n} \varphi\left(\frac{x_1}{5^n}, \frac{x_2}{5^n}, \frac{x_3}{5^n}, \frac{x_4}{5^n}\right) = 0 \end{aligned}$$

for all  $x_1, x_2, x_3, x_4 \in G$ . So

$$\left\| \varrho(2x_1 \pm x_2 \pm x_3 \pm x_4) - \varrho(x_1 \pm x_3 \pm x_4) - \varrho(x_1 \pm x_2 \pm x_3) - \varrho(x_1 \pm x_2 \pm x_4) - \varrho(\pm x_1) + \varrho(\pm x_2) + \varrho(\pm x_3) + \varrho(\pm x_4) \right\| = 0$$

for all  $x_1, x_2, x_3, x_4 \in G$ . By Theorem 2.3, the mapping  $\varrho : G \rightarrow X$  is quadratic.

**Corollary 4.2** Let  $r < 5$  and  $\lambda$  be non negative real numbers and let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and

$$\begin{aligned} \left\| f(2x_1 \pm x_2 \pm x_3 \pm x_4) - f(x_1 \pm x_3 \pm x_4) - f(x_1 \pm x_2 \pm x_3) - f(x_1 \pm x_2 \pm x_4) + f(\pm x_1) - f(\pm x_2) - f(\pm x_3) - f(\pm x_4) \right\| \\ \leq \lambda \left( \|x_1\|^r + \|x_2\|^r + \|x_3\|^r + \|x_4\|^r \right) \end{aligned} \quad (4.5)$$

for all  $x_1, x_2, x_3, x_4 \in G$ . Then there exists a unique quadratic mapping  $\varrho : G \rightarrow X$  such that

$$\|f(x) - \varrho(x)\| \leq \frac{25\delta\eta(\|x\|)}{6|5|^{2-r}}$$

for all  $x \in G$ .

**Proof.** The proof follows from Theorem 4.1 by taking  $\varphi(x_1, x_2, x_3, x_4) = \lambda (\|x_1\|^r + \|x_2\|^r + \|x_3\|^r + \|x_4\|^r)$

for all  $x_1, x_2, x_3, x_4 \in G$ . Then we can choose  $L = |5|^{2-r}$  and we get the required result.

## V. CONCLUSION

Throughout this paper, we introduced the following results:

- (i) In the section II, we established the general solution for the functional equation (1.4)
- (ii) In section III, we investigated the stability of the Quadratic functional equation (1.4) in Non-Archimedean Banach Space using Direct Method and also the output of the stability results exposed in Corollary 3.2.
- (iii) In the section IV, we estimated the stability of the Quadratic functional equation (1.4) in Non-Archimedean Banach Space using fixed Method and also the output of the stability results exposed in Corollary 4.2.

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