Single Counter Markovian Queuing Model with Multiple Inputs

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Abstract

This paper analyzes a M/M/1 queueing model with multiple input, where the rate of arrival and service capacity follow Poisson distribution. The arrival process of the model remains in three stages said to be state I, II, and III. The system remains in three state for a random time which is exponentially distributed. The queue discipline is first-in-first-out. Laplace transforms of the various probability generating functions are obtained and the steady state results are derived. The probability that the arrival process (input) will be. In state I, II and III is also analyzed.

Key Words - *Queuing Theory, Markovian Process, Exponential Distribution, Poisson distribution and Probability Generating Function*.

I. INTRODUCTION

Queuing theory is the mathematical study of waiting lines, or queues [1]. In queuing theory, a model is constructed so that queue lengths and waiting times can be predicted [1]. Queuing theory is generally considered a branch of operations research because the results are often used when making business decisions about the resources needed to provide service. Queuing theory explored with research by Agner Krarup Erlang when he created models to describe the Copenhagen telephone exchange [1]. The word queue comes via French, from the Latin cavda, meaning tail. The spelling "queuing" over "queuing" is typically encountered in academic research field. In fact, one of the flagship journals of the profession is named Queuing Systems.

In queuing system, utilization plays the crucial role and is defined as the proportion of the system's resources which is used by the traffic which arrives at it. It should be strictly less than one for the system to function well. It is usually denoted by the symbol ρ . If $\rho \geq 1$, then the queue will continue to grow as time goes on. In the simplest case of an M/M/1 queue (Poisson arrivals and a single Poisson server) then it is given by the mean arrival rate over the mean service rate, that is,

$\rho = \dfrac{\lambda}{\mu}$, where λ is the mean arrival rate and μ is the mean service rate.

More generally:

$\rho = \dfrac{\lambda}{\mu \times c}$, where λ is the mean arrival rate, μ is the mean service rate and c is the number of servers, such as in an M/M/c queue.

In general, a lower utilization corresponds to less queuing for customers but means that the system is more idle, which may be considered inefficient.

A useful queuing model represents a real life system with sufficient accuracy and is analytically tractable. A queuing model based on the Poisson process and its companion exponential probability distribution often meets two requirements. A Poisson process models random events as maintaining from a memoryless process. That is, the length of time interval from the current time to the occurrence of the next event does not depend upon the time occurrence of the last event.

Applications of queuing with customer impatience can be seen in business, traffic modeling and industries, computer-communication, health sector and medical science etc.,. A set of queuing problems by Avi-Itzhak and Naor [2] and Gaver [4] may be considered as possessing characteristics of service heterogeneity, to wit, when the service station is subject to breakdown. Krishanamoorthy [6] considers a Poisson queue with two-heterogeneous servers with modified queue disciplines. The steady-state solution, transient solution and busy period distribution for the first discipline and the steady-state solution for the second discipline are obtained. Singh [8] extends the work of Krishanamoorthy [6] on heterogeneous servers by incorporating balking to compares results with homogeneous servers queue to show that the conditions under which the heterogeneous system is better than corresponding homogeneous system. Heterogeneous researches in the area of queuing theory have been studied be Yechiali and Naor [9], Neuts [11], Murari and Agarwal [12]. In the queuing model studied in [9], the arrival pattern at a service station is Poisson, the service time distribution is taken as negative exponential and the parameters depend on the environment. In the queuing model studied by Murari and Agarwal [12], the arrival process breaks down with two arrival intensities viz., λ and 0. The system with zero arrival rate is repaired and then brought to stage I having λ as mean arrival rate. In a research paper by I. M. Premachandra and Liliana Gongalez [14], the service mechanism consists of three stages performed in sequence by two servers. Kumar and Sharma [19] study the two-heterogeneous server Markovian queuing model with discouraged arrivals, reneging and retention of reneged customers. The steady-state probabilities of system size are obtained explicitly using iterative method and also discussed some useful measures of effectiveness.

In the ensuing problem, the arrival pattern is non-homogeneous, i.e., there exists three different arrival rates, viz., λ_1 (when input source in state I), λ_2 (when input source in state II) and zero (when input in state III). Further, it is assumed that λ_2 lower than λ_1 (i.e., $\lambda_2 < \lambda_1$). The input source is operative in one state at a time. The queue discipline is first in first out which states that customers are served one at a time and that the customer that has been waiting the longest is served first. The service time of customers is exponentially distributed with Poissonian service rate μ corresponding to arrival rates λ_1, λ_2 and 0 (zero). The system starts with input source in the state I. The time interval during which the inputs function at any one level is an exponentially distributed random variable. The state I can jump to state II and state III with Poisson intensity η_{12} and η_{13} respectively.

The model under consideration possesses many practical use, e.e., consider a machine which can be in one of three states, i.e., state I (machine working with full capacity), state II (machine working with low capacity), and state III (machine not in working condition i.e., broken down). The state I can jump in state II by reasons like power shortage, go slow practice of workers, or shortage of raw material and in state III by reasons like power failure, strike of workers, or machinery failure etc., . Let us assume that a machine in state I will remain in this way for a random time and will go either state II or state III. A machine in state II will remain in that way for a random time and will then go to either state I or state III. A broken machine (state III) will be repaired and when repaired it may go to state I. If the machine is manufacturing some items then it may be identified as an input source. The behavior of input source exactly the same as that of machine. Thus, the input source remains in three states.

II. FORMULATION OF MODEL

(i) A stream of Poisson-type unit arrives at a single service station. There exists three different arrival rates λ_1, λ_2 and 0 (zero), only one of which is operative at any instant.

(ii) The system has Poissonian rate μ to arrival rates λ_1, λ_2 and 0 (zero). The state of the system, operating with arrival rate λ_1 is designated as P, operating with arrival rate λ_2 is designated as L, and operating with arrival rate zero is designated as M.

(iii) The queue discipline is first-in-first-out.

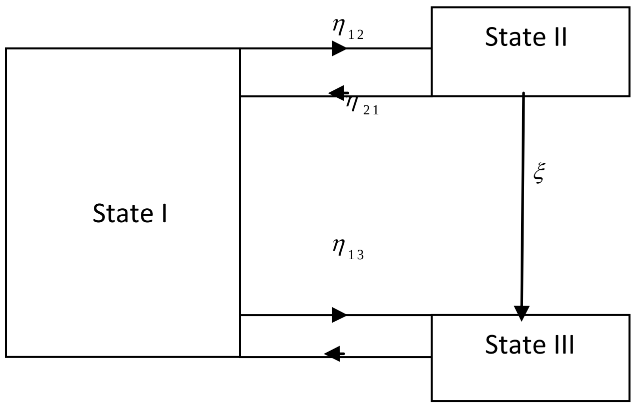
(iv) The Poisson rate at which the input source moves from state P to L or L to P and from state P to M or M to P and from state L to M are denoted by η_{12}, η_{21} and η_{13}, η_{31} and ξ respectively.

(v) The system starts with input source in the state I.

(vi) If at any instant the queue length is N, then the arriving customers will be considered lost for the system.

(vii) The service is instantaneous.

The transition rate from one state to another state is as shown in the following diagram 1:



Further, service time is assumed to be exponentially distributed with parameter μ for all states of the input. The stochastic processes involved, viz., interarrival time of units and service time of customers are independent of each other.

The paper divided in two sections (A and B). The following results have been analyzed.

SECTION A (TIME DEPENDENT SOLUTION)

(i) $L.T.'^{s}$ of the probability generating function distribution of the number of units in the system for different states of the input in both limited and unlimited space.

(ii) $L.T.'^{s}$ of the probabilities for different states of the input.

SECTION B (STEADY STATE SOLUTION)

(i) Explicit steady state probabilities for different states of the input.

(ii) Explicit steady state probability generating function of the distribution of the number of units in the system for different states of the input.

III. SOLUTION OF QUEUING MODEL

In this section, the mathematical framework of the queuing model is presented. The time dependent and steady state solution of the problem have been discussed.

A. Time Dependent Solution

Define,

$P_n(t)$ = Probability that at time t, the input is in the state I and n units are in the system, including one in service.

$L_n(t)$ = Probability that at time t, the input is in the state II and n units are in the system, including one in service.

$M_n(t)$ = Probability that at time t, the input is in the state III and n units are in the system, including one in service.

$R_n(t)$ = Probability that n units are in the system at time t, including one in service.

Obviously,

$$R_n(t) = P_n(t) + L_n(t) + M_n(t)$$

Initial conditions:

Let the time be reckoned from the instant when the queue length is zero and the input source is in the state I.

$$P_n(0) = \begin{cases} 1 & , \quad n = 0 \\ 0 & , \quad othrewise \end{cases}$$

$$L_n(0) = 0, \qquad \forall n \geq 0$$

$$M_n(0) = 0, \qquad \forall n \geq 0$$

Define the following probability generating functions of $P_n(t)$, $L_n(t)$, $M_n(t)$ and $R_n(t)$ by:

$$P(z,t) = \sum_{n=0}^{N} z^n P_n(t)$$

$$L(z,t) = \sum_{n=0}^{N} z^n L_n(t)$$

$$M(z,t) = \sum_{n=0}^{N} z^n M_n(t)$$

$$R(z,t) = \sum_{n=0}^{N} z^n R_n(t)$$

These must converge within the unit circle $|z| = 1$.

KQL MOGORV'S forward differential equations lead to the model:

For State I

$$\frac{d}{dt} P_o(t) = -(\lambda_1 + \eta_{12} + \eta_{13}) P_0(t) + \mu P_1(t) + \eta_{21} L_0(t) + \eta_{31} M_0(t) \tag{1.1}$$

$$\frac{d}{dt} P_n(t) = f P_n(t) + \lambda_1 P_{n-1}(t) + \mu P_{n+1}(t) + \eta_{21} L_n(t) + \eta_{31} M_n(t), \quad 0<n<N \tag{1.2}$$

$$\frac{d}{dt} P_N(t) = \lambda_1 P_{N-1}(t) + (f + \lambda_1) P_N(t) + \eta_{21} L_N(t) + \eta_{31} M_N(t) \tag{1.3}$$

Where $f = -(\lambda_1 + \mu + \eta_{12} + \eta_{13})$

For State II

$$\frac{d}{dt} L_o(t) = -(\lambda_2 + \eta_{21} + \xi) L_0(t) + \mu L_1(t) + \eta_{12} P_0(t) \tag{1.4}$$

$$\frac{d}{dt} L_n(t) = g L_n(t) + \lambda_2 L_{n-1}(t) + \mu L_{n+1}(t) + \eta_{12} P_n(t) \quad 0<n<N \tag{1.5}$$

$$\frac{d}{dt} L_N(t) = \lambda_2 L_{N-1}(t) + (g + \lambda_2) L_N(t) + \mu L_{N+1}(t) + \eta_{12} L_N(t) \tag{1.6}$$

Where $g = -(\lambda_2 + \mu + \eta_{21} + \xi)$

For State III

$$\frac{d}{dt}M_0(t) = -\eta_{31}M_0(t) + \mu M_1(t) + \eta_{13}P_0(t) + \xi L_0(t) \tag{1.7}$$

$$\frac{d}{dt}M_n(t) = hM_n(t) + \mu M_{n+1}(t) + \eta_{12}P_n(t) + \xi L_n(t), \quad 0<n<N \tag{1.8}$$

$$\frac{d}{dt}M_N(t) = hM_N(t) + \eta_{13}P_N(t) + \xi L_N(t), \tag{1.9}$$

Where $h = -(\mu + \eta_{12})$

Multiplying (1.1 – 1.9) by appropriate powers of z, using their respective probability generations function, taking L.T.'s and using initial conditions, we have

$$k_1(z,s)\overline{P}(z,s) = z + \mu(z-1)\overline{p}_0(s) + z\eta_{21}\overline{L}(z,s) + z\eta_{31}\overline{M}(z,s) - z^{N+1}(z-1)\lambda_1\overline{P}_N(s) \tag{1.10}$$

$$k_2(z,s)\overline{L}(z,s) = \mu(z-1)\overline{L}_0(s) + z\eta_{12}\overline{P}(z,s) - z^{N+1}(z-1)\lambda_2\overline{L}_N(s) \tag{1.11}$$

$$k_3(z,s)\overline{M}(z,s) = \mu(z-1)\overline{M}_0(s) + z\eta_{13}\overline{P}(z,s) + z\xi\overline{L}(z,s) \tag{1.12}$$

We have

$$k_1(z,s) = \left[z\left\{s + \lambda_1(1-z) + \mu + \eta_{12} + \eta_{13}\right\} - \mu\right]$$

$$k_2(z,s) = \left[z\left\{s + \lambda_2(1-z) + \mu + \eta_{21} + \xi\right\} - \mu\right]$$

$$k_3(z,s) = \left[z\left\{s + \mu + \eta_{31}\right\} - \mu\right]$$

Solving equations (1.10 – 1.12), we get

$$\overline{P}(z,s) = \frac{\begin{bmatrix}\left\{z + \mu(z-1)\overline{p}_0(s)\right\}k_2(z,s)k_3(z,s) + z\mu(z-1)\left\{\eta_{21}k_3(z,s) + z\eta_{31}\xi\right\}\overline{L}_0(s) + z\eta_{31}\mu(z-1)k_2(z,s)\overline{M}_0(s) \\ -z^{N+1}(z-1)\left\{\lambda_1 k_2(z,s)k_3(z,s)\overline{P}_N(s) + z\lambda_2\left(\eta_{31}\right)k_3(z,s) + z\eta_{31}\xi\overline{L}_N(z,s)\right\}\end{bmatrix}}{k_1(z,s)k_2(z,s)k_3(z,s) - z^2\left\{\eta_{12}\eta_{21}k_3(z,s) + \eta_{13}\eta_{31}k_2(z,s)\right\} - z^3\xi\eta_{12}\eta_{31}} \tag{1.13}$$

$$\overline{L}(z,s) = \frac{\begin{matrix}z^2\eta_{12}k_3(z,s) + \mu(z-1)\left[\left\{k_1(z,s)k_2(z,s) - z^2\eta_{31}\eta_{13}\right\}\overline{L}_0(s)z\eta_{12}k_3(z,s)\overline{p}_0(s) + z^2\eta_{12}\eta_{31}\overline{M}_0(s)\right] \\ -z^{N+1}(z-1)\left[z\left(\eta_{12}\lambda_1 k_3(z,s)\overline{p}_N(s)\right) + \lambda_2\left\{k_1(z,s)k_3(z,s) - z^2\eta_{13}\eta_{31}\right\}\overline{L}_N(s)\right]\end{matrix}}{k_1(z,s)k_2(z,s)k_3(z,s) - z^2\left\{\eta_{12}\eta_{21}k_3(z,s) + \eta_{13}\eta_{31}k_2(z,s)\right\} - z^3\xi\eta_{12}\eta_{31}}$$

(1.14) $$\overline{M}(z,s)=\frac{z^{3}\eta_{12}\xi+\mu(z-1)\begin{bmatrix}\left\{z\eta_{13}k_{2}(z,s)+z^{2}\eta_{12}\xi\right\}\overline{p}_{0}(s)+\left\{z\xi k_{1}(z,s)+z^{2}\eta_{13}\eta_{21}\right\}\overline{L}_{0}(s)+\\ \left\{k_{1}(z,s)k_{2}(z,s)-z^{2}\eta_{12}\eta_{21}\right\}\overline{M}_{0}(s)\end{bmatrix} -z^{N+1}(z-1)\left[z\lambda_{1}\left\{z\eta_{12}\xi+\eta_{13}k_{1}(z,s)\right\}\overline{P}_{N}(s)+z\lambda_{2}\left\{z\eta_{13}\eta_{21}+\xi k_{1}(z,s)\overline{L}_{N}(s)\right\}\right]}{k_{1}(z,s)k_{2}(z,s)k_{3}(z,s)-z^{2}\left\{\eta_{12}\eta_{21}k_{3}(z,s)+\eta_{13}\eta_{31}k_{2}(z,s)\right\}-z^{3}\xi\eta_{12}\eta_{31}}$$

(1.15)

Equations (1.13 -1.14) give the values of L.T's of p.g.f. of the distributions of the number of units for different states of the input source. These are explicitly known if the five unknowns, viz., $\overline{P}_{0}(s)$, $\overline{L}_{0}(s)$, $\overline{M}_{0}(s)$, $\overline{P}_{N}(s)$ and $\overline{L}_{N}(s)$ involved in these equations are determined. We observed the highest power of z in the numerator of each of the above mentioned equations in (N+5), and highest power of z in their denominators in five. Moreover, $\overline{P}(z,s)$ etc. are polynomial of N degree. Hence, five zeros in the denominators of (1.13 -1.15) must vanish their numerators giving rise to five equations, we find their numerators giving rise to five equations in five above written unknowns. Solving this set of five equations, we find their values, therefore, $\overline{P}(z,s)$, $\overline{L}(z,s)$ and $\overline{M}(z,s)$ are known. Thus,

$$\overline{R}(z,s)=\overline{p}(z,s)+\overline{L}(z,s)+\overline{M}(z,s) \tag{1.16}$$

Letting $N\to\infty$, equations (1.13 -1.15) in unlimited space,

$$\overline{P}(z,s)=\frac{\left[\left\{z+\mu(z-1)\overline{p}_{0}(s)\right\}k_{2}(z,s)k_{3}(z,s)+z\mu(z-1)\left\{\eta_{21}k_{3}(z,s)+z\eta_{31}\xi\right\}\overline{L}_{0}(s)+z\eta_{31}\mu(z-1)k_{2}(z,s)\overline{M}_{0}(s)\right]}{k_{1}(z,s)k_{2}(z,s)k_{3}(z,s)-z^{2}\left\{\eta_{12}\eta_{21}k_{3}(z,s)+\eta_{13}\eta_{31}k_{2}(z,s)\right\}-z^{3}\xi\eta_{12}\eta_{31}} \tag{1.17}$$

$$\overline{L}(z,s)=\frac{z^{2}\eta_{12}k_{3}(z,s)+\mu(z-1)\left[\left\{k_{1}(z,s)k_{3}(z,s)-z^{2}\eta_{31}\eta_{13}\right\}\overline{L}_{0}(s)+z\eta_{12}k_{3}(z,s)\overline{p}_{0}(s)+z^{2}\eta_{12}\eta_{31}\overline{M}_{0}(s)\right]}{k_{1}(z,s)k_{2}(z,s)k_{3}(z,s)-z^{2}\left\{\eta_{12}\eta_{21}k_{3}(z,s)+\eta_{13}\eta_{31}k_{2}(z,s)\right\}-z^{3}\xi\eta_{12}\eta_{31}} \tag{1.18}$$

$$\overline{M}(z,s)=\frac{z^{3}\eta_{12}\xi+\mu(z-1)\begin{bmatrix}\left\{z\eta_{13}k_{2}(z,s)+z^{2}\eta_{12}\xi\right\}\overline{p}_{0}(s)+\left\{z\xi k_{2}(z,s)+z^{2}\eta_{12}\xi\right\}\overline{L}_{0}(s)+\\ \left\{k_{1}(z,s)k_{2}(z,s)-z^{2}\eta_{12}\eta_{21}\right\}\overline{M}_{0}(s)\end{bmatrix}}{k_{1}(z,s)k_{2}(z,s)k_{3}(z,s)-z^{2}\left\{\eta_{12}\eta_{21}k_{3}(z,s)+\eta_{13}\eta_{31}k_{2}(z,s)\right\}-z^{3}\xi\eta_{12}\eta_{31}} \tag{1.19}$$

Substituting the values $\overline{P}(z,s)$ etc. in (1.16), $\overline{R}(z,s)$ is also known in terms of three unknowns, viz.,

$\overline{p}_{0}(s), \overline{L}_{0}(s)\, and\, \overline{M}(s)$ for determining $\overline{R}(z,s)$, completely, we proceed as follows:

We now prove that each of $k_{1}(z,s), k_{2}(z,s)\, and k_{3}(z,s)$ has a zero inside the unit circle $|z|=1$.

Write
$$g(z)=z\left\{s+\lambda_{1}(1-z)+\mu+\lambda_{12}+\lambda_{13}\right\}$$
$$f(z)=\mu$$

To apply Rouche's theorem, we observe,

(i) Both $g(z)$ and $f(z)$ are analytic inside and on the contour $|z| = 1$

(ii) On $|z| = 1$

Therefore, we see that $g(z)$ and $k_1(z,s) = g(z) - f(z)$ have the same number of zeros inside the unit circle $|z| = 1$. Since $g(z)$ has one zero inside the unit circle,

$\left| z\{s + \lambda_1(1-z) + \mu + \eta_{12} + \eta_{13}\} \right| > \mu$, for Rl $(s) > 0$

Ie, $|g(z)| > |f(z)|$

Therefore $k_1(z,s)$ also has one zero inside the unit circle.

Similarly it can be proved that each of $k_1(z,s)$ and $k_3(z,s)$ also has one zero inside the unit circle.

The denominator of each equations (1.17-1.19) is of five degree in z. it is given by

$$k_1(z,s)k_2(z,s)k_3(z,s) - z^2\{k_2(z,s)\eta_{13}\eta_{31} + \eta_{12}\eta_{21}k_3(z,s)\} - z^3\xi\eta_{12}\eta_{31}.$$

It is also the denominator of $\overline{R}(z,s)$

Taking $G(z) = \left[k_1(z,s)k_2(z,s)k_3(z,s) - z^2\{\eta_{12}\eta_{21}k_3(z,s) + \eta_{13}\eta_{31}k_2(z,s)\}\right]$

$F(z) = z^3\xi\eta_{12}\eta_{31}$

(i) Both $G(z)$ and $F(z)$ are analytic inside and on the contour $|z| = 1$

(ii) On $|z| = 1$

Hence $|G(z)| > |F(z)|$

Both the conditions of Rouche's theorem are true.

Therefore $G(z)$ and $G(z) - F(z)$ both will have same number of zeros inside the unit circle $|z| = 1$, $G(z) - F(z)$ will also have three zeros inside the $|z| = 1$.

Since $\overline{R}(z,s)$, in unlimited space, is a power series. It is differentiable inside the unit circle $|z| = 1$.

Hence three zeros of denominator in $\overline{R}(z,s)$, which are inside $|z| = 1$ must vanish its numerator giving rise to three equations in three unknowns, viz, $\overline{P}_0(s), \overline{L}_0(s)$ and $\overline{M}_0(s)$. Solving these three equations one can determine these three unknowns occurring in the numerator of $\overline{R}(z,s)$.

Thus $\overline{R}(z,s)$ is obtained completely.

B. Different States of Input Source

Setting $z = 1$ in equation (1.10-1.12).

$$(s + \eta_{12} + \eta_{13})\overline{P}(1, s) = 1 + \eta_{21}(1, s) + \eta_{31}\overline{M}(1, s) \tag{1.20}$$

$$(s + \eta_{12} + \xi)\overline{L}(1, s) = \eta_{12}\overline{P}(1, s) \tag{1.21}$$

$$(s + \eta_{31})\overline{M}(1, s) = \eta_{13}\overline{P}(1, s) + \xi\overline{L}(1, s) \tag{1.22}$$

Solving (1.20-1.22), we have

$$\overline{P}(1, s) = \frac{(s + \xi + \eta_{21})}{s(s + \eta_{12} + \eta_{21} + \xi)} \tag{1.23}$$

$$\overline{L}(1, s) = \frac{\eta_{12}}{s(s + \eta_{12} + \eta_{21} + \xi)} \tag{1.24}$$

$$\overline{M}(1, s) = \frac{\{\eta_{13}(s + \xi + \eta_{21}) + \xi\eta_{12}\}}{s(s + \eta_{31})(s + \xi + \eta_{12} + \eta_{21})} \tag{1.25}$$

Taking inverse L.T. Of equations (1.23-1.25)

$$p(1, t) = \frac{1}{(\xi + \eta_{12} + \eta_{21})}\left[(\xi + \eta_{21}) + \eta_{12}e^{-(\xi+\eta_{12}+\eta_{21})t}\right]$$

$\equiv$ Probability that at time t the input in the state I.

$$L(1, t) = \frac{\eta_{12}}{(\xi + \eta_{12} + \eta_{21})}\left[1 - e^{-(\xi+\eta_{12}+\eta_{21})t}\right]$$

$\equiv$ Probability that at time t the input will be in state II.

$$M(1, t) = \left[\frac{\{\eta_{13}(\xi + \eta_{21}) + \xi\eta_{12}\}}{\eta_{31}(\xi + \eta_{12} + \eta_{21})}\right]$$

$$-\left[\frac{\{\eta_{13}(\eta_{31} - \xi - \eta_{21}\} - \xi\eta_{12}}{\eta_{31}(\xi + \eta_{12} + \eta_{21} - \eta_{31}}\right]e^{-(\eta_{31})t}$$

$$-\left[\frac{\{\eta_{13}\eta_{21}(\eta_{21} - \eta_{12}) + \eta_{12}\eta_{31}(\eta_{13} - \xi)\}}{\eta_{31}(\xi + \eta_{12} + \eta_{21})(\xi + \eta_{12} + \eta_{21} - \eta_{31})}\right]e^{-(\xi+\eta_{12}+\eta_{21})t}$$

$\equiv$ Probability that at the time t the input will be in the state III.

C. Steady State Solution

The steady state solution can be obtained by the well – known properly of the L, T., viz.,

$$\lim_{s \to 0} s\overline{F}(s) = \lim_{t \to \infty} F(t) \tag{1.26}$$

If the limit on the right exists.

Thus, if

$$\lim_{t \to \infty} P_n(t) = P_n$$

We have,

$$\lim_{s \to 0} s\overline{P}_n(s) = P_n \quad \text{etc.}$$

Using property (1.26) to equations (1.10 -1.12) and letting N tend to infinity, we have.

$$k_1(z)P(z) = \mu(z-1)P_0 + z\eta_{21}L(z) + z\eta_{31}M(z) \tag{1.27}$$

$$k_2(z)L(z) = \mu(z-1)L_0 + z\eta_{12}P(z) \tag{1.28}$$

$$k_3(z)M(z) = \mu(z-1)M_0 + z\eta_{13}P(z) + z\xi L(z) \tag{1.29}$$

Where $k_1(z) = \left[z\left\{\lambda_1(1-z) + \mu + \eta_{12} + \eta_{13}\right\} - \mu\right]$

$$k_2(z) = \left[z\left\{\lambda_2(1-z) + \mu + \eta_{21} + \xi\right\} - \mu\right]$$

$$k_3(z) = \left[z(\mu + \eta_{31}) - \mu\right]$$

Setting Z=1 in equations (1.27 -1.29).

$$(\eta_{12} + \eta_{13})P(1) = \eta_{21}L(1) + \eta_{31}M(1) \tag{1.30}$$

$$(\xi + \eta_{21})L(1) = \eta_{12}P(1) \tag{1.31}$$

$$\eta_{31}M(1) = \eta_{13}P(1) + \xi L(1) \tag{1.32}$$

Equation (1.30 -1.32) give,

$P(1) \equiv$ The Steady state probability that the system will be in the state I

$$= \frac{\xi + \eta_{21}}{(\xi + \eta_{12} + \eta_{21})} \tag{1.33}$$

$L(1) \equiv$ The Steady state probability that the system will be in the state II

$$= \frac{\eta_{12}}{(\xi + \eta_{12} + \eta_{21})} \tag{1.34}$$

$M(1) \equiv$ The Steady state probability that the system will be in the state III.

$$= \frac{\left\{\eta_{13}(\xi + \eta_{21}) + \xi\eta_{12}\right\}}{\eta_{31}(\xi + \eta_{12} + \eta_{21})} \tag{1.35}$$

Equation (1.27 -1.29) give

$$P(z) = \frac{\mu(z-1)\left[\left\{k_2(z)\,\mathrm{k}_3(z)P_0\right\}P_0 + z\left\{\eta_{21}k_3(z) + z\xi\eta_{31}\right\}L_0 + \left\{z\eta_{31}k_2(z)\right\}M_0\right]}{k_1(z)\,\mathrm{k}_2(z)\,\mathrm{k}_3(z) - z^2\left\{\eta_{12}\eta_{21}k_3(z) + \eta_{13}\eta_{31}k_2(z)\right\} - z^3\xi\eta_{12}\eta_{31}} \tag{1.36}$$

$$L(z) = \frac{\mu(z-1)\left[\left\{k_1(z)\,\mathrm{k}_3(z)\right\} - \left\{z^2\eta_{31}\eta_{13}\right\}L_0 + \left\{z\eta_{12}k_2(z)\right\}P_0 + \left\{z^2\eta_{12}\eta_{31}\right\}M_0\right]}{k_1(z)\,\mathrm{k}_2(z)\,\mathrm{k}_3(z) - z^2\left\{\eta_{12}\eta_{21}k_3(z) + \eta_{13}\eta_{31}k_2(z)\right\} - z^3\xi\eta_{12}\eta_{31}} \tag{1.37}$$

$$M(z) = \frac{\mu(z-1)\left[\left\{z\eta_{13}k_2(z) + z^2\eta_{12}\xi\right\}P_0 + \left\{z\xi k_1(z) + z^2\eta_{13}\eta_{21}\right\}L_0 + \left\{k_1(z)\,\mathrm{k}_2(z) - z^2\eta_{12}\eta_{21}\right\}M_0\right]}{k_1(z)\,\mathrm{k}_2(z)\,\mathrm{k}_3(z) - z^2\left\{\eta_{12}\eta_{21}k_3(z) + \eta_{13}\eta_{31}k_2(z)\right\} - z^3\xi\eta_{12}\eta_{31}} \tag{1.38}$$

The denominator of $p(z), L(z)$ and $M(z)$,

c is of 5th degree in z. So this must have five Zeros. We now prove that it has three zeros inside and two zeros outside the unit circle $|z| = 1$.

$k_1(z) = \left[z\left\{\lambda_1(1-z) + \mu + \eta_{12} + \eta_{13}\right\} - \mu\right]$ has two zeros, viz., α_1 and α_2 whose values are given by

$$\alpha_1 = \frac{1}{2\lambda_1}\left[(\lambda_1 + \mu + \eta_{12} + \eta_{13}) - \sqrt{\left\{(\lambda_1 + \mu + \eta_{12} + \eta_{13})^2 - 4\lambda_1\mu\right\}}\right]$$

$$\alpha_2 = \frac{1}{2\lambda_1}\left[(\lambda_1 + \mu + \eta_{12} + \eta_{13}) + \sqrt{\left\{(\lambda_1 + \mu + \eta_{12} + \eta_{13})^2 - 4\lambda_1\mu\right\}}\right]$$

As proved earlier $k_1(z,s)$ has two real zeros, one inside and other outside of unit circle $|z| = 1$.

Therefore, we say α_1 is inside and α_2 is outside of $|z| = 1$.

$k_2(z) = \left[z\left\{\lambda_2(1-z) + \mu + \xi + \eta_{21}\right\} - \mu\right]$, has two real zeros, viz., α_3 and α_4 whose values are given by;

$$\alpha_3 = \frac{1}{2\lambda_2}\left[(\lambda_2 + \mu + \xi + \eta_{21}) - \sqrt{\left\{(\lambda_2 + \mu + \xi + \eta_{21})^2 - 4\lambda_2\mu\right\}}\right]$$

$$\alpha_4 = \frac{1}{2\lambda_2}\left[(\lambda_2 + \mu + \xi + \eta_{21}) + \sqrt{\left\{(\lambda_2 + \mu + \xi + \eta_{21})^2 - 4\lambda_2\mu\right\}}\right]$$

By the gives earlier α_3 is inside and α_4 is outside $|z| = 1$.

$k_3(z) = \left\{(\mu + \eta_{31}) - \mu\right\}$ has one zero, viz., α_5

So, $\alpha_5 = \dfrac{\mu}{\mu + \eta_{31}}$

Which is clearly inside $|z| = 1$.

So, we Conclude that the zeros α_1, α_3 and α_5 are inside of $|z| = 1$. and α_2, α_4 are outside $|z| = 1$.

Observing (1.37) a factor (z-1) is common in numerator and denominator of $L(z)$. We cancel this factor. The denominator of $L(z)$ will now have four zeros, two inside and two outside of $|z| = 1$. We Now proceed to prove that denominator of $L(z)$ has two real zeros outside the unit circle $|z| = 1$.

Let $f(z) = k_1(z)\,k_2(z)\,k_3(z) - z^2\{\eta_{12}\eta_{21}k_3(z) + \eta_{13}\eta_{31}k_2(z)\} - z^3\xi\eta_{12}\eta_{31}$

$$\equiv (z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)(z-\alpha_5) - \frac{z^3\xi\eta_{12}\eta_{31}}{\lambda_1\lambda_2(\mu+\eta_{31})} \qquad (1.39)$$

Dividing $f(z)$ by $(z-1)$ and taking limit of z tends to infinity, we find that

$$\lim_{z\to\infty}\frac{f(z)}{z-1} > 0$$

If we take limit as z tends to 1. Then

$$\lim_{z\to 1}\frac{f(z)}{z-1} = \left[\mu\{\eta_{31}(\xi+\eta_{21}) + \eta_{12}\eta_{31} + \eta_{13}(\xi+\eta_{21}) + \xi\eta_{12}\}\right] - \eta_{31}\{(\xi+\eta_{21})\lambda_1 + \lambda_2\eta_{12}\}$$

This is obtained by using L' Hospital's rule.

For $\frac{f(z)}{(z-1)}$ to have even number of real zeros between 1 and ∞, $\lim_{z\to 1}\frac{f(z)}{z-1} > 0$, i.e.,

$$\left[\mu\{\eta_{31}(\xi+\eta_{21}) + \eta_{12}\eta_{31} + \eta_{13}(\xi+\eta_{21}) + \xi\eta_{12}\}\right] - \eta_{31}\{(\xi+\eta_{21})\lambda_1 + \lambda_2\eta_{12}\} > 0 \qquad (1.40)$$

and this must be true, as this is the condition of ergodicity ,

Which is proved as below?

Effective arrival rate of units is $\{\lambda_1 P(1) + \lambda_2 L(1)\}$, as it represents the total number of arrivals in one unit of time when the input is in working state (I and II).

Total number of units served by the system in one unit of time are $\left[\mu\{P(1) + L(1) + M(1)\}\right]$.

Condition of ergodicity demands that effective arrival rate be less them effective service rate. Therefore,

$\{\lambda_1 P(1) + \lambda_2 L(1)\} < \mu\{P(1) + L(1) + M(1)\}$.

Substituting the values of $P(1), L(1)\, and\, M(1)$ from equations (1.30 -1.32), We obtain,

$$\left\{\lambda_1(\xi+\eta_{21})+\lambda_2\eta_{12}\right\} < \mu\left[(\xi+\eta_{21})+\eta_{12}+\frac{\left\{\eta_{13}(\xi+\eta_{21})+\xi\eta_{12}\right\}}{\eta_{31}}\right]$$

Or $$\mu\left[\eta_{31}(\xi+\eta_{21})+\eta_{12}\eta_{31}+\left\{\eta_{13}(\xi+\eta_{21})+\xi\eta_{12}\right\}\right]-\eta\left\{\lambda_1(\xi+\eta_{21})+\lambda_2\eta_{12}\right\}>0 \qquad (1.41)$$

We find that (1.40) and (1.41) are identical and this gives the condition of ergodicity.

This Concludes that $\lim_{z\to\infty}\frac{f(z)}{z-1}$ and $\lim_{z\to 1}\frac{f(z)}{z-1}$ have like signs, so an even number of zeros of $f(z)$ lie in between 1 and ∞.

We proceed to prove that $\frac{f(z)}{z-1}$ has two zeros say z_1 *and* z_2, which lie outside $|z|=1$

Considering $\alpha_4 > \alpha_2$, we have from (1.39).

$$\lim_{z\to\alpha_2}\frac{f(z)}{(z-1)}=\frac{\alpha_2^3\xi\eta_{12}\eta_{31}}{(\alpha_2-1)\lambda_1\lambda_2(\mu+\eta_{31})}<0$$

Sign changes between 1 and α_2. So there is a real zero, say z_1, in between 1 and α_2.

$$\lim_{z\to\alpha_4}\frac{f(z)}{(z-1)}=\frac{\alpha_4^3\xi\eta_{12}\eta_{31}}{(\alpha_4-1)\lambda_1\lambda_2(\mu+\eta_{31})}<0$$

Like sign between α_2 and α_4. But there is a change of sign in between α_4 and ∞. So there is a real zero, say z_2, is between α_4 and ∞. This conclude that two real zeros of $\frac{f(z)}{z-1}$, z_1 and z_2 lie in the interval

$[1,\alpha_2)$ and $[\alpha_4,\infty)$ respectively.

The two zeros of the denominator in (1.37) which are inside $|z|=1$. Must vanish its numerator, $L(z)$ is a well defined functions inside the unit circle. Thus, cancelling two factors in the numerator and in the denominator corresponding to these zeros, then equations (1.33) reduces to the following form:

$$L(z)=\frac{A}{(z-z_1)}+\frac{B}{(z-z_2)}$$

Where A and B are to determined.

Setting $z=1$.

$$L(1)=\frac{A}{1-z_1}+\frac{B}{1-z_2}$$

Using (1.31), $A = -\left[\frac{B(z_1-1)}{(z_2-1)} + \frac{(z_1-1)\eta_{12}}{(\xi+\eta_{12}+\eta_{31})}\right]$

Therefore, $L(z)$ in term of B is

$$L(z) = \frac{B(z_2-z_1)(z-1)}{(z_2-1)(z-z_2)(z-z_1)} - \frac{(z_1-1)\eta_{12}}{(z-z_1)(\xi+\eta_{12}+\eta_{21})} \tag{1.42}$$

$$L_0 = \frac{B(z_1-z_2)}{z_1z_2(z_2-1)} + \frac{(z_1-1)}{z_1(\xi+\eta_{12}+\eta_{21})} \tag{1.43}$$

Substituting the values of $L(z)$ and L_0 from (1.42) and (1.43) in equation (1.28).

$$P(z) = \frac{B(z_2-z_1)(z-1)}{(z_2-1)\eta_{12}}\left[\frac{\{\lambda_2(1-z)+\mu+\xi+\eta_{21}\}}{(z-z_1)(z-z_2)} - \frac{\mu}{z} + \frac{\mu}{zz_1z_2}\right]$$

$$-\frac{(z_1-1)}{(z-z_1)(\xi+\eta_{12}+\eta_{21})}\left[\{\lambda_2(1-z)+\mu+\xi+\eta_{21}\} - \frac{\mu}{z} + \frac{\mu(z-1)(z-z_1)}{z_1}\right] \tag{1.44}$$

Setting z=0 in (1.44)

$$P_0 = \frac{B(z_1-z_2)}{\eta_{12}(z_2-1)}\left[\frac{(\lambda_2+\xi+\eta_{21}+\mu)}{z_1z_2}\right] + \frac{(z_1-1)}{z_1(\xi+\eta_{12}+\eta_{21})}(\lambda_2+\mu+\xi+\eta_{21}) \tag{1.45}$$

Substituting the values of $L(z), P(z)$ and P_0 from equations (1.42) ,(1.44) and (1.45) in equation (1.27),

$$M(z) = \frac{B(z_2-z_1)(z-1)}{(z_2-1)z\eta_{12}\eta_{31}}\left[\begin{array}{l}\dfrac{\{z(\lambda_1(1-z)+\mu+\eta_{12}+\eta_{13})-\mu\}\times\{\lambda_1(1-z)+\mu+\xi+\eta_{21}\}}{(z-z_1)(z-z_2)} \\ -\dfrac{\mu}{z}\left(1-\dfrac{1}{z_1z_2}\right)\{z(\lambda_1(1-z)+\mu+\eta_{12}+\eta_{13})-\mu\} \\ +\dfrac{\mu(\lambda_2+\xi+\eta_{21}+\mu)}{z_1z_2} - \dfrac{z\eta_{12}\eta_{21}}{(z-1)(z-z_1)(z-z_2)}\end{array}\right]$$

$$-\frac{(z_1-1)}{z\eta_{31}(z-z_1)(\xi+\eta_{12}+\eta_{21})}\left[\begin{array}{l}\{z(\lambda_1(1-z)+\mu+\eta_{12}+\eta_{13})-\mu\}\times\{\lambda_2(1-z)+\mu+\xi+\eta_{21}\} \\ -\dfrac{\mu}{z}+\dfrac{\mu(z-1)(z-z_1)}{z_1}+\dfrac{\mu(z-1)(z-z_1)}{z_1}\times\{(\lambda_2+\mu+\xi+\eta_{21})+z\eta_{21}\eta_{12}\}\end{array}\right]$$

(1.46)

Setting z=0 in (1.46),

$$M_0 = \frac{B(z_1 - z_2)}{\eta_{12}\eta_{31}(z_2 - z_1)}\left[\frac{(\lambda_2 + \mu + \xi + \eta_{21})(\lambda_1 + \mu + \eta_{12} + \eta_{13})}{z_1 z_2} + \frac{\eta_{12}\eta_{21}}{z_1 z_2}\right]$$

$$+ \frac{(z_1 - 1)}{\eta_{31} z_1 (\xi + \eta_{12} + \eta_{21})} \times \left[(\lambda_1 + \mu + \eta_{12} + \eta_{13})(\lambda_2 + \mu + \xi + \eta_{21}) + \eta_{21}\eta_{12}\right] \quad (1.47)$$

Equations (1.42 -1.47) give the values of $L(z), L_0, P(z), P_0$ *and* $M(z), M_0$ respectively in terms of B, if B is known, these all are obtained explicitly.

Setting $z = \alpha_3$ in (1.28), we get

$$P(\alpha_3) = \frac{\mu(1 - \alpha_3)}{\alpha_3 \eta_{12}} L_0 \quad (1.48)$$

Substituting the value of L_0 from (1.43),

$$P(\alpha_3) = \frac{\mu(1 - \alpha_3) B(z_1 - z_2)}{\alpha_3 \eta_{12} z_1 z_2 (z_2 - 1)} + \frac{\mu(1 - \alpha_3)(z_1 - 1)}{z_1 \alpha_3 (\xi + \eta_{12} + \eta_{21})}$$

Substituting $z = \alpha_3$ in $P(z)$ gives by (1.44) and equations two values of $P(\alpha_3)$, thus we obtained

$$\frac{B(z_2 - z_1)(\alpha_3 - 1)}{\eta_{12}(z_2 - 1)}\left[-\frac{\{\lambda_2(1 - \alpha_3) + \mu + \xi + \eta_{21}\}}{(\alpha_3 - z_2)(\alpha_3 - z_1)} + \frac{\mu}{\alpha_3}\right]$$

$$= -\frac{(z_1 - 1)}{(\xi + \eta_{12} + \eta_{21})}\left[\frac{\mu(1 - \alpha_3)}{\alpha_3 z_1} + \frac{\{\lambda_2(1 - \alpha_3) + \mu + \xi + \eta_{21}\}}{(\alpha_3 - z_1)} - \frac{\mu}{\alpha_3(\alpha_3 - z_1)} + \frac{\mu(\alpha_3 - 1)}{z_1}\right] \quad (1.49)$$

IV. CONCLUSION

The time dependent and steady state results are analyzed explicitly as assumptions mentioned in the formulation of model.

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