Some Best Proximity Points for Generalized α-Ø- Proximal Penta - Contractive Mappings

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Abstract

In this manuscript,the existence of the generalized α-Ø-Penta-Contractive Mappings is investigated.Our Result Generalized Mohamed Ladh Ayari[5].We illustrate our work by an example and also shown applications to fixed point results.

Keywords

Best proximity points; α-Ø-Proximal Penta -Contractive Mappings on Metric Spaces

I. INTRODUCTION

Let A and B two nonvoid subsets of metric spaces (X, d).Fact that a non self mapping $T : A \to B$ does not necessarily have a fixed point.It is of considerable significance to explore the existence of an element x that is close to Tx as possible.Best proximity point analysis has been developed in this direction. The Best proximity points of T are the points $x \in$A satisfying $d(x, Tx) = d(A, B)$. Recently Mohamed Ladh Ayari([5]) introduced a novel class of contractive mappings called α-β contractive type mappings.They provide some interesting results to obtain existence of fixed points for self mappings.

The main objective of this paper is to generalize the results of Mohamed Ladh Ayari([5]) by introducing the proximal α-ϕ-penta -contractive mappings on metric spaces involving β comparison functions.we have dervied some theorems on best proximity points for a specific class of proximal α-ϕ-penta -contractive mappings .The present result generalize the theorem of Mohamed Ladh Ayari and many results existing in the literature

II. PRELIMINARIES AND DEFINITIONS

Let (A,B) be pair of nonvoid subsets of a metric space(X,d).We adopt the following notations:

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$$
$$A_0 = \{a \in A : \exists b \in B \ni d(a,b) = d(A,B)\};$$
$$B_0 = \{b \in B : \exists a \in A \ni d(a,b) = d(A,B)\}$$

Definition 2.1. *([12])Let $T : A \to B$ be a mapping. An element x^* is said to be a best proximity point of T if $d(x^*, Tx^*) = d(A,B)$*

Definition 2.2. *([13]) Let $\beta \in (0,+\infty)$. A β-comparsion function is a map $\phi : [0,+\infty) \to [0,+\infty)$ fulfilling the following properties:*
(1) ϕ is increasing;
(2)$lim_{n\to\infty}\ \phi^n_\beta(t) = 0$ for all $t > 0$ where ϕ^n_β denotes the nth iterate of ϕ_β and $\phi_\beta(t)$=$\phi(\beta t)$
(3)there exists $s \in (0,+\infty)$ such that $\Sigma^\infty_{n=1}\ \phi^n_\beta(s) < \infty$.
The set of all β -comparsion functions ϕ satisfying (1)-(3) will be denoted by ϕ_β

Remark : Let $\alpha, \beta \in (0,+\infty)$ .If $\alpha < \beta$,then $\Phi_\beta \subset \Phi_\alpha$

Lemma 2.1. *([13])Let $\beta \in (0,+\infty)$ and $\phi \in \Phi_\beta$.Then*
(1)ϕ_β is increasing;
(2)$\phi_\beta(t) < t$ for all t>0.;
(3)$\Sigma^\infty_{n=1}\ \phi^n_\beta(t) < \infty$ for all t>0.

Definition 2.3. *([7])Let (A,B) be a pair of nonvoid subsets of a metric space (X,d) such that A_0 is nonvoid.Then the pair (A,B) is said to have the P-property iff $d(x_1,y_1) = d(x_2,y_2) = d(A,B) \implies d(x_1,x_2) = d(y_1,y_2)$ where $x_1,x_2 \in A$ and $y_1,y_2 \in B$.*

Definition 2.4. *([9])Let $T : A \to B$ and $\alpha : A \times A \to [0,+\infty)$. We say that T is α-proximal admissible if $\alpha(x_1,x_2) \geq 1$ and $d(u_1,Tx_1) = d(u_2,Tx_2) = d(A,B) \implies \alpha(u_1,u_2) \geq 1$ for all $x_1,x_2,u_1,u_2 \in A$*

Definition 2.5. *([9])A non-self mapping $T : A \to B$ is said to be a generalized $\alpha - \psi$-proximal contraction ,where $\alpha : A \times A \to [0,+\infty)$ and ψ is a (c)-comparison function if*
$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y))\ \forall x,y \in A$ where

$$M(x,y) = max\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)] - d(A,B), \frac{1}{2}[d(y,Tx) + d(x,Ty)] - d(A,B)\}$$

Definition 2.6. *([9]) A non-self -mapping $T : A \to B$is said to be (α, d) regular,where $\alpha : A\times A \to [0,+\infty)$ if for all (x,y) such that $0 \leq \alpha(x,y) < 1$there exists $u_0 \in A_0$ such that*

$$\alpha(x,u_0) \geq 1 \text{ and } \alpha(y,u_0) \geq 1$$

Definition 2.7. *([5])Let (X,d) be a metric space and (A,B) be a pair of nonempty subsets of X.Let $\beta \in (0, +\infty)$..A non-self mapping $T : A \to B$ is said to be a generalized $\alpha - \beta$-Proximal quasi-contractive,where $\alpha : A \times A \to [0, +\infty)$ iff there exists $\varphi \in \Phi_\beta$ and positive numbers $\alpha_0, \alpha_1, ...\alpha_4$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \varphi(M_T(x, y)) \ \forall x, y \in A$$

where

$$M_T(x, y) = max\{\alpha_0 d(x, y), \alpha_1[d(x, Tx) - d(A, B)], \alpha_2[d(y, Ty) - d(A, B)], \alpha_3[d(y, Tx) - d(A, B)], \alpha_4[d(x, Ty) - d(A, B)]\}$$

III. MAIN RESULTS

In this section we define generalized $\alpha - \phi$-proximal-penta contractive and state our main results.

Definition 3.1. *Let (X,d) be a metric space and (A,B) be a pair of nonvoid subsets of X.Let $\beta \in (0, +\infty)$.A non-self mapping $T : A \to B$ is said to be a generalized $\alpha - \phi$-Proximal penta-contractive,where $\alpha : A \times A \to [0, +\infty)$ iff there exists $\phi \in \Phi_\beta$ and positive numbers $\lambda_0, \lambda_1, ..., \lambda_5$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \phi(M_R(x, y)) \ \forall x, y \in A$$

where

$$M_R(x, y) = max\{\lambda_0 d(x, y), \frac{\lambda_1}{2}[d(x, Tx) + d(y, Ty) - d(A, B)], \frac{\lambda_2}{2}[d(y, Tx) + d(x, Ty) - d(A, B)], \lambda_3[d(y, Ty) - d(A, B)], \frac{\lambda_4}{2}[d(y, Tx) - d(A, B)], \frac{\lambda_5}{2}[d(x, Ty) - d(A, B)]\}$$

Theorem 3.1. *Let $T : A \to B$ be a non self-mapping and $\alpha : A \times A \to [0, +\infty)$ satisfying the following conditions:*
(1) $T(A_0) \subset B_0$
(2)T is a α-proximal admissible;
(3) there exist element $x_0, x_1 \in A$ such that $d(x_1, Tx_0)$=d(A,B) and $\alpha(x_0, x_1) \geq 1$;
then there exists a sequence $\{x_n\} \subset A_0$ such that $d(x_{n+1}, Tx_n) = d(A, B)$ and $\alpha(x_n, x_{n+1}) \geq 1$
such a sequence $\{x_n\}$ is a cauchy sequence.

Proof. By hypothesis of (3),there exist element $x_0, x_1 \in A$ such that $d(x_1, Tx_0)$=d(A,B) and $\alpha(x_0, x_1) \geq 1$;
As $T(A_0) \subset B_0$ there exists $x_2 \in A_0$ such that $d(x_2, Tx_1)$=d(A,B).
As T is α-proximal admissible and using $\alpha(x_0, x_1) \geq 1$
therefore $d(x_1, Tx_0)$= $d(x_2, Tx_1)$=d(A,B)
which implies $\alpha(x_1, x_2) \geq 1$

By proceeding in this fashion so by induction ,
we can build a sequence $\{x_n\} \subset A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A,B) \;\; and \;\; \alpha(x_n, x_{n+1}) \geq 1 \;\; for\ every \;\; n \in N \cup \{0\} \quad (1)$$

To prove :The sequence $\{x_n\}$ is a Cauchy sequence
Using the P-property ,we deduce from (1) that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \;\; for \;\; all \;\; n \in N \quad (2)$$

Since T is generalized $\alpha - \phi$ proximal penta-contractive ,there exists a function $\phi \in \Phi_\beta$ such that

$$\alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \leq \phi(M_R(x_{n-1}, x_n)) \;\; \forall n \in N \quad (3)$$

Using the equations (1) ,(2) and the triangular inequality,we get

$$M_R(x_{n-1}, x_n) = max\{\lambda_0 d(x_{n-1}, x_n), \frac{\lambda_1}{2}[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) - d(A,B)],$$
$$\frac{\lambda_2}{2}[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n) - d(A,B)], \lambda_3[d(x_n, Tx_n) - d(A,B)],$$
$$\frac{\lambda_4}{2}[d(x_n, Tx_{n-1}) - d(A,B)], \frac{\lambda_5}{2}[d(x_{n-1}, Tx_n) - d(A,B)]\}$$

$$\leq max\{\lambda_0 d(x_{n-1}, x_n), \frac{\lambda_1}{2}[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) + d(x_{n+1}, Tx_n) - d(A,B)],$$
$$\frac{\lambda_2}{2}[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n) + d(x_{n+1}, Tx_n) - d(A,B)],$$
$$\lambda_3[d(x_n, Tx_n) + d(x_{n+1}, Tx_n) - d(A,B)], \frac{\lambda_4}{2}[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n) - d(A,B)],$$
$$\frac{\lambda_5}{2}[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) - d(A,B)]\}$$

$$= max\{\lambda_0 d(x_{n-1}, x_n), \frac{\lambda_1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \frac{\lambda_2}{2}[d(x_{n-1}, x_{n+1})],$$
$$\lambda_3[d(x_n, x_{n+1})], \frac{\lambda_4}{2}[d(x_{n-1}, x_{n+1})], \frac{\lambda_5}{2}[d(x_{n-1}, x_{n+1})]\}$$
$$\leq max\{\lambda_0 d(x_{n-1}, x_n), \lambda_1[d(x_n, x_{n+1})], \frac{\lambda_2}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})],$$
$$\lambda_3[d(x_n, x_{n+1})], \frac{\lambda_4}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \frac{\lambda_5}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\}$$

$$\leq max\{\lambda_0 d(x_{n-1},x_n),\lambda_1[d(x_n,x_{n+1})],\lambda_2[d(x_n,x_{n+1})],$$
$$\lambda_3[d(x_n,x_{n+1})],\lambda_4[d(x_n,x_{n+1})],\lambda_5[d(x_n,x_{n+1})]\}$$

$$\leq \beta max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}$$

$Hence,$

$$M_R(x_{n-1},x_n) \leq \beta max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}$$

where $\beta \geq max_{0\leq k\leq 5}\{\lambda_k\}$
Using the inequalities (2),(3)and(4) we get,
Since ϕ is increasing,

$$\begin{aligned} d(x_{n+1},x_n) &\leq \phi(\beta max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}) \\ &= \phi_\beta max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} \end{aligned}$$

Suppose that for some n, we have $d(x_{n-1},x_n) \leq d(x_n,x_{n+1})$
It follows that $d(x_{n+1},x_n) \leq \phi_\beta d(x_{n+1},x_n) < d(x_{n+1},x_n)$
which is a contradiction.
Therefore ,$\forall n \geq 0$ we have necessary the inequality $d(x_{n-1},x_n) > d(x_n,x_{n+1})$
it shows that

$$d(x_{n+1},x_n) \leq \phi_\beta d(x_{n-1},x_n) \qquad \forall n \in N \tag{4}$$

By induction,we get

$$d(x_{n+1},x_n) \leq \phi_\beta^n d(x_1,x_0) \qquad \forall n \in N \cup \{0\} \tag{5}$$

Using the triangular inequality and the above inequality (5), we obtain

$$\begin{aligned} d(x_n,x_m) &\leq \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \phi_\beta^k d(x_1,x_0) \to 0 \qquad as n,m \to \infty \end{aligned}$$

Since the series $\sum_{n=1}^{\infty} \phi_\beta^n(d(x_1,x_0))$ converges.
Thus,The sequence $\{x_n\}$ is a Cauchy sequence in the metric space (X.d). □

Theorem 3.2. *Let (A,B) be a pair of nonvoid closed subsets of a complete metric space (X,d) such that A_0 is nonvoid.Let $\alpha : A\times A \to [0,+\infty)$ and $\phi \in \Phi_\beta$ .Consider a nonself mapping $T : A \to B$ satisfying the following assertions:*
(1) $T(A_0) \subset B_0$ and the pair (A,B) satisfies the P-property;

(2)T is a α-proximal admissible;
(3) there exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0)$=d(A,B) and $\alpha(x_0, x_1) \geq 1$;
(4) if $\{x_n\}$ a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$and $\lim_{n\to\infty} x_n = x_* \in A$ then there exists a subsequence $\{x_{n_k}\}$of $\{x_n\}$ such that $\alpha(x_{n_k}, x_*) \geq 1$ $\forall k$
(5)there exists $\beta \geq max_{0\leq k\leq 5}\{\lambda_k\}$ such that T is generalized $\alpha - \phi$-proximal penta-contractive.
Moreover,suppose that one of the following conditions holds:
(i)ϕ is continuous and
(ii)$\beta > max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}$.
Then T ha a best proximity point $x_* \in A$ such that $d(x_*, Tx_*) = d(A, B)$.

Proof. By the hypothesis that (X,d) is complete and A is closed assures that the sequence $\{x_n\}$ converges to some element $x_* \in A$
By the hypothesis of theorem (4),there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_*) \geq 1$ for all k.
Since T is generalized $\alpha - \phi$ -proximal penta -contractive ,then we have

$$\begin{aligned} d(T(x_{n_k}, Tx_*)) &\leq \alpha(x_{n_k}, x_*)d(T(x_{n_k}, Tx_*)) \\ &\leq \phi(M_R(x_{n_k}, x_*)) \qquad \forall k \end{aligned} \tag{6}$$

where

$$\begin{aligned} M_R(x_{n_k}, x_*) = max\{&\lambda_0 d(x_{n_k}, x_*), \frac{\lambda_1}{2}[d(x_{n_k}, Tx_{n_k}) + d(x_*, Tx_*) \\ &- d(A,B)], \frac{\lambda_2}{2}[d(x_*, Tx_{n_k}) + d(x_{n_k}, Tx_*) \\ &- d(A,B)], \lambda_3[d(x_*, Tx_*) - d(A,B)], \frac{\lambda_4}{2}[d(x_*, Tx_{n_k}) - d(A,B)], \\ &\frac{\lambda_5}{2}[d(x_{n_k}, Tx_*) - d(A,B)]\} \end{aligned} \tag{7}$$

By the triangular inequality and (2)we have,

$$\begin{aligned} d(x_*, Tx_*) &\leq d(x_*, x_{n_{k+1}}) + d(x_{n_{k+1}}, Tx_{n_k}) + d(Tx_{n_k}, Tx_*) \\ &= d(x_*, x_{n_{k+1}}) + d(A,B) + d(Tx_{n_k}, Tx_*) \end{aligned}$$

Therefore,we obtain that

$$d(Tx_{n_k}, Tx_*) \geq d(x_*, Tx_*) - d(A,B) - d(x_*, x_{n_{k+1}}) \tag{8}$$

From(6)and (8),we get

$$d(x_*, Tx_*) - d(A,B) - d(x_*, x_{n_{k+1}}) \leq \phi(M_R(x_{n_k}, x_*)) \qquad \forall k \tag{9}$$

By the triangular inequality and (2) on (7) we get

$$\begin{aligned} M_R(x_{n_k}, x_*) \leq max\{\lambda_0 d(x_{n_k}, x_*), \frac{\lambda_1}{2}[d(x_{n_k}, x_{n_{k+1}}) + d(x_*, Tx_*) \\ - d(A,B)], \frac{\lambda_2}{2}[d(x_*, x_{n_{k+1}}) + d(x_{n_k}, Tx_*) \\ - d(A,B)], \lambda_3[d(x_*, Tx_*) - d(A,B)], \frac{\lambda_4}{2}[d(x_*, x_{n_{k+1}})], \\ \frac{\lambda_5}{2}[d(x_{n_k}, Tx_*) - d(A,B)]\} \end{aligned} \tag{10}$$

As ϕ is increasing ,combining inequalities and (2)and (10),we get

$$\begin{aligned} & d(x_*, Tx_*) - d(A,B) - d(x_*, x_{n_{k+1}}) \\ & \leq \phi(max\{\lambda_0 d(x_{n_k}, x_*), \frac{\lambda_1}{2}[d(x_{n_k}, x_{n_{k+1}}) + d(x_*, Tx_*) \\ & - d(A,B)], \frac{\lambda_2}{2}[d(x_*, x_{n_{k+1}}) + d(x_{n_k}, Tx_*) \\ & - d(A,B)], \lambda_3[d(x_*, Tx_*) - d(A,B)], \frac{\lambda_4}{2}[d(x_*, x_{n_{k+1}})], \\ & \frac{\lambda_5}{2}[d(x_{n_k}, Tx_*) - d(A,B)]\}) \end{aligned} \tag{11}$$

Assume $\gamma = d(x_*, Tx_*) - d(A,B) > 0$
we consider two separate cases as follows
If ϕ is continuous, as $k \to \infty$ we get

$$\begin{aligned} \gamma &\leq \phi(max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}\gamma) \\ &\leq \phi(\beta\gamma) < \gamma, \end{aligned}$$

which is a contradiction.
If $\beta > max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}$,we claim that $\gamma = 0$.
Suppose by contradiction that $\gamma > 0$
Letting $k \to \infty$ in(7),we get $M_R(x_{n_k}, x_*) \to max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}\gamma$.
Then there exists $\varepsilon > 0$ and $N > 0$ such that
$M_R(x_{n_k}, x_*) < (max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\} + \varepsilon)\gamma$
and $\beta > max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\} + \varepsilon \qquad \forall n > N$
Therefore,

$$\begin{aligned} d(x_*, Tx_*) - d(A,B) - d(x_*, x_{n_{k+1}}) &\leq \phi(M_R(x_{n_k}, x_*)) \\ &\leq ((max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\} + \varepsilon)\gamma) \\ &= \phi_\beta \frac{max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\} + \varepsilon}{\beta}\gamma \\ &< \frac{max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\} + \varepsilon}{\beta}\gamma < \gamma \end{aligned}$$

By the consequence, letting $k \to \infty$,we get

$\gamma < \frac{max\{\lambda_1,\lambda_2,\lambda_3,\lambda_5\}+\varepsilon}{\beta}\gamma < \gamma$,
which is a contradiction .
Hence,$\gamma = 0$.Thus,it shows that x_* is a best proximity point of T that is

$$d(x_*, Tx_*) = d(A, B) \tag{12}$$

□

Theorem 3.3. *Let (A,B) be a pair of nonvoid closed subsets of a complete metric space (X,d) such that A_0 is nonvoid.Let $\alpha : A \times A \to [0, +\infty)$ and $\phi \in \Phi_\beta$ .Consider a nonself mapping $T : A \to B$ satisfying the following assertions:*
(1) $T(A_0) \subset B_0$ and the pair (A,B) satisfies the P-property;
(2)T is α-proximal admissible;
(3) there exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0)$=d(A,B) and $\alpha(x_0, x_1) \geq 1$;
(4) if $\{x_n\}$ a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$and $\lim_{n\to\infty} x_n = x_* \in A$ then there exists a subsequence $\{x_{n_k}\}$of $\{x_n\}$ such that $\alpha(x_{n_k}, x_*) \geq 1\ \forall k$
(5)there exists $\beta \geq max_{0\leq k\leq 5}\{\alpha_k\}$ such that T is generalized $\alpha - \phi$-proximal penta-contractive.
(6)suppose that one of the following conditions holds:
(i) ϕ is continuous and (ii)$\beta > max\{2\lambda_1, 3\lambda_2, \lambda_4, 2\lambda_5\}$.
Moreover,Suppose that T is a (α, d) regular and Then T has a unique best proximity point $x_* \in A$ such that $d(x_*, Tx_*) = d(A, B)$.

Proof. Claim:T has a unique best proximity point
Suppose that x_* and y_* are two distinct best proximity points of T.
Let $s = d(x_*, y_*) > 0$. By the P-property , we obtain
$d(Tx_*, Ty_*) = d(x_*, y_*) = s$.We arise two cases
Case (i) If $\alpha(x_*, y_*) \geq 1$
Since T is a generalized $\alpha - \phi$-proximal penta-contractive,
therefore

$$d(x_*, y_*) = s \leq \alpha(x_*, y_*)\phi(M_R(x_*, y_*)), \tag{13}$$

where

$$\begin{aligned} M_R(x_*, y_*) = max\{&\lambda_0 d(x_*, y_*), \frac{\lambda_1}{2}[d(x_*, Tx_*) + d(y_*, Ty_*) - d(A, B)], \\ &\frac{\lambda_2}{2}[d(y_*, Tx_*) + d(x_*, Ty_*) - d(A, B)], \lambda_3[d(y_*, Ty_*) - d(A, B)], \\ &\frac{\lambda_4}{2}[d(y_*, Tx_*) - d(A, B)], \frac{\lambda_5}{2}[d(x_*, Ty_*) - d(A, B)]\} \end{aligned} \tag{14}$$

Using the triangular inequality in (14),we get

$$M_R(x_*, y_*) \leq max\{\lambda_0, \lambda_2, \lambda_4, \lambda_5\}s \tag{15}$$

Combining the equations (13)and (15)and using the increasing property of the function ϕ,we get that

$$s \leq \phi_\beta(s) < s,$$

Which is a contraction .So $s = d(x_*, y_*) = 0$
Hence , we get $x_* = y_*$.

Case(ii) If $\alpha(x_*, y_*) < 1$
Since T is (α, d) regular,there exists $u_0 \in A_0$ such that $\alpha(x_*, u_0) \geq 1$ and $\alpha(y_*, u_0) \geq 1$
Since $T(A_0) \subset B_0$,there exists $u_1 \in A_0$such that $d(u_1, Tu_0) = d(A, B)$.
we have $d(x_*, Tx_*) = d(u_1, Tu_0) = d(A, B)$ and $\alpha(x_*, u_0) \geq 1$
Using the Fact that T is α-proximal admissible ,we get$\alpha(x_*, u_1) \geq 1$
By induction we obtain that to find $\{u_n\} \in A_0$ such that

$$\begin{aligned} &d(u_{n+1}, Tu_n) = d(A, B) and \\ &\alpha(x_*, u_n) \geq 1 \qquad \forall n \in N \cup \{0\} \end{aligned} \tag{16}$$

Using the P-property and (16),we have

$$d(u_{n+1}, x_*) = d(Tu_n, Tx_*) \qquad \forall n \in N \cup \{0\} \tag{17}$$

As T is generalized $\alpha - \phi$ penta-contractive ,then we get

$$\alpha(u_{n+1}, x_*)d(Tu_{n+1}, Tx_*) \leq \phi(M_R(u_n, x_*)) \qquad \forall n \in N \cup \{0\} \tag{18}$$

Using (16)and (18),we get

$$\alpha(u_{n+1}, x_*)d(u_{n+1}, x_*) \leq \phi(M_R(u_n, x_*)) \qquad \forall n \in N \cup \{0\} \tag{19}$$

Therefore ,from the equation (16),we get that

$$d(u_{n+1}, x_*) \leq \phi(M_R(u_n, x_*)) \qquad \forall n \in N \cup \{0\} \tag{20}$$

Now, let using that (12),$\forall n \in N \cup \{0\}$ we obtain

$$\begin{aligned}M_R &= max\{\lambda_0 d(u_n,x_*), \frac{\lambda_1}{2}[d(u_n,Tu_n)+d(x_*,Tx_*)-d(A,B)],\\ &\frac{\lambda_2}{2}[d(x_*,Tu_n)+d(u_n,Tx_*)-d(A,B)], \lambda_3[d(x_*,Tx_*)-d(A,B)],\\ &\frac{\lambda_4}{2}[d(x_*,Tu_n)-d(A,B)], \frac{\lambda_5}{2}[d(u_n,Tx_*)-d(A,B)]\}\\ &= max\{\lambda_0 d(u_n,x_*), \frac{\lambda_1}{2}[d(u_n,Tu_n)], \frac{\lambda_2}{2}[d(x_*,Tu_n)+d(u_n,Tx_*)\\ &-d(A,B)], \frac{\lambda_4}{2}[d(x_*,Tu_n)-d(A,B)], \frac{\lambda_5}{2}[d(u_n,Tx_*)-d(A,B)]\} \qquad (21)\end{aligned}$$

Using the triangular inequality and consider these equations(12),(17),(21) we get

$$\begin{aligned}M_R(u_n,x_*) \leq\; &max\{\lambda_0 d(u_n,x_*), \frac{\lambda_1}{2}[d(u_n,u_{n+1})+d(u_{n+1},Tu_n)],\\ &\frac{\lambda_2}{2}[d(x_*,Tx_*)+d(Tx_*,Tu_n)+d(u_n,u_{n+1})+d(u_{n+1},Tx_*)\\ &-d(A,B)], \frac{\lambda_4}{2}[d(x_*,Tx_*)+d(Tx_*,Tu_n)-d(A,B)],\\ &\frac{\lambda_5}{2}[d(u_n,u_{n+1})+d(u_{n+1},Tx_*)-d(A,B)],\}\\ &\leq \beta max\{d(u_n,x_*), d(u_{n+1},x_*)\} \qquad (22)\end{aligned}$$

Since $\alpha(u_{n+1},x_*) \geq 1$,combining the equations(20)and(22),we get that

$$d(u_{n+1},x_*) \leq \phi_\beta(max\{d(u_n,x_*), d(u_{n+1},x_*)\}) \qquad \forall n \in N \cup \{0\} \qquad (23)$$

where $\beta > max\{2\lambda_1, 3\lambda_2, \lambda_4, 2\lambda_5\}$
Assume that,for some n,we have$d(u_n,x_*) \leq d(u_{n+1},x_*)$
we have from the equation (23),we obtain,that
$d(u_{n+1},x_*) \leq \phi_\beta(max\{d(u_{n+1},x_*)\}) < d(u_{n+1},x_*)$,
which is a contradiction
now, for every $n \geq 0$,we have $d(u_{n+1},x_*) < d(u_n,x_*)$
from(23),we have $d(u_{n+1},x_*) \leq \phi_\beta(d(u_n,x_*))$ for every n
By induction,we obtain

$d(u_n,x_*) \leq \phi_\beta^n(d(u_0,x_*))$ $\forall n \in N \cup \{0\}$
Thus,by letting $n \to \infty$ in the above inequality,we obtain that $\{u_n\}$ converges to x_*.
Similarly ,we can prove that $\{u_n\}$ converges to y_*.
Therefore the uniqueness of limit,we conclude that $x_* = y_*$. □

Example Consider the complete Euclidian space $X = R^2$ with the metric $d((x_1,x_2),(y_1,y_2)) = 2(|x_1-x_2|+|y_1-y_2|)$. Let $A = \{(\rho,0) : \rho \in [0,1]\}$and

$B = \{(\delta, 1) : \delta \in [0,1]\}$.

Now,let $T : A \to B$ be defined by $T(\rho, 0) = (\frac{\rho}{8}, 1)$.Then it is easy to see that $d(A, B) = 1$ and $A_0 = A$,$B_0 = B$. To show that T is an $\alpha - \phi$-proximal penta contractive mapping with $\phi(t) = \frac{7}{8}t$,$\alpha \equiv 1$ and $\beta_1 = \frac{7}{8}$ and $\lambda_i = \frac{1}{2^{n+1}}$ for n=0,1,2,3,4,5.
Let $x, y \in A$,where $x = (\rho_1, 0)$ and $y = (\rho_2, 0)$.

$$\begin{aligned} d(Tx, Ty) &= d((\frac{\rho_1}{8}, 0), (\frac{\rho_2}{8}, 0)) \\ &= \frac{1}{8}|\rho_1 - \rho_2| \\ &= \frac{1}{8}d(x, y) \\ &= \frac{7}{8}(\frac{1}{7}d(x, y)) \\ &\leq \frac{7}{8}max\{\frac{1}{2}d(x, y), \frac{1}{4}[d(x, Tx) + d(y, Ty) - d(A, B)], \frac{1}{8}[d(y, Tx) + d(x, Ty) - d(A, B)], \\ &\frac{1}{16}[d(y, Ty) - d(A, B)], \frac{1}{32}[d(y, Tx) - d(A, B)], \frac{1}{64}[d(x, Ty) - d(A, B)]\} \end{aligned}$$

So, T is an $\alpha - \beta$-proximal penta contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in A$ and $\phi(t) = \frac{7}{8}t$, $\beta = \frac{7}{8}$ and $\lambda_i = \frac{1}{2^{n+1}}$ for n=0,1,2,3,4,5.
Since $\beta = \frac{7}{8} \geq max_{0 \leq k \leq 5}\{\lambda_k\}$.
It is easy to see that the pair (A,B) satisfies the P-property.Since $\alpha(x, y) = 1$ for all $x, y \in A$, then the mapping T is α-admissible.Also the fact that $\beta = \frac{7}{8} \geq max\{\frac{1}{2}, \frac{3}{8}, \frac{1}{32}, \frac{1}{32}\}.=max\{2\lambda_1, 3\lambda_2, \lambda_4, 2\lambda_5\}$,$=\frac{1}{2}$ and T is (α, d) regular since $\alpha \equiv 1$ assures the uniqueness of the proximity point of T. Therefore ,all the conditions of theorem 3.2 and 3.3 are satisfied and so T has a unique proximity point which is $x_* = (0, 0) \in A$
d((0,0),T(0,0)=d((0,0),(0,1))=1=d(A,B).

IV. APPLICATIONS TO FIXED POINT RESULTS

Let us recall the following definitions

Definition 4.1. *Let A be a nonempty set of a metric space (X,d).A self mapping $T : A \to A$ is called a generalized $\alpha - \phi$ proximal- penta-contractive if there exist two functions $\alpha : A \times A \to [0, +\infty)$ and $\phi \in \Phi_\beta$,where $\beta > 0$,such that for all $x, y \in A$,we have*

$$\alpha(x, y)d(Tx, Ty) \leq \phi(M_R(x, y)) \ \forall x, y \in A$$

where

$$M_R(x,y) = max\{\lambda_0 d(x,y), \frac{\lambda_1}{2}[d(x,Tx)+d(y,Ty)], \frac{\lambda_2}{2}[d(y,Tx)+d(x,Ty)], \lambda_3[d(y,Ty)], \frac{\lambda_4}{2}[d(y,Tx)], \frac{\lambda_5}{2}[d(x,Ty)]\}$$

with $\alpha_k \geq 0$ *for k=0,1,2,...5.*

By considering the particular case ,A=B in Theorem 3.2 and 3.3,the fixed point results were deduced as follows.

Corollary 4.1. *Let A be a nonvoid closed subset of a complete metric space (X,d).Consider a self mapping* $T : A \to A$ *be an* $\alpha - \phi$*-penta-contractive mapping,where* $\beta \geq max_{0\leq k\leq 5}\{\lambda_k\}$*satisfying the following assertions:*
(1)T is a α*-proximal admissible;*
(2) there exist elements $x_0, x_1 \in A$ *such that* $\alpha(x_0, x_1) \geq 1$*;*
(3) if $\{x_n\}$ *a sequence in A such that* $\alpha(x_n, x_{n+1}) \geq 1$*and* $\lim_{n\to\infty} x_n = x_* \in A$ *then there exists a subsequence* $\{x_{n_k}\}$*of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x_*) \geq 1\ \forall k$
Moreover,suppose that one of the following conditions holds:
(i)ϕ *is continuous and*
(ii)$\beta > max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}$.
Then T has a fixed point

Corollary 4.2. *Let A be a nonvoid closed subset of a complete metric space (X,d).Consider a self mapping* $T : A \to A$ *be an* $\alpha - \phi$*-penta-contractive mapping,where* $\beta \geq max_{0\leq k\leq 5}\{\lambda_k\}$*satisfying the following assertions:*
(1)T is a α*-proximal admissible;*
(2) there exist elements $x_0, x_1 \in A$ *such that* $\alpha(x_0, x_1) \geq 1$*;*
(3) if $\{x_n\}$ *a sequence in A such that* $\alpha(x_n, x_{n+1}) \geq 1$*and* $\lim_{n\to\infty} x_n = x_* \in A$ *then there exists a subsequence* $\{x_{n_k}\}$*of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x_*) \geq 1\ \forall k$
(4)suppose that one of the following conditions holds:
(i)ϕ *is continuous and*
(ii)$\beta > max\{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}$.
Moreover, suppose that T is (α, d) *regular and* $\beta > max\{2\lambda_1, 3\lambda_2, \lambda_4, 2\lambda_5\}$ *Then T has a unique fixed point.*

V. CONCLUSION

In this paper we given some improvements to the best proximity point theorems previously made by Mohamed Ladh Ayari([5])for $\alpha - \beta$-proximal quasi-contractive mappings.This improvement was obtained by introducing the proximal $\alpha - \phi$-penta-contractive mappings on metric spaces involving β comparsion functions.The applications established not only the existence but the uniqueness of best proximity point results .

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