Generalization of S - Normal Matrices

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Abstract

The properties of matrices namely bi-normal and bi-unitary matrices and also the generalization of normal and s-normal and s-unitary matrices. Further to extend the polynomial normal matrices.

Keywords: *Bi-normal, S-normal, polynomial matrix.*

I. INTRODUCTION

In matrix theory, the special types of matrices namely bi-normal matrices and bi-unitary matrices as a generalization of normal and s-normal and s-unitary matrices. Some of its basic properties are studied. Numerical examples are provided. Further, we shall define polynomial s-normal matrices with an example, with reference to Ramesh, Sudha and Gajalakshmi [1,2].

II. MAIN RESULTS

Definition: 2.1

Let $A \in C^{n\times n}$ is said to be bi-normal if A is normal as well as s-normal.

Example:

$A = \begin{bmatrix} i & 1+i \\ 1+i & i \end{bmatrix}$ is bi-normal.

Remark:

Any normal matrix need not be s-normal. This can be seen from the following example,

$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ is normal, but not s-normal.

Similarly, any s-normal matrix need not be normal. This can be seen from the following example,

$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ is s-normal but it is not a normal matrices.

In the following we shall see some basic properties of bi-normal matrices analogous to that of s-normal matrices.

Theorem: 2.2

Any matrix $A \in C^{n\times n}$ is bi-normal if and only if A^T is bi-normal.

Proof:

A is bi-normal □ $AA^* = A^*A$ & $AA^\theta = A^\theta A$.

$(AA^*)^T$□ $= (A^*A)^T$□ & $(AA^\theta)^T$□ $= (A^\theta A)^T$□(Taking transpose on both sides)

$(A^*)^T(A)^T$□ $= (A^T)(A^*)^T$□ □ & $(A^\theta)^T(A)^T$□ $= (A^T)(A^\theta)^T$

$(A^T)^*\,(A)^T$□ $= (A^T)(A^T)^*$ & $(A^T)^\theta(A)^T$□ $= (A^T)(A^T)^\theta$

A^T is bi-normal.

Hence it is proved.

Theorem: 2.3

Any matrix $A \in C^{n\times n}$ is bi-normal if and only if A^* is bi-normal.

Proof:

A is bi-normal □ $AA^* = A^*A$ & $AA^\theta = A^\theta A$.

$(AA^*)^* = (A^*A)^*$ & $(AA^\theta)^* = (A^\theta A)^*$

$(A^*)^*(A)^* = (A^*)(A^*)^*$ & $(A^\theta)^*(A)^* = (A^*)(A^\theta)^*$

$(A^*)^*\,(A)^* = (A^*)(A^*)^*$ & $(A^\theta)^*\,(A)^* = (A^*)(A^*)^\theta$

A^* is bi-normal. Hence it is proved.

Theorem: 2.4

Any non-singular matrix $A \in C^{n\times n}$ is bi-normal if and only if A^{-1} is bi-normal.

Proof:

A is bi-normal □ $AA^* = A^*A$ & $AA^\theta = A^\theta A$.

$(AA^*)^{-1}$□$= (A^*A)^{-1}$□& $(AA^\theta)^{-1}$□ $= (A^\theta A)^{-1}$□ □ □ □(Taking inverses)

$(A^*)^{-1}(A)^{-1} = (A^{-1})(A^*)^{-1}$ & $(A^\theta)^{-1}(A)^{-1} = (A^{-1})(A^\theta)^{-1}$

$(A^{-1})^* (A)^{-1} = (A^{-1})(A^{-1})^*$ & $(A^{-1})^{\theta}(A)^{-1} = (A^{-1})(A^{-1})^{\theta}$
 A^{-1} is bi-normal. Hence it is proved.

Theorem: 2.5

If $A \in C^{n\times n}$ is bi-normal matrix then (iA) is bi-normal.

Proof:

A is bi-normal implies $AA^* = A^*A$ and $AA^{\theta} = A^{\theta}A$.

$$\begin{aligned} \text{Now, } (iA)(iA)^* &= (iA)(-iA^*) \\ &= i(-i)AA^* \\ &= i(-i)A^*A \quad (\text{ since A is normal }) \\ &= (iA)^*(iA) \end{aligned}$$

$$\begin{aligned} \text{Also, } (iA)(iA)^{\theta} &= (iA)(-iA^{\theta}) \\ &= i(-i)AA^{\theta} \\ &= i(-i)A^{\theta}A \quad (\text{ since A is s-normal }) \\ &= (iA)^{\theta}(iA) \end{aligned}$$

Hence (iA) is bi-normal.

In general sum and product of bi-normal matrices is not a bi-normal matrix. In the following theorem, we shall see that sum and product of bi-normal matrices is a bi-normal matrix under certain conditions.

Proof:

Since A is bi-normal implies, $AA^* = A^*A$ & $AA^{\theta} = A^{\theta}A$ and

B is bi-normal implies, $BB^* = B^*B$ & $BB^{\theta} = B^{\theta}B$.

$$\begin{aligned} \text{Now, } (A+B)(A+B)^* &= (A+B)(A+B)^* \\ &= (A+B)(A^*+B^*) \\ &= AA^*+AB^*+BA^*+BB^* \\ &= AB^*+BB^*+AA^*+BA^* \\ &= B^*A+B^*B+A^*A+A^*B \\ &= B^*(A+B)+A^*(A+B) \\ &= (B^*+A^*)(A+B) \\ &= (A^*+B^*)(A+B) \\ &= (A+B)^*(A+B) \end{aligned}$$

$$\begin{aligned} \text{Also, } (A+B)(A+B)^{\theta} &= (A+B)(A+B)^{\theta} \\ &= (A+B)(A^{\theta}+B^{\theta}) \\ &= AA^{\theta}+AB^{\theta}+BA^{\theta}+BB^{\theta} \\ &= AB^{\theta}+BB^{\theta}+AA^{\theta}+BA^{\theta} \\ &= B^{\theta}A+B^{\theta}B+A^{\theta}A+A^{\theta}B \\ &= B^{\theta}(A+B)+A^{\theta}(A+B) \\ &= (B^{\theta}+A^{\theta})(A+B) \\ &= (A^{\theta}+B^{\theta})(A+B) \\ &= (A+B)^{\theta}(A+B). \end{aligned}$$

Hence (A+B) is binormal.

Theorem: 2.6

Let A & B $\in C^{n\times n}$ be bi-normal. If $AB^* = B^*A$ and $AB^{\theta} = B^{\theta}A$, then AB is bi-normal.

Proof:

Since A is bi-normal implies, $AA^* = A^*A$ & $AA^{\theta} = A^{\theta}A$ and

B is bi-normal implies, $BB^* = B^*B$ & $BB^{\theta} = B^{\theta}B$.

$$\begin{aligned} \text{Now, } (AB)(AB)^* &= ABB^*A^* = AB^*BA^* \quad (\text{since } BB^* = B^*B) \\ &= B^*AA^*B \\ &= B^*A^*AB \\ &= (AB)^*AB \end{aligned}$$

$$\begin{aligned} \text{Also, } (AB)(AB)^{\theta} &= ABB^{\theta}A^{\theta} \\ &= AB^{\theta}BA^{\theta} \quad (\text{since } BB^{\theta} = B^{\theta}B) \\ &= B^{\theta}AA^{\theta}B \\ &= B^{\theta}A^{\theta}AB \end{aligned}$$

Hence (AB) is binormal.

Now we shall define polynomial s-normla matrices with an example.

Definition: 2.7

A polynomial s-normal matrices is a polynomial matrix whose coefficient matrix are s-normal matrices.

Example:

$$A\lambda = \begin{bmatrix} i\lambda^2 + i\lambda + \lambda & \lambda i + i \\ i\lambda + i & i\lambda^2 + i\lambda + \lambda \end{bmatrix}$$

$$= \lambda^2 \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} + \lambda \begin{bmatrix} i+1 & i \\ i & i+1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$= A_2\lambda^2 + A_1\lambda + A_0$$

Where A_0, A_1, A_2 are s-normal matrices some results on polynomial s-normal matrices.

In general product of polynomial s-normal matrices need not be polynomial s-normal matrices. In the following it is shown that product of polynomial s-normal matrices is a polynomial s-normal matrix when they are commuting.

Theorem: 2.8

If $A(\lambda)$ and $B(\lambda)$ are polynomial s-normal matrices and $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$. then $A(\lambda)B(\lambda)$ is a polynomial s-normal matrix**.**

Proof:

Let $A(\lambda) = A_0 + A_1\lambda + \ldots + A_n\lambda^n$ and

$B(\lambda) = B_0 + B_1\lambda + \ldots + B_n\lambda^n$ be polynomial s-normal matrices,

$A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$

$A(\lambda)B(\lambda) = (A_0 + A_1\lambda + \ldots\ldots\ldots + A_n\lambda^n)(B_0 + B_1\lambda + \ldots\ldots\ldots + B_n\lambda^n)$

$= (A_0B_0 + A_0B_1\lambda + \ldots\ldots + A_0B_n\lambda^n) + (A_1B_0\lambda + A_1B_1\lambda^2 + \ldots\ldots + A_1B_n\lambda^{n+1})$

$+ \ldots\ldots\ldots\ldots\ldots + (A_nB_0\lambda^n + A_nB_1\lambda^{n+1} + \ldots\ldots + A_nB_n\lambda^{2n})$

$= (A_0 + B_0) + (A_0B_1 + A_1B_0)\lambda + \ldots\ldots\ldots + (A_0B_n + A_1B_{n-1} + \ldots\ldots\ldots + A_nB_0)\lambda^n$

$B(\lambda)A(\lambda) = B_0A_0 + (B_0A_1 + B_1A_0)\lambda + \ldots\ldots\ldots\ldots\ldots + (B_0A_n + B_1A_{n-1} + \ldots\ldots + B_nA_0)\lambda^n$

Here each coefficient of λ and constants terms are equal.

(i.e) $A_0B_0 = B_0A_0$

$A_0B_1 + A_1B_0 = B_0A_1 + B_1A_0$

$\Rightarrow A_0B_1 = B_0A_1$ & $A_1B_0 = B_1A_0$

$A_nB_0 = B_0A_n$, $A_1B_{n-1} = B_1A_{n-1}$, ………., $A_0B_n = B_nA_0$

Now we have to prove $A(\lambda)B(\lambda)$ is s-normal**.**

$$A(\lambda)B(\lambda)\,[A(\lambda)B(\lambda)]^\theta = A(\lambda)B(\lambda)[B(\lambda)A(\lambda)]^\theta$$
$$= A(\lambda)B(\lambda)\,A(\lambda)^\theta B(\lambda)^\theta$$
$$= A(\lambda)A(\lambda)^\theta B(\lambda)B(\lambda)^\theta$$
$$= A(\lambda)^\theta A(\lambda)B(\lambda)^\theta B(\lambda)$$
$$= A(\lambda)^\theta B(\lambda)^\theta A(\lambda)\,B(\lambda)$$
$$= [B(\lambda)A(\lambda)]^\theta[A(\lambda)\,B(\lambda)]$$
$$= [A(\lambda)\,B(\lambda)]^\theta[A(\lambda)\,B(\lambda)]$$

Hence $A(\lambda)\,B(\lambda)$ is s-normal.

Definition: 2.9

A polynomial s-unitary matrix is a polynomial matrix whose coefficients matrices are s-unitary matrices.

Definition: 2.10

Let $A(\lambda), B(\lambda) \in C^{n\times n}(\lambda)$, set of all n×n polynomial matrices. where A_i's , B_i's $\in C^{n\times n}$, $i = 1,2,3,\ldots.m$. The polynomial matrix $B(\lambda)$ is said to be s-unitarily equivalent to $A(\lambda)$. If there exists a polynomial s-unitary matrix $U(\lambda)$ such that $B(\lambda) = U(\lambda)^\theta\, A(\lambda)\, U(\lambda)$.

Theorem: 2.11

If $A(\lambda)$ is a polynomial s-normal matrix if and only if every polynomial s-unitarily equivalent to $A(\lambda)$ is polynomial s-normal matrix.

Proof:

Suppose $A(\lambda)$ is polynomial s-normal matrix and $B(\lambda) = U(\lambda)^\theta A(\lambda)U(\lambda)$, where $U(\lambda)$ is polynomial s-unitary matrix.

Now we show $B(\lambda)$ is polynomial s-normal matrix.

$$B(\lambda)^\theta B(\lambda) = [U(\lambda)^\theta A(\lambda)U(\lambda)]^\theta[U(\lambda)^\theta A(\lambda)U(\lambda)]$$
$$= U(\lambda)^\theta A(\lambda)^\theta U(\lambda)U(\lambda)^\theta A(\lambda)U(\lambda) \quad \text{(since } U(\lambda) \text{ is s-unitary matrix)}$$
$$= U(\lambda)^\theta A(\lambda)^\theta A(\lambda)U(\lambda)$$
$$= U(\lambda)^\theta A(\lambda)A(\lambda)^\theta U(\lambda)$$
$$= [U(\lambda)^\theta A(\lambda)U(\lambda)][U(\lambda)^\theta A(\lambda)^\theta U(\lambda)]$$
$$= B(\lambda)B(\lambda)^\theta$$

Hence $B(\lambda)$ is polynomial s-normal matrix.

Conversely, assume $B(\lambda)$ is polynomial s-normal matrix.

We shall show that $A(\lambda)$ is polynomial s-normal matrix.

$B(\lambda) = U(\lambda)^{\theta}A(\lambda)U(\lambda)$ is s-normal matrix.

Now,

$$B(\lambda)B(\lambda)^{\theta} = B(\lambda)^{\theta}B(\lambda)$$

$\Rightarrow [U(\lambda)^{\theta}A(\lambda)U(\lambda)][U(\lambda)^{\theta}A(\lambda)^{\theta}U(\lambda)] = [U(\lambda)^{\theta}A(\lambda)^{\theta}U(\lambda)]\,[U(\lambda)^{\theta}A(\lambda)U(\lambda)]$

$\Rightarrow U(\lambda)^{\theta}A(\lambda)[U(\lambda)U(\lambda)^{\theta}]A(\lambda)^{\theta}U(\lambda) = U(\lambda)^{\theta}A(\lambda)^{\theta}[U(\lambda)U(\lambda)^{\theta}]A(\lambda)U(\lambda)$

$\Rightarrow U(\lambda)^{\theta}A(\lambda)A(\lambda)^{\theta}U(\lambda) = U(\lambda)^{\theta}A(\lambda)^{\theta}A(\lambda)U(\lambda)$

Pre and post multiply by $U(\lambda)$ and $U(\lambda)^{\theta}$, we get

$\Rightarrow A(\lambda)A(\lambda)^{\theta} = A(\lambda)^{\theta}A(\lambda)$

Hence $A(\lambda)$ is a polynomial s-normal matrix.

Theorem: 2.12

Let $A(\lambda) = A_0 + A_1\lambda + \ldots\ldots + A_m\lambda^m$ and
$B(\lambda) = B_0 + B_1\lambda + \ldots\ldots. + B_m\lambda^m$ be polynomial matrix ,
where A_i's B_i's $\in C^{n\times n}$, $i = 0,1,2,\ldots\ldots.$ m are s-normal matrices and
$A_0B_0 = A_1B_1 = \ldots\ldots\ A_mB_m = 0$. Then $A_0^{\theta}B_0 = A_1^{\theta}B_1 = \ldots\ldots = A_m^{\theta}B_m = 0$.

Proof:

Let $A(\lambda) = A_0A_1\lambda + A_2\lambda^2 + \ldots\ldots. + A_m\lambda^m$ is polynomial s-normal matrix.
Where $A_0, A_1, \ldots\ldots\ A_m$ are s-normal matrices and
$B(\lambda) = B_0B_1\lambda + B_2\lambda^2 + \ldots\ldots. + B_m\lambda^m$ is a polynomial matrix and
Also given $A_0B_0 = A_1B_1 = \ldots\ldots. = A_mB_m = 0$.
We know that " If A is s-normal and AB = 0 then $A^{\theta}B = 0$"
Thus $A_0^{\theta}B_0 = A_1^{\theta}B_1 = \ldots\ldots. = A_m^{\theta}B_m = 0$.

III. CONCLUSION

In this paper, we have proved the bi-normal matrices and bi-unitary matrices for normal, s-normal and s-unitary matrices. Similarly, the polynomial S- normal and polynomial matrices are also discussed.

REFERENCES

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