The Upper Monophonic Hull Number of a Graph

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Abstract

For a connected graph $G = (V,E)$, the monophonic hull number $mh(G)$ of a graph G is the minimum cardinality of a set of vertices whose monophonic convex hull contains all vertices of G. A monophonic hull set S in a connected graph G is called a minimal monophonic hull set of G if no proper subset of S is a monophonic hull set of G. The upper monophonic hull number $mh^+(G)$ of G is the maximum cardinality of a minimal monophonic hull set of G. The upper monophonic hull number of certain classes of graphs are determined. Connected graphs of order p with upper monophonic hull number p or p-1 are characterized. It is shown that for every integer $a \geq 2$, there exists a connected graph G with $mh(G) =a$ and $mh^+(G) =2a$

Keywords - *hull number, monophonic hull number, upper monophonic hull number.*

I. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1, 9]. A convexity on a finite set V is a family C of subsets of V, convex sets which is closed under intersection and which contains both V and the empty set. The pair (V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph $G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected sub graph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W[2,8]. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u–v path in G. An u–v path of length $d(u,v)$ is called an u–v geodesic. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $I[u,v]$ denotes the set of all vertices which lie on $u - v$ geodesic. For a set S of vertices, let $I[S] = \cup_{u,v\in} I[u, v]$. The set S is convex if $I[S] = S$. Clearly if $S = \{v\}$ or $S = V$, then S is convex. The convexity number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex set containing S is denoted by $I_h(S)$ and called the convex hull of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h(S) \subseteq V$. A subset $S \subseteq V$ is called a geodetic set if $I[S] = V$ and a hull set if $I_h(S) = V$. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a minimum geodetic set or simply a g- set of G. Similarly, the hull number $h(G)$ of G is the minimum order of its hull sets and any hull set of order $h(G)$ is a minimum hull set or simply a h- set of G. The geodetic number of a graph is studied in [1,4,10] and the hull number of a graph is studied in [1,6]. A chord of a path $u_o, u_1, u_2, \ldots, u_n$ is an edge u_iu_j with $j \geq i + 2$. $(0 \leq i, j \leq n)$. A $u - v$ path P is called monophonic path if it is a chordless path. A vertex x is said to lie on a $u - v$ monophonic path P if x is a vertex of P including the vertices u and v. For two vertices u and v, let J $[u,v]$ denotes the set of all vertices which lie on a u – v monophonic path. For a set M of vertices, let $J[M] =\cup_{u,v} \in J[u, v]$. The set M is monophonic convex or m-convex if $J[M] = M$. Clearly if $M = \{v\}$or $M = V$, then M is m-convex. The m-convexity number, denoted by $C_m(G)$, is the cardinality of a maximum proper m-convex subset of V. The smallest m-convex set containing M is denoted by $J_h(M)$ and called the monophonic convex hull or m-convex hull of M. Since the intersection of two m-convex set is m-convex, the m-convex hull is well defined. Note that $M\subseteq J[M] \subseteq J_h(M) \subseteq V$. A subset $M \subseteq V$ is called a monophonic set if $J[M] = V$ and a m-hull set if $J_h(M) = V$. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a m- set of G. Similarly, the monophonic hull number $mh(G)$ of G is the minimum order of its m-hull sets and any m-hull set of order $m_h(G)$ is a minimum monophonic set or simply a *mh- set* of G. The monophonic number of a graph is studied in [5,7, 11,13] and the monophonic hull number of a graph is studied in [12,13]. A vertex v of G is said to be a monophonic vertex of a graph G if v belongs to every minimum monophonic set of G. A vertex v is an extreme vertex of a graph G if the sub graph induced by its neighbours is complete.. Throughout the following G denotes a connected graph with at least two vertices.

***Theorem 1.1.*[12]** Let G be a connected graph. Then each extreme vertex of G belongs to every monophonic hull set of G.

***Theorem 1.2* .[12]** For a connected graph G ,$mh(G)=p$ if and only if $G=K_p$

***Theorem 1.3.* [12]** For a connected graph G , $mh(G)=p-1$ if and only if $G=K_1+\cup m_j k_j$,where $\sum m_j \geq 2$

II. THE UPPER MONOPHONIC HULL NUMBER

Definition 2.1. A monophonic hull set S in a connected graph G is called a *minimal monophonic hull set* of G if no proper subset of S is a monophonic hull set of G. The *upper monophonic hull number* $mh^+(G)$ of G is the maximum cardinality of a minimal monophonic hull set of G.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_7\}$, $S_2 = \{v_3, v_7\}$, $S_3 = \{v_4, v_7\}$ are the only three mh-sets of G so that $mh(G) = 2$. Also the set $M= \{v_2, v_5, v_7\}$ is a monophonic hull set of G. Since no proper subsets of M is a monophonic hull set of G, M is a minimal monophonic hull set of G so that $mh^+(G) \geq 3$. It is easily verified that there is no minimal monophonic hull sets S of G with $|S| \geq 4$. Hence $mh^+(G) = 3$.

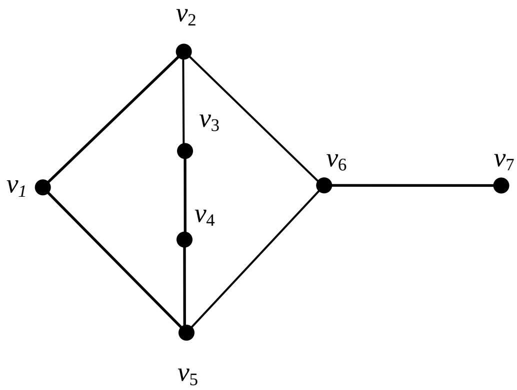

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Figure 2.1

Remark 2.3. Every minimum monophonic hull set of G is a minimal monophonic hull set of G and the converse is not true. For the graph G given in Figure 2.1, $M = \{v_2, v_5, v_7\}$ is a minimal monophonic hull set but not a minimum monophonic hull set of G.

Theorem 2.4. Each extreme vertex of G belongs to every minimal monophonic hull set of G.

Proof. Let S be a minimal monophonic hull set of G and v be an extreme vertex of G. Suppose that $v \notin S$. Then v is an internal vertex of a monophonic path, which is a contradiction to v an extreme vertex of G. ■

Theorem 2.5. Let G be a connected graph with cut-vertices and let S be a minimal monophonic hull set of G. If v is a cut-vertex of G, then every component of $G – v$ contains an element of S.

Proof. Suppose that there is a component B of $G- v$ such that B contains no vertex of S. Let $u \in V(B)$. Since S is a minimal monophonic hull set, there exists a pair of vertices x and y in S such that u lies on some $J^k[x , y]$; $k \geq 1$. Let x - y be a monophonic path $P : x = u_0, u_1, \ldots, u, \ldots, u_n = y$ in G. Since v is a cut-vertex of G, the $x – u$ sub path of P and the $u – y$ sub path of P both contain v, it follows that P is not a path, contrary to assumption. ■

The proof of the following theorem is straight forward so we omit it.

Theorem 2.6. For any connected graph G, no cut-vertex of G belongs to any minimal monophonic hull set of G.

Proof. Let v be any cut-vertex of G and let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - v$. Let S be any minimal monophonic hull set of G. Then by Theorem 2.5, S contains at least one element from each $G_i (1 \leq i \leq r)$. Since $\langle S \rangle$ is connected, it follows that $v \notin S$. ■

Corollary 2.7. For any non-trivial tree T, $mh^+(T) = k$, where k is the number of end vertices of T.

Proof. Since all the vertices of T is either a cut vertex or an extreme vertex, the result follows from Theorems 2.4 and 2.6. ■

Corollary 2.8. For a complete graph K_p, $p \geq 2$, $mh^+(K_p) = p$.

Proof. Since all the vertices of G are extreme vertices, the result follows from Theorem 2.4. ■

Theorem 2.9. For a complete bipartite graph $G = K_{m,n}(m, n \geq 2)$, $mh^+(G) = 2$.

Proof. Let $U = \{x_1, x_2, \ldots, x_m\}$ and $W = \{y_1, y_2, \ldots, y_n\}$ be a bipartition of G. Let $S = \{x, y\}$, where x and y are two independent vertices of G. Then S is a monophonic hull set of G, so that $mh(G) = 2$. We prove that $mh^+(G) = 2$. If not, let S_1 be a minimal monophonic hull set of G. with $|S_1| \geq 3$. Then S_1 consists of at least two independent vertices of G, it follows that $S \subsetneq S_1$, which is a contradiction. Hence $mh^+(G) = 2$. ■

Theorem 2.10. For a connected graph G, $2 \leq mh(G) \leq mh^+(G) \leq p$.

Proof. Any monophonic hull set needs at least two vertices and so $mh(G) \geq 2$. Since every minimal monophonic hull set is a monophonic hull set, $mh(G) \leq mh^+(G)$. Also, since $V(G)$ is a monophonic hull set of G, it is clear that $mh^+(G) \leq p$. Thus $2 \leq mh(G) \leq mh^+(G) \leq p$.

■***Remark 2.11.*** The bounds in Theorem 2.10 are sharp. For the graph G given in Figure 2.1, $mh(G) = 2$. For any non-trivial tree T, $mh(T) = mh^+(T)$ and for the complete graph $G = K_p$, $mh^+(G) = p$. Also, all the inequalities in Theorem 2.10 are strict. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_6, v_7\}$, $S_2 = \{v_2, v_6, v_7\}$, $S_3 = \{v_8, v_6, v_7\}$, $S_4 = \{v_9, v_6, v_7\}$and are the only four mh-sets of G so that $mh(G) = 3$. Also $M = \{v_3, v_5, v_6, v_7\}$ is a minimal monophonic hull set of G and so $mh^+(G) \geq 4$. It is easily verified that there is no minimal hull set S of G with $|S| \geq 5$ and hence $mh^+(G) = 4$. Thus $2 < mh(G) < mh^+(G) < p$.

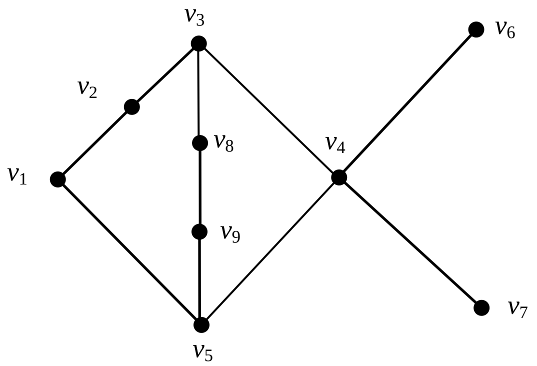

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Figure 2.2

Theorem 2.12. For a connected graph G, $mh(G) = p$ if and only if $mh^+(G) = p$.

Proof. Let $mh^+(G) = p$. Then $S = V(G)$ is the unique minimal monophonic hull set of G. Since no proper subset of S is a monophonic hull set, it is clear that S is the unique minimum monophonic hull set of G and so $mh(G) = p$. The converse follows from Theorem 2.10. ■

Corollary 2. 13. For a connected graph G of order p, the following are equivalent:

(i) $mh(G) = p$

(ii) $mh^+(G) = p$

(iii) $G = K_p$

Proof. This follows from Theorem 2.12 and Theorem 1.2. ■

Theorem 2.14. Let G be a non complete connected graph without cut vertices. Then $mh^+(G) \leq p–2$.

Proof. Suppose that $mh^+(G) \geq p - 1$. Then by Corollary 2.13, $mh^+(G) = p - 1$. Let v be a vertex of G and let $S = V(G) - \{v\}$ be a minimal monophonic hull set of G. By Theorem 2.4, v is not an extreme vertex of G. Then there exist $x, y \in N(v)$ such that $xy \notin E(G)$. Since v is not a cut vertex of G, $< G - v >$ is connected and also $< G - v >$ contains a monophonic path of length at least two. Let $x, x_1, x_2, \ldots, x_n, y$ be a monophonic path in $< G - v >$ of length at least two. Then $S_1 = S - \{x_1, x_2, \ldots, x_n\}$ is a monophonic hull set of G . Since $S_1 \subsetneq S$, S_1 is not a minimal monophonic hull set of G, which is a contradiction. Therefore $mh^+(G) \leq p–2$. ■

Theorem 2.15. For a connected graph G, $mh(G) = p-1$ if and only if $mh^{+}(G) = p-1$.

Proof. Let $mh(G) = p-1$. Then it follows from Theorem 2.10, $mh^{+}(G) = p$ or $p-1$. If $mh^{+}(G) = p$, then by Theorem 2.12, $mh(G) = p$, which is a contradiction. Hence $mh^{+}(G) = p-1$. Conversely, let $mh^{+}(G) = p-1$. Then it follows from Corollary 2.13 that G is non-complete. Hence by Theorem2.14, G contains a cut vertex, say v. Since $mh^{+}(G) = p-1$, it follows from Theorem 2.5 that $S = V(G) - \{v\}$ is the unique minimal monophonic hull set of G. Therefore $mh(G) = p-1$. ■

Corollary 2. 16. For a connected graph G of order p, the following are equivalent:

(i) $mh(G) = p-1$

(ii) $mh^{+}(G) = p-1$

(iii) $G = K_1 + \cup\, m_j K_j$, where $\sum m_j \geq 2$

Proof. This follows from Theorem 2.15 and Theorem 1.3. ■

In view of Theorem 2.10, we have the following realization result.

Theorem 2.17. For any positive integer $a \geq 2$, there exists a connected graph G such that $mh(G) = a$ and $mh^{+}(G) = 2a$.

Proof. Let $Q_i : u_i, v_i, x_i, y_i, w_i, u_i$ $(1 \leq i \leq a)$ be a copy of cycle C_5. Let G be the graph given in Figure 2.3 is obtained from Q_i by adding a new vertex x and joining the edges xv_i and xw_i $(1 \leq i \leq a)$. Let $H_i = \{x_i, y_i, u_i\}$. Since x is a cut vertex of G , by Theorem 2.5 every monophonic hull set of G contain at least one vertex from each component of $G - x$ so that $mh(G) \geq a$. Let $M_1 = \{x_1, x_2, x_3, \ldots x_a\}$. Then $J_h[M_1] = V(G)$ and so M_1 is a monophonic hull set of G so that $mh(G) = a$. Now, $S = \{w_1, w_2, \ldots, w_a, v_1, v_2, \ldots, v_a\}$is a monophonic hull set of G. We show that S is a minimal monophonic hull set of G. Let M be any proper subset of S. Then there exist at least one vertex, say $u \in S$ such that $u \notin M$. First assume that $u = w_i$ for some i $(1 \leq i \leq a)$. Then $J_h[M] \neq V(G)$ and so M is not a monophonic hull set of G. Next assume that $u = v_j$ for some j $(2 \leq j \leq a)$. Then also $J_h[M] \neq V(G)$ and so M is not a monophonic hull set of G. Hence S is a minimal monophonic hull set of G so that $mh^{+}(G) \geq 2a$. Since every minimum monophonic hull set contains exactly one vertex from each H_i $(1 \leq i \leq a)$, it follows that there is no minimal monophonic hull set X of G with $|X| \geq 2a + 1$. Thus $mh^{+}(G) = 2a$. ■

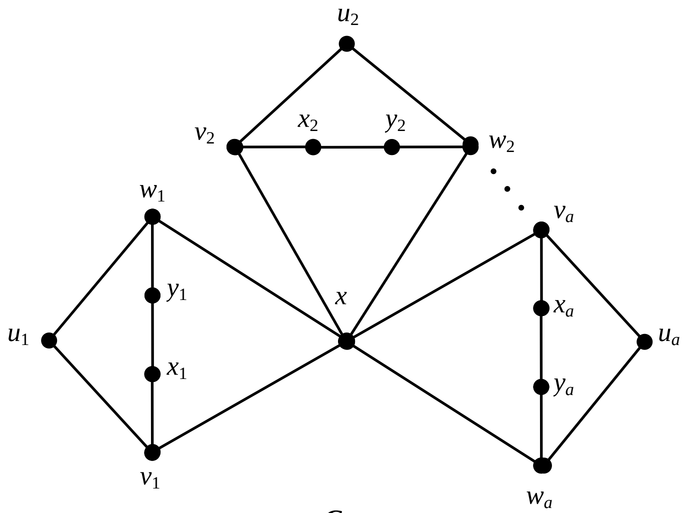

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Figure 2.3

III. CONCLUSIONS

In this paper, the upper monophonic hull number of certain classes of graphs are determined. Connected graphs of order p with upper monophonic hull number p or p-1 are characterized. The bounds for certain graphs are also determined. The upper bounds for certain graphs can also be reduced in the future.

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