Strong Forms of Mixed Continuity

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Abstract

Multivalued functions between topological spaces have applications to the fixed point theory which in turn has applications to social science, science and engineering. The authors have recently studied upper mixed continuous, lower mixed continuous multifunctions and their weak forms between topological spaces. The purpose of this paper is to introduce and characterize some strong forms of upper and lower mixed continuous multifunctions.

Keywords: *Multifunctions, mixed continuity, almost continuity, clopen continuity.*

MSC(2010): 54C60

I. INTRODUCTION AND PRELIMINARIES

Functions between two sets play a significant role in mathematics. In particular multivalued functions have applications in applied mathematics such as optimal control, calculus of variations, probability, statistics, different inclusions, fixed point theorems and so on.

Let X and Y be any two non empty sets. A multivalued function from X to Y, is a function $F:X\to\wp(Y)$ such that $F(x)\neq\varnothing$ for every $x\in X$ where $\wp(Y)$ denotes the power set of Y. A multifunction F from X to Y is denoted by $F:X\twoheadrightarrow Y$. If $F:X\twoheadrightarrow Y$ then for every subset A of Y, $F^+(A)=\{x\in X:F(x)\subseteq A\}$ is called the upper inverse of A and $F^-(A)=\{x\in X:F(x)\cap A\neq\varnothing\}$ is called the lower inverse of A. For every subset A of X, $F(A)=\bigcup_{x\in A}F(x)$.

For the basic results in multivalued analysis one may consult [1, 5, 7, 11,]. The applications of multivalued functions are found in [4, 6, 9, 10, 14, 15, 17, 19, 21, 22, 27]. The properties of multifunctions are discussed in [12, 13, 16]. The continuity of multifunctions is studied in [2,3,,8,9,12,13,20,21,24,26,30,31,32,33,34,]. Throughout this paper $F:X\twoheadrightarrow Y$ is a multi valued function, A , B are the subsets of X and 'iff' denotes ' if and only if '. The following lemmas will be useful to study the continuity of multifunctions. For the basic concepts and results on mixed continuity and their weak forms the readers may consult[29]. The following definitions will be useful in sequel.

Definition 1.1: Let $F:X\twoheadrightarrow Y$ and $G:X\twoheadrightarrow Y$. The multifunctions $F\cup G$ and $F\cap G$ are defined as $(F\cup G)(x) = F(x)\cup G(x)$ and $(F\cap G)(x) = F(x)\cap G(x)$ for every $x\in X$.

Definition 1.2: Let $F:X\twoheadrightarrow Y$ and $G:X\twoheadrightarrow Z$. The multifunction $F\times G:X\twoheadrightarrow Y\times Z$ is defined as $(F\times G)(x) = F(x)\times G(x)$ for every $x\in X$.

Definition 1.3: Let $F:X\twoheadrightarrow Y$ and $G:Y\twoheadrightarrow Z$. The multifunction $G°F :X\twoheadrightarrow Z$ is defined as $(G°F)(x)=\bigcup_{y\in F(x)}G(y)$ for every $x\in X$.

Let (X, τ) be a topological spaces and A be a subset of X.

Definition 1.4: A is regular open [25] if A = *Int Cl* A and is regular closed if A = *Cl Int* A. A is clopen if it is both open and closed.

Let $RO(X, \tau)$ denote the collection of all regular open sets in (X, τ). Since the union of regular open sets is not regular open, $RO(X, \tau)$ is not topology on X. Since the intersection of two regular open sets is regular open, $RO(X, \tau)$ is a base for some topology on X. This topology is denoted by τ^δ and the members of τ^δ are called δ-open sets is (X, τ). δ-open sets were studied by Velicko [35]. This τ^δ is called the semi-regularization of τ.

$$RO(X, \tau)\subseteq \tau^\delta \subseteq \tau$$

Definition 1.5: A is θ-open[35] if for all $x\in A$ there is an open set U with $x\in U\subseteq Cl\,U\subseteq A$.

The complement of **a** θ-open set is θ-closed in (X,τ). The collection of θ-open sets in (X,τ) is a topology on X, denoted by τ^θ. More over $\tau^\theta\subseteq\tau^\delta\subseteq\tau$.

Definition 1.6: F: (X, τ)$\twoheadrightarrow$ (Y, σ) is

(i) upper continuous (briefly U.C) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists $U\in\tau$ containing x such that $F(U)\subseteq V$.

(ii) upper clopen continuous (briefly U.cloC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists a clopen set U containing x such that $F(U)\subseteq V$.

(iii) upper almost continuous (briefly U.aC) if for all $x\in X$ and for every regular open set V in (Y, σ) with $F(x)\subseteq V$ there exists $U\in\tau$ containing x such that $F(U)\subseteq V$.

(iv) upper regular continuous (briefly U.rC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists a regular open set U containing x such that $F(U)\subseteq V$.

(v) upper δ-continuous (briefly U.δC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists $U\in\tau$ containing x such that $F(\mathit{Int\,Cl}\, U)\subseteq \mathit{Int\,Cl}\, V$.

(vi) upper θ-continuous (briefly U.θC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists an open set U in (X, τ) containing x such that $F(\mathit{Cl}\, U)\subseteq V$.

(vii) upper super continuous (briefly U. superC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists an open set U in (X,τ) containing x such that $F(\mathit{Int\,Cl}\, U)\subseteq V$.

Definition 1.7: F: (X, τ)$\twoheadrightarrow$ (Y, σ) is

(i) lower continuous (briefly L.C) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V\neq\emptyset$ there exists $U\in\tau$ containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$.

(ii) lower clopen continuous (briefly L.cloC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V\neq\emptyset$ there exists a clopen set U containing x such that $F(u)\cap V\neq\emptyset$ for all $u\in U$

(iii) lower almost continuous (briefly L.aC) if for all $x\in X$ and for every regular open set Vin (Y, σ) with $F(x)\cap V\neq\emptyset$ there exists $U\in\tau$ containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$.

(iv) lower regular continuous (briefly L.rC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V\neq\emptyset$ there exists $U\in\tau$ containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$

(v) lower continuous (briefly L.δC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V\neq\emptyset$ there exists an open set U containing x such that $F(u)\cap \mathit{Int\,Cl}\, V\neq\emptyset$ for every $u\in U$.

(vi) lower θ-continuous (briefly L.θC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V\neq\emptyset$ there exists an open set U in (X, τ) containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in \mathit{Cl}U$.

(vii) lower super continuous (briefly L.superC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V\neq\emptyset$, there exists an open set U in (X,τ) containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in \mathit{Int\,Cl}U$.

Definition 1.8: F: (X, τ)$\twoheadrightarrow$ (Y, σ) is upper mixed continuous(briefly U.M.C) [28] if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists $U\in\tau$ containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$ and is lower mixed continuous (briefly L.M.C)[28] if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V\neq\emptyset$ there exists $U\in\tau$ containing x such that $F(U)\subseteq V$.

Lemma 1.9:

(i) F: (X, τ)$\twoheadrightarrow$ (Y, σ) is U.rC iff it is U.superC.

(ii) F: (X, τ)$\twoheadrightarrow$ (Y, σ) is L.rC iff it is L.superC.

II. STRONG FORMS OF UPPER MIXED CONTINUOUS MULTIFUNCTIONS

Definition 2.1: F:(X, τ)$\twoheadrightarrow$ (Y, σ) is

(i)upper mixed clopen continuous (U.M.cloC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists a clopen set U in X,τ) containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$,

(ii)upper mixed almost continuous (U.M.aC) if for all $x\in X$ and for every regular open set in (Y, σ) with $F(x)\subseteq V$ there exists $U\in\tau$ containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$,

(iii)upper mixed regular continuous (U.M.rC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists a regular open set U in X,τ) containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$,

(iv)upper mixed δ-continuous (U.M.δC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists a δ-open set U in X,τ) containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$,

(v)upper mixed θ-continuous (U.M.θC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\subseteq V$ there exists a θ-open set U in X,τ) containing x such that $F(u)\cap V\neq\emptyset$ for every $u\in U$,

Proposition 2.2 : Let $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ be a multifunction. Then

(i) If F is U.M.cloC then it is U.M.rC and U.M.θC
(ii) If F is U.M.rC then it is U.M.aC and U.M.δC
(iii) If F is U.M.θC then it is U.M.δC
(iv) If F is U.M.δ C then it is U.M. C
(v) If F is U.M.C then it is U.M.aC

Proof: Suppose $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ is U.M.cloC. Let $V \in \sigma$ and $F(x) \subseteq V$. Since F is U.M.cloC at x, there exists a clopen set U in (X,τ) containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$. Since every clopen set is regular open and θ-open it follows that F is U.M.rC and U.M.θC. This proves (i). Now suppose $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ is U.M.rC. If $V \in \sigma$ and $F(x) \subseteq V$ then there exists a regular open set U in (X,τ) containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$. Since every regular open set is open and δ-open it follows that F is both U.M.aC and U.M.δC. This shows that $u \in F^-(V)$ for every $u \in U$ that implies $x \in U \subseteq F^-(V)$. Since $F^+(V) \subseteq F^-(V)$, the above arguments show that $F^+(V) \subseteq Int_\delta F^-(V)$. This proves (i) and (ii). (iii) follows from the fact that every θ-open set is δ-open and (iv) follows from the fact that every δ-open set is open. Similarly since every regular open set is open , it follows that U.M.C $\Rightarrow$ U.M.aC.

Diagram 2.3:

$$\begin{array}{ccccc} \text{U.M.cloC} & \Rightarrow & \text{U.M.rC} & \Rightarrow & \text{U.M.aC.} \\ \Downarrow & & \Downarrow & & \Uparrow \\ \text{U.M.}\theta\text{C} & \Rightarrow & \text{U.M.}\delta\text{C} & \Rightarrow & \text{U.M. C} \end{array}$$

Proposition 2.4 : Let $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ be a multifunction. Then

(i) If F is U.M.rC then $F^+(V) \subseteq Int_\delta\, F^-(V)$ for every $V \in \sigma$.
(ii) If F is U.M.cloC then $F^+(V) \subseteq Int_\delta\, F^-(V)$ for every $V \in \sigma$.
(iii) If F is U.M.δC then $F^+(V) \subseteq Int_\delta\, F^-(V)$ for every $V \in \sigma$.
(iv) If F is U.M.θC then $F^+(V) \subseteq Int_\theta F^-(V)$ for every $V \in \sigma$.
(v) F is U.M.aC iff $F^+(V) \subseteq Int\, F^-(V)$ for every $V \in RO(Y,\sigma)$
(vi) If F is U.M.rC then $Cl_\delta F^+(V) \subseteq F^-(V)$ for every closed set V in (Y, σ).
(vii) If F is U.M.cloC then $Cl_\delta F^+(V) \subseteq F^-(V)$ for every closed set V in (Y, σ).
(viii) F is U.M.aC iff $Cl F^+(V) \subseteq F^-(V)$ for every $V \in RC(Y,\sigma)$.

Proof: Suppose $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ is U.M.rC. Let $x \in F^+(V)$ and $V \in \sigma$. Then $F(x) \subseteq V$. Since F is U.M.rC at x, there exists a regular open set U in (X,τ) containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$. This shows that $u \in F^-(V)$ for every $u \in U$ that implies $x \in U \subseteq F^-(V)$. Since $F^+(V) \subseteq F^-(V)$, the above arguments show that $F^+(V) \subseteq Int_\delta F^-(V)$. This proves (i). Since every U.M.cloC is U.M.rC, it follows that (ii) is also true. The proof for (iii) and (iv) is similar.

Suppose $F: (X, \tau) \to (Y, \sigma)$ is U.M.aC. Let $x \in F^+(V)$ and $V \in RO(Y,\sigma)$. Then $F(x) \subseteq V$. Since F is U.M.aC at x, there exists an open set U in (X,τ) containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$. This shows that $u \in F^-(V)$ for every $u \in U$ that implies $x \in U \subseteq F^-(V)$. Since $F^+(V) \subseteq F^-(V)$, the above arguments show that every point of $F^+(V)$ is an interior point of $F^-(V)$ that implies $F^+(V) \subseteq Int\, F^-(V)$. Conversely suppose $F^+(V) \subseteq Int\, F^-(V)$ for every $V \in RO(Y,\sigma)$. Let $V \in RO(Y,\sigma)$ with $F(x) \subseteq V$. Then $x \in F^+(V) \subseteq Int\, F^-(V)$ that implies there is an open set U in (X, τ) containing x such that $x \in U \subseteq F^-(V)$. This shows that $F(u) \cap V \neq \emptyset$ for every $u \in U$ that proves that F is U.M.aC. This proves (v).

F is U.M.rC $\Rightarrow$ $F^+(Y \backslash B) \subseteq Int_\delta F^-(Y \backslash B)$ for every closed set B in (Y, σ).
$\Rightarrow$ $X \backslash F^-(B) \subseteq Int_\delta\, (X \backslash F^+(B)) = X \backslash Cl_\delta(F^+(B))$
$\Rightarrow$ $Cl_\delta\, (F^+(B)) \subseteq F^-(B)$ for every closed set B in (Y, σ).
F is U.M.aC iff $F^+(Y \backslash V) \subseteq Int\, F^-(Y \backslash V)$ for every $V \in RC(Y,\sigma)$.
iff $X \backslash F^-(V) \subseteq Int\, (X \backslash F^+(V)) = X \backslash Cl\, (F^+(V))$
iff $Cl\, (F^+(V)) \subseteq F^-(V)$ for every $V \in RC(Y,\sigma)$.
F is U.M.cloC $\Rightarrow$ F is U.M.rC $\Rightarrow$ $Cl_\delta\, (F^+(B)) \subseteq F^-(B)$ for every closed set B in (Y, σ).
This proves (vi), (vii) and (viii).

Proposition 2.5 :

(i) If $F: (X, \tau) \twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau) \twoheadrightarrow (Y , \sigma)$ are U.M.cloC then $F \cup G$ is U.M.cloC.
(ii) If $F: (X, \tau) \twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau) \twoheadrightarrow (Y , \sigma)$ are U.M.aC then $F \cup G$ is U.M.aC
(iii) If $F: (X, \tau) \twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau) \twoheadrightarrow (Y , \sigma)$ are U.M.rC then $F \cup G$ is U.M.rC.
(iv) If $F: (X, \tau) \twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau) \twoheadrightarrow (Y , \sigma)$ are U.M.δC then $F \cup G$ is U.M.δC.
(v) If $F: (X, \tau) \twoheadrightarrow (Y , \sigma)$ and $G: (X, \tau) \twoheadrightarrow (Y , \sigma)$ are U.M.θC then $F \cup G$ is U.M.θC

Proof: Suppose F: (X, τ)↠ (Y, σ) and G: (X, τ)↠ (Y, σ) are U.M.cloC. Let $x\in X$ and $V\in\sigma$ with $(F\cup G)(x)\subseteq V$. Then $F(x)\subseteq V$ and $G(x)\subseteq V$. Since F:(X, τ)↠ (Y, σ) and G:(X, τ)↠ (Y, σ) are U.M.cloC at x, there are clopen sets U_1 and U_2 in X such that $F(x')\cap V\neq\emptyset$ for every $x'\in U_1$ and $G(x'')\cap V\neq\emptyset$ for every $x''\in U_2$. Then $U = U_1\cap U_2$ is the required clopen set satisfying $(F\cup G)(u)\cap V\neq\emptyset$ for $u\in U$. This shows (i). Now suppose F:(X, τ)↠ (Y, σ) and G:(X, τ)↠ (Y, σ) are U.M.aC. Let $x\in X$ and $V\in RO(Y,\sigma)$ with $(F\cup G)(x)\subseteq V$. Then $F(x)\subseteq V$ and $G(x)\subseteq V$. Since F: (X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are U.M.aC at x, there are open sets U_1 and U_2 in X such that $F(x')\cap V\neq\emptyset$ for every $x'\in U_1$ and $G(x'')\cap V\neq\emptyset$ for every $x''\in U_2$. Then $U = U_1\cap U_2$ is the required open set satisfying $(F\cup G)(u)\cap V\neq\emptyset$ for $u\in U$. This shows (ii). The proof for the rest is analogous.

Proposition 2.6:
(i)If F:(X, τ)↠(Y,σ) is U.M.cloC and G:(Y,σ)↠(Z, η) is U.C then G°F:(X, τ)↠(Z , η) is U.M.cloC .
(ii)If F:(X, τ)↠(Y,σ) is U.M.rC and G:(Y,σ)↠(Z , η) is U.C then G°F: (X, τ)↠ (Z , η) is U.M.rC.
(iii)If F: (X, τ)↠ (Y,σ) is U.M.δC and G:(Y,σ) ↠ (Z , η) is U.C then G°F: (X, τ)↠ (Z , η) is U.M.δC.
(iv)If F: (X, τ)↠ (Y,σ) is U.M.θC and G:(Y,σ) ↠ (Z , η) is U.C then G°F: (X, τ)↠ (Z , η) is U.M.θC.
(v)If F:(X, τ)↠(Y,σ) is U.M.C and G:(Y, σ)↠ (Z, η) is U.aC then G°F:(X, τ)↠ (Z , η) is U.M.aC.
Proof: Suppose F: (X, τ)↠ (Y,σ) is U.M.cloC and G:(Y,σ) ↠ (Z, η) is U.C. Let $x\in X$ and $V\in\eta$ with $G°F(x)\subseteq V$. Then $G(y)\subseteq V$ for every $y\in F(x)$. That is $y\in G^+(V)$ for every $y\in F(x)$. Since G is U.C, $G^+(V)$ is open in Y. Clearly $F(x)\subseteq G^+(V)$. Since F is U.M.cloC, there is a clopen set U in X containing x such that $F(u)\cap G^+(V)\neq\emptyset$ for every $u\in U$. That is $G(F(u)\cap G^+(V))\neq\emptyset$ for every $u\in U$ that implies $G(F(u))\cap G(G^+(V)))\neq\emptyset$ for every $u\in U$. This shows that $G(F(u))\cap V\neq\emptyset$ for every $u\in U$ that implies G°F is U.M.cloC. This proves (i). The proof for (ii), (iii) and (iv) is analog. Suppose F: (X, τ)↠ (Y, σ) is U.M.C and G:(Y,σ) ↠ (Z, η) is U.aC. Let $x\in X$ and $V\in RO(Z, \eta)$ with $G°F(x)\subseteq V$. Then $G(y)\subseteq V$ for every $y\in F(x)$. That is $y\in G^+(V)$ for every $y\in F(x)$. Since G is U.C, $G^+(V)$ is open in Y. Clearly $F(x)\subseteq G^+(V)$. Since F is U.M.C, there is an open set U in X containing x such that $F(u)\cap G^+(V)\neq\emptyset$ for every $u\in U$. That is $G(F(u)\cap G^+(V))\neq\emptyset$ for every $u\in U$ that implies $G(F(u))\cap G(G^+(V)))\neq\emptyset$ for every $u\in U$. This shows that $G(F(u))\cap V\neq\emptyset$ for every $u\in U$ that implies G°F is U.M.aC. This proves (v).

Proposition 2.7: If F:(X,τ) ↠ (Y, σ) and G:(X,τ) ↠ (Z , η) are U.M. cloC then F×G: (X,τ) ↠ (Y×Z , $\sigma\times\eta$) is U.M. cloC where $\sigma\times\eta$ is the product topology on Y×Z.

Proof: Suppose F: (X,τ) ↠ (Y, σ) and G: (X,τ) ↠ (Z, η) are U.M.cloC. Let $x\in X$, $V\in\sigma$ and $W\in\eta$ with $F(x)\times G(x)\subseteq V\times W$. Then $F(x)\subseteq V$ and $G(x)\subseteq W$. Since F and G are U.M.cloC at x, there are clopen sets U_1 and U_2 in X containing x such that $F(x')\cap V\neq\emptyset$ for every $x'\in U_1$ and $G(x'')\cap W\neq\emptyset$ for every $x''\in U_2$. Taking $U = U_1\cap U_2$ we see that $(F(u)\times G(u))\cap(V\times W)\neq\emptyset$ for every $u\in U$. This shows that F×G is U.M. cloC.

Proposition 2.8: If F:(X,τ) ↠ (Y, σ) and G:(X,τ) ↠ (Z , η) are U.M. aC then F×G: (X,τ) ↠ (Y×Z , $\sigma\times\eta$) is U.M. aC .

Proof: Suppose F:(X,τ) ↠ (Y, σ) and G:(X,τ) ↠ (Z, η) are U.M. rC. Let V×W be regular open in (Y×Z , $\sigma\times\eta$) with $F(x)\times G(x)\subseteq V\times W$. Then $V\in RO(Y,\sigma)$ and $W\in RO(Z,\eta)$ so that $F(x)\subseteq V$ and $G(x)\subseteq W$. Since F and G are U.M.aC at x, there are open sets U_1 and U_2 in X containing x such that $F(x')\cap V\neq\emptyset$ for every $x'\in U_1$ and $G(x'')\cap W\neq\emptyset$ for every $x''\in U_2$. Taking $U = U_1\cap U_2$ we see that $(F(u)\times G(u))\cap (V\times W)\neq\emptyset$ for every $u\in U$. This shows that F×G is U.M. aC.

The proof for the next two propositions is analog.

Proposition 2.9: If F:(X,τ) ↠ (Y, σ) and G:(X,τ) ↠ (Z , η) are U.M. δC then F×G: (X,τ) ↠ (Y×Z , $\sigma\times\eta$) is U.M. δC.

Proposition 2.10: If F:(X,τ) ↠ (Y, σ) and G:(X,τ) ↠ (Z , η) are U.M. θC then F×G: (X,τ) ↠ (Y×Z , $\sigma\times\eta$) is U.M. θC.

III. STRONG FORMS OF LOWER MIXED CONTINUOUS MULTIFUNCTIONS

Definition 3.1: A multifunction F:(X, τ)↠ (Y, σ) is
(i)lower mixed clopen continuous (briefly L.M.cloC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V\neq\emptyset$ there exists a clopen set U in (X,τ) containing x such that $F(U)\subseteq V$.

(ii)lower mixed almost continuous (briefly L.M.aC) if for all $x\in X$ and for every $V\in RO(Y,\sigma)$ with $F(x)\cap V \neq\emptyset$ there exists $U\in\tau$ containing x such that $F(U)\subseteq V$.

(iii)lower mixed regular continuous (briefly L.M.rC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V \neq\emptyset$ there exists $U\in RO(X,\tau)$ containing x such that $F(U)\subseteq V$.

(iv)lower mixed δ-continuous (briefly L.M.δC) if for all $x\in X$ and for every $V\in,\sigma$ with $F(x)\cap V \neq\emptyset$ there exists a δ-open set U containing x such that $F(U)\subseteq V$.

(v)lower mixed θ-continuous (briefly L.M.θC) if for all $x\in X$ and for every $V\in\sigma$ with $F(x)\cap V \neq\emptyset$ there exists a θ-open set U containing x such that $F(U) \subseteq V$.

Proposition 3.2: Let $F: (X, \tau)\twoheadrightarrow (Y, \sigma)$ be a multifunction and $x\in X$.

(i)If F is L.M.cloC and $V \in\sigma$ then $F(x)\cap V \neq\emptyset \Leftrightarrow F(x) \subseteq V$.

(ii) If F is L.M.aC and $V\in RO(Y,\sigma)$ then $F(x)\cap V \neq\emptyset \Leftrightarrow F(x) \subseteq V$.

(iii) If F is L.M.rC and $V\in\sigma$ then $F(x)\cap V \neq\emptyset \Leftrightarrow F(x) \subseteq V$.

(iv) If F is L.M.δC and $V\in\sigma$ then $F(x)\cap V \neq\emptyset \Leftrightarrow F(x) \subseteq V$.

(v) If F is L.M. θC and $V\in\sigma$ then $F(x)\cap V \neq\emptyset \Leftrightarrow F(x) \subseteq V$.

Proof: Suppose $F:(X, \tau)\twoheadrightarrow (Y, \sigma)$ is L.M.cloC and $V\in\sigma$. If $F(x)\cap V\neq\emptyset$ then there exists a clopen set U in (X,τ) containing x such that $F(u)\subseteq V$ for every $u\in U$ that implies $F(x) \subseteq V$. The reverse part $F(x) \subseteq V \Rightarrow F(x)\cap V\neq\emptyset$ is obviously true. This proves (i) and the rest of the cases can be analogously proved.

Proposition 3.3 : Let $F: (X, \tau)\twoheadrightarrow (Y, \sigma)$ be a multifunction. Then

(i) If F is L.M.cloC then it is L.M.rC and L.M.θC
(ii) If F is L.M.rC then it is L.M.aC and L.M.δC
(iii) If F is L.M.θC then it is L.M.δC
(iv) If F is L.M.δ C then it is L.M. C
(v) If F is L.M.C then it is L.M.aC

Proof: Suppose $F:(X, \tau)\twoheadrightarrow (Y, \sigma)$ is L.M.cloC. Let $V\in\sigma$ and $F(x)\cap V\neq\emptyset$. Since F is L.M.cloC at x, there exists a clopen set U in (X,τ) containing x such that $F(u)\subseteq V$ for every $u\in U$. Since every clopen set is regular open and θ-open it follows that F is L.M.rC and L.M.θC. This proves (i). Now suppose $F:(X, \tau)\twoheadrightarrow (Y, \sigma)$ is L.M.rC. If $V\in\sigma$ and $F(x)\cap V\neq\emptyset$. then there exists a regular open set U in (X,τ) containing x such that $F(u)\subseteq V$ for every $u\in U$. Since every regular open set is open and δ-open it follows that F is both L.M.aC and L.M.δC. This proves (ii). (iii) follows from the fact that every θ-open set is δ-open and (iv) follows from the fact that every δ-open set is open. Similarly since every regular open set is open , it follows that U.M.C $\Rightarrow$ U.M.aC.

Proposition 3.4 : Let $F: (X, \tau)\twoheadrightarrow (Y, \sigma)$ be a multifunction. Then

(i) If F is L.M.cloC then $F^{-}(V)$ is δ-open in (X, τ) for every $V\in\sigma$.
(ii) If F is L.M.aC then $F^{-}(V)$ is open in (X, τ) for every $V\in RO(Y,\sigma)$.
(iii) If F is L.M.rC then $F^{-}(V)$ is δ-open in (X, τ) for every $V\in\sigma$.
(iv) If F is L.M.δC then $F^{-}(V)$ is δ-open in (X, τ) for every $V\in\sigma$
(v) If F is L.M.θC then $F^{-}(V)$ is θ-open in (X, τ) for every $V\in\sigma$

Proof: Suppose $F: (X, \tau)\twoheadrightarrow (Y, \sigma)$ is L.M.cloC. Let $x\in F^{-}(V)$ and $V\in\sigma$. Then $F(x)\cap V \neq\emptyset$. Since F is L.M.clo at x, there exists a clopen set U in (X,τ) containing x such that $F(u)\subseteq V$ for every $u\in U$. This shows that $u\in F^{+}(V)$ for every $u\in U$ that implies $U\subseteq F^{+}(V)$. Since $F^{+}(V)\subseteq F^{-}(V)$, the above arguments show that $U\subseteq Int_{\delta}F^{-}(V)$ that implies $F^{-}(V)$ is δ-open in (X, τ). This proves (i). Suppose $F:(X, \tau)\twoheadrightarrow (Y, \sigma)$ is U.M.aC. Let $x\in F^{-}(V)$ and $V\in RO(Y,\sigma)$. Then $F(x)\cap V \neq\emptyset$. Since F is L.M.aC at x, there exists an open set U in (X,τ) containing x such that $F(u)\subseteq V$ for every $u\in U$. This shows that $u\in F^{+}(V)$ for every $u\in U$ that implies $x\in U\subseteq F^{+}(V)$. Since $F^{+}(V)\subseteq F^{-}(V)$, the above arguments show that $x\in U\subseteq F^{-}(V)$ that implies $F^{-}(V)$ is open in (X,τ). This proves (ii) and the proof for the rest is analogous.

Proposition 3.5: Let $F: (X, \tau)\twoheadrightarrow (Y, \sigma)$ be a multifunction. Then

(i) If F is L.M.cloC then $F^{+}(V)$ is δ-closed in (X,τ) for every closed set V in (Y, σ).
(ii) If F is L.M.aC then $F^{+}(V)$ is closed in (X,τ) for every $V\in RC(Y,\sigma)$..
(iii) If F is L.M.rC then $F^{+}(V)$ is δ-closed in (X,τ) for every closed set V in (Y, σ).
(iv) If F is L.M.rC then $F^{+}(V)$ is δ-closed in (X,τ) for every closed set V in (Y, σ)
(v) If F is L.M.rC then $F^{+}(V)$ is δ-closed in (X,τ) for every closed set V in (Y, σ)

Proof: F is L.M.cloC$\Rightarrow F^{-}(Y\backslash B)$ is δ-open in (X,τ) for every closed set B in (Y, σ).
$\Rightarrow X\backslash F^{+}(B)$ is δ-open in (X,τ) for every closed set B in (Y, σ).
$\Rightarrow F^{+}(B)$ is δ-closed in (X,τ) for every closed set B in (Y, σ).

F is L.M.aC $\Rightarrow F^{-}(Y\backslash V)$ is open in (X,τ) for every $V\in RC(Y,\sigma)$.

$\Rightarrow$ X\ $F^+(V)$ is open in (X,τ) for every $V\in RC(Y,\sigma)$.

$\Rightarrow$ $F^+(V)$ is closed in (X,τ) for every $V\in RC(Y,\sigma)$.

The proof for the rest is similar.

Proposition 3.6 :

(i) If F:(X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.cloC then $F\cap G$ is L.M.cloC.

(ii) If F:(X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.aC then $F\cap G$ is L.M.aC

(iii) If F:(X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.rC then $F\cap G$ is L.M.rC.

(iv) If F:(X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.δC then $F\cap G$ is L.M.δC.

(v) If F:(X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.θC then $F\cap G$ is L.M.θC.

Proof: Suppose F:(X, τ)↠ (Y, σ) and G: (X, τ)↠ (Y , σ) are L.M.cloC. Let $x\in X$ and $V\in\sigma$ with $(F\cap G)(x)\cap V\neq\varnothing$. Then $F(x)\cap V\neq\varnothing$ and $G(x)\cap V\neq\varnothing$. Since F: (X, τ)→(Y, σ) and G: (X, τ)→(Y , σ) are L.M.cloC at x, there are clopen sets U_1 and U_2 in X such that $F(x')\subseteq V$ for every $x'\in U_1$ and $G(x'')\subseteq V$ for every $x''\in U_2$. Then $U = U_1\cap U_2$ is the required clopen set satisfying $(F\cap G)(u)\subseteq V$ for $u\in U$. This shows (i). Suppose F:(X, τ)↠ (Y , σ) and G: (X, τ)↠ (Y , σ) are L.M.aC. Let $x\in X$ and $V\in RO(Y, \sigma)$ with $(F\cap G)(x)\cap V\neq\varnothing$. Then $F(x)\cap V\neq\varnothing$ and $G(x)\cap V\neq\varnothing$. Since F: (X, τ)↠ (Y, σ) and G:(X, τ)↠ (Y , σ) are L.M.aC at x, there are open sets U_1 and U_2 in X such that $F(x')\subseteq V$ for every $x'\in U_1$ and $G(x'')\subseteq V$ for every $x''\in U_2$. Then $U = U_1\cap U_2$ is the required open set satisfying $(F\cap G)(u)\subseteq V$ for $u\in U$. This shows (ii). Analogous proof can be given to (iii), (iv) and (v).

Proposition 3.7: Let G:(Y,σ) ↠ (Z , η) be such that $G\,G^-(V)\subseteq V$ for every open set V in (Y,σ).

(i)If F: (X, τ)↠ (Y,σ) is L.M.cloC and G:(Y,σ) ↠ (Z , η) is L.C then
G°F: (X, τ)↠ (Z , η) is L.M.cloC.

(ii)If F:(X, τ)↠ (Y, σ) is L.M.C and G:(Y, σ)↠ (Z, η) is L.aC then
G°F:(X, τ)↠ (Z , η) is L.M.aC.

(iii)If F: (X, τ)↠ (Y,σ) is L.M.rC and G:(Y,σ) ↠ (Z , η) is L.C then
G°F: (X, τ)↠ (Z , η) is L.M.rC.

(iv)If F: (X, τ)↠ (Y,σ) is L.M.δC and G:(Y,σ) ↠ (Z , η) is L.C then
G°F: (X, τ)↠ (Z , η) is L.M.δC.

(v)If F: (X, τ)↠ (Y,σ) is L.M.θC and G:(Y,σ) ↠ (Z , η) is L.C then
G°F: (X, τ)↠ (Z , η) is L.M.θC.

Proof: Suppose F:(X, τ)↠(Y,σ) is L.M.cloC and G:(Y,σ) ↠ (Z , η) is L.C. Let $x\in X$ and $V\in\eta$ with $G°F(x)\cap V\neq\varnothing$. Then $G(y)\cap V\neq\varnothing$ for some $y\in F(x)$. That is $y\in G^-(V)$ for some $y\in F(x)$. Since G is L.C, $G^-(V)$ is open in Y. Clearly $F(x)\cap G^-(V)\neq\varnothing$. Since F is L.M.cloC, there is a clopen set U in X containing x such that $F(u)\subseteq G^-(V)$ for every $u\in U$. That is $G(F(u)\subseteq GG^-(V)\subseteq V$ for every $u\in U$ This implies G°F is L.M.cloC. This proves (i). Suppose F: (X, τ)→(Y, σ) is L.M.C and G:(Y, σ)→ (Z, η) is L.aC. Let $x\in X$ and $V\in RO(Z,\eta)$ with $G°F(x)\cap V\neq\varnothing$. Then $G(y)\cap V\neq\varnothing$ for some $y\in F(x)$. That is $y\in G^-(V)$ for some $y\in F(x)$. That is $y\in G^-(V)$ for some $y\in F(x)$. Since G is L.C, $G^-(V)$ is open in Y. Clearly $F(x)\cap G^-(V)\neq\varnothing$. Since F is L.M.C, there is an open set U in X containing x such that $F(u)\cap G^-(V)\neq\varnothing$ for every $u\in U$. That is $G(F(u)\cap G^+(V))\neq\varnothing$ for every $u\in U$ that implies $G(F(u))\subseteq G(G^+(V)))\subseteq V$ for every $u\in U$. This shows that G°F is L.M.aC.

Proposition 3.8: If F:(X,τ) ↠ (Y, σ) and G:(X,τ) ↠ (Z , η) are L.M. cloC then F×G: (X,τ) ↠ (Y×Z , σ×η) is L.M. cloC .

Proof: Suppose F: (X,τ) ↠ (Y, σ) and G: (X,τ) ↠ (Z, η) are L.M. cloC. Let $x\in X$, $V\in\sigma$ and $W\in\eta$ with $F(x)\times G(x)\cap V\times W\neq\varnothing$. Then $F(x)\cap V\neq\varnothing$ and $G(x)\cap W\neq\varnothing$. Since F and G are L.M.cloC at x, there are clopen sets U_1 and U_2 in X containing x such that $F(x')\subseteq V$ for every $x'\in U_1$ and $G(x'')\subseteq W$ for every $x''\in U_2$. Taking $U = U_1\cap U_2$ we see that $F(u)\times G(u) \subseteq V\times W$ for every $u\in U$. This shows that F× G is L.M. cloC.

Proposition 3.9: If F:(X,τ) ↠ (Y, σ) and G:(X,τ) ↠ (Z , η) are L.M. aC then F×G: (X,τ) ↠ (Y×Z , σ×η) is L.M. aC.

Proof: Suppose F: (X,τ) ↠ (Y, σ) and G: (X,τ) ↠ (Z, η) are L.M. aC. Let $x\in X$ and V×W be regular open in (Y×Z , σ×η) , $V\in\sigma$ and $W\in\eta$ with $F(x)\times G(x)\cap V\times W\neq\varnothing$. Then $F(x)\cap V\neq\varnothing$ and $G(x)\cap W\neq\varnothing$. Since F and G are L.M.aC at x, there are open sets U_1 and U_2 in X containing x such that $F(x')\subseteq V$ for every $x'\in U_1$ and $G(x'')\subseteq W$ for every $x''\in U_2$. Taking $U = U_1\cap U_2$ we see that $F(u)\times G(u) \subseteq V\times W$ for every $u\in U$. This shows that F× G is L.M. aC.

The proof for the next three propositions is analogous.

Proposition 3.10: If $F:(X,\tau) \twoheadrightarrow (Y, \sigma)$ and $G:(X,\tau) \twoheadrightarrow (Z , \eta)$ are L.M. rC then $F\times G: (X,\tau) \twoheadrightarrow (Y\times Z , \sigma\times\eta)$ is L.M. rC.

Proposition 3.11: If $F:(X,\tau) \twoheadrightarrow (Y, \sigma)$ and $G:(X,\tau) \twoheadrightarrow (Z , \eta)$ are L.M. δC then $F\times G: (X,\tau) \twoheadrightarrow (Y\times Z , \sigma\times\eta)$ is L.M. δC.

Proposition 3.12: If $F:(X,\tau) \twoheadrightarrow (Y, \sigma)$ and $G:(X,\tau) \twoheadrightarrow (Z , \eta)$ are L.M. θC then $F\times G: (X,\tau) \twoheadrightarrow (Y\times Z , \sigma\times\eta)$ is L.M. θC.

Proposition 3.13 : Let $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ be a multifunction. Then

(i) If F is L.M.cloC then it is U.M.cloC
(ii) If F is L.M.rC then it is U.M.rC
(iii) If F is L.M.θC then it is U.M. θC
(iv) If F is L.M.δC then it is U.M. δC

Proof: Suppose $F: (X, \tau) \twoheadrightarrow (Y, \sigma)$ is L.M.cloC. Let $V\in\sigma$ with $F(x)\subseteq V$. Then $F(x)\cap V\neq\varnothing$. Since F is L.M.cloC at x, there exists a clopen set U in (X,τ) containing x such that $F(u)\subseteq V$ for every $u\in U$ that implies $F(u)\cap V\neq\varnothing$ $F(x)\cap V\neq\varnothing$. This proves (i). The proof for the rest is analog.

Diagram 3.14:

$$\begin{array}{ccccc}
\text{L.M.cloC} & \Rightarrow & \text{L.M.rC} & \Rightarrow & \text{L.M.aC} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{U.M.cloC} & \Rightarrow & \text{U.M.rC} & \Rightarrow & \text{U.M.aC.} \\
\Downarrow & & \Downarrow & & \Uparrow \\
\text{U.M.}\theta\text{C} & \Rightarrow & \text{U.M.}\delta\text{C} & \Rightarrow & \text{U.M. C} \\
\Uparrow & & \Uparrow & & \Uparrow \\
\text{L.M.}\theta\text{C} & \Rightarrow & \text{L.M.}\delta\text{C} & \Rightarrow & \text{L.M. C} \\
\Uparrow & & \Uparrow & & \Downarrow \\
\text{L.M.cloC} & \Rightarrow & \text{L.M.rC} & \Rightarrow & \text{L.M.aC.}
\end{array}$$

IV. CONCLUSION

The strong forms of lower and upper mixed continuous multifunctions such as lower and upper mixed delta, theta and clopen continuous multifunctions have been discussed in this chapter and the links among them are investigated. Further the union, intersection, Cartesian product and composition of such functions are studied.

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