The Application of Generalized Prolate Spheroidal Wave Function, Class of Multivariable Polynomials, and the Multivariable Gimel-Function in Heat Conduction

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ABSTRACT

Gupta [5] and Gupta et al. [6] have studied the application of generalized prolate spheroidal wave function, generalized polynomials, and the multivariable H-function in heat conduction. In this paper we employ the generalized prolate spheroidal wave function, generalized hypergeometric function, class of multivariable polynomials and the multivariable Gimel-function in heat conduction in obtaining the formal solution of partial differential equation related to a problem of heat conduction in an anisotropic material. This problem occurs mainly in mechanics of solids, physics and applied mathematics.

Keywords : Modified multivariable Gimel-function, , classes of multivariable polynomials, generalized hypergeometric function, generalized prolate spheroidal wave function, expansion formula

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1. Introduction and preliminaries.

The generalized prolate spheroidal wave functions has been recently defined by Gupta [4] as the solution of the differential equation

$$(1 - x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + [\xi(s) - s^2x^2]y = 0$$
(1.1)

in the form of an infinite sum :

$$\phi_n^{(\alpha,\beta)}(s,x) = \sum_{p=0}^{\infty} R_{p,n}^{\alpha,\beta}(s) P_{n+p}^{(\alpha,\beta)}(x)$$
(1.2)

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomials, where s = 0 and $\xi(0) = (n+p)(\alpha + \beta + n + p + 1), p \ge 0$

In this paper we employ the generalized prolate spheroidal wave function, generalized hypergeometric , a class of multivariable polynomials and the multivariable Gimel-function in applied mathematics, we shall consider the problem of obtaining a solution of a problem of heat conduction. Consider the partial differential equation related to a problem of heat conduction in an anisotropic material which has been obtained by Saxena and Nageria [10] and given below :

$$\frac{\partial}{\partial x} \left[(1 - x^2) \frac{\partial U}{\partial x} \right] - \frac{pcv}{\lambda} \frac{\partial U}{\partial x} + \frac{Q(x)}{\lambda} = \frac{pc}{\lambda} \frac{\partial U}{\partial t}$$
(1.3)

With the law of conductivity $K = \lambda(1 - x^2)$, Q(x) is the intensity of a continuous source of heat situated inside this solid . Let initial temperature of the rod be given by

$$U(x,o) = F(x) \tag{1.4}$$

Gupta([5], page 107) took $v = \frac{\alpha - \beta}{q}$, $Q(x) = -(\alpha + \beta)\lambda x \frac{\partial U}{\partial x} - s^2 x^2 \lambda U$, $q = \frac{pc}{\lambda}$, in equation (1.3) and obtained the equation (1.3) to the form (1.1).

http://www.ijmttjournal.org

Page 51

International Journal of Mathematics Trends and Technology (IJMTT) - Volume 61 Number 1 - September 2018

We consider a generalized transcendental function called Gimel function of several complex variables.

$$\exists (z_1, \cdots, z_r) = \exists_{X; p_{i_r}, q_{i_r}, \tau_{i_r}: R_r: Y}^{U; 0, n_r: V} \begin{pmatrix} z_1 & \mathbb{A}; \mathbf{A}: \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}: \mathbf{B} \end{pmatrix} = \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^r \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r (1.1)$$

with $\omega = \sqrt{-1}$

The following quantities $\mathbb{A}, \mathbf{A}, A, \mathbb{B}, \mathbf{B}, B, X, Y, U, V, \psi(s_1, \dots, s_r)$ and $\theta_k(s_k)(k = 1, \dots, r)$ are defined by Ayant [2].

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$$\begin{split} &\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}) , max(|z_1|, \cdots, |z_r|) \to 0 \\ &\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}) , min(|z_1|, \cdots, |z_r|) \to \infty \text{ where } i = 1, \cdots, r: \end{split}$$

 $\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] \text{ and } \beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re\left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$

In this paper we shall assume that

$$F(x) = {}_{M}F_{N} \begin{bmatrix} (e_{M}) \\ \cdot \\ (f_{N}) \end{bmatrix} y(1-x)^{g}(1+x)^{w} \end{bmatrix} S_{N_{1},\cdots,N_{u}}^{M_{1},\cdots,M_{u}} \begin{pmatrix} y_{1}(1-x)^{f_{1}}(1+x)^{w_{1}} \\ \cdot \cdots \\ y_{u}(1-x)^{f_{u}}(1+x)^{w_{u}} \end{pmatrix}$$

$$H \begin{pmatrix} Z_{1}(1-x)^{h_{1}}(1+x)^{k_{1}} \\ \cdot \cdots \\ Z_{r}(1-x)^{h_{r}}(1+x)^{k_{r}} \end{pmatrix}$$
(1.2)

The class of multivariable polynomials defined by Srivastava [11], is given in the following manner :

$$S_{N_{1},\cdots,N_{u}}^{M_{1},\cdots,M_{u}}[y_{1},\cdots,y_{u}] = \sum_{K_{1}=0}^{[N_{1}/M_{1}]} \cdots \sum_{K_{u}=0}^{[N_{u}/M_{u}]} \frac{(-N_{1})_{M_{1}K_{1}}}{K_{1}!} \cdots \frac{(-N_{u})_{M_{u}K_{u}}}{K_{u}!} A[N_{1},K_{1};\cdots;N_{u},K_{u}]$$

$$A[N_{1},K_{1};\cdots;N_{u},K_{u}]y_{1}^{K_{1}}\cdots y_{u}^{K_{u}}$$

$$(1.3)$$

We shall use the following notation

$$B' = \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_u)_{M_uK_u}}{K_u!} A[N_1, K_1; \cdots; N_u, K_u]$$
(1.4)

On suitably specializing the above coefficients, $S_{N_1,\dots,N_u}^{M_1,\dots,M_u}[y_1,\dots,y_u]$ yields some of known polynomials, these ISSN: 2231 - 5373 <u>http://www.ijmttjournal.org</u> Page 52

International Journal of Mathematics Trends and Technology (IJMTT) - Volume 61 Number 1 - September 2018

include the Jacobi polynomials, Laguerre polynomials, and others polynomials ([14],p. 158-161)

The orthogonality property of the generalized prolate spheroidal wave function (Gupta [3], page 107, eq.(3.1)). We have the following integral.

Lemma

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \phi_{n}^{(\alpha,\beta)}(c,x) \phi_{m}^{(\alpha,\beta)}(c,x) \mathrm{d}x = N_{n,m}^{\alpha,\beta} \delta_{m,n} \quad \text{,where}$$
(1.5)

$$N_{n,n}^{\alpha,\beta} = 2^{\alpha+\beta+1} \sum_{p=0}^{\infty} \left[R_{p,n}^{\alpha,\beta} \right]^2 \frac{\Gamma(\alpha+p+1)\Gamma(\beta+p+1)}{(1+\alpha+\beta+2n+2p)\Gamma(n+p+1)\Gamma(1+\alpha+\beta+n+p)}$$
(1.6)

where $\delta_{m,n}$ is the Kronecker symbol.

2. Main integral

We have the following integral :

Theorem 1

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} \phi_{n}^{(\alpha,\beta)}(s,x)_{M} F_{N} \begin{bmatrix} (e_{M}) \\ \cdot \\ (f_{N}) \end{bmatrix} y(1-x)^{g} (1+x)^{w} \\ S_{N_{1},\cdots,N_{u}}^{M_{1},\cdots,M_{u}} \begin{pmatrix} y_{1}(1-x)^{f_{1}}(1+x)^{w_{1}} \\ \cdot \cdot \cdot \\ \vdots \\ y_{u}(1-x)^{f_{u}}(1+x)^{w_{u}} \end{pmatrix}$$

$$R_{p,n}^{\alpha,\beta}(s) \frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!(\alpha+1)_m} y_1^{K_1} \cdots y_u^{K_u} 2^{gq+wq+\sum_{i=1}^u (f_i+w_i)K_i}$$

$$\exists_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r;Y}^{U;0,n_r+2:V} \begin{pmatrix} Z_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ Z_r 2^{h_r+k_r} \end{pmatrix} \mathbb{A}; (1\text{-m-p-gq-}\sum_{i=1}^u f_i K_i:h_1,\cdots,h_r;1),$$

$$; (-\sigma - wq - \sum_{i=1}^{u} w_i K_i : k_1, \cdots, k_r; 1) : \mathbf{A}, A \\ \vdots \\ \mathbb{B}; \mathbf{B}, (-1 - m - \sigma - \rho - (g + w)q - \sum_{i=1}^{u} (f_i + w_i)K_i : h_1 + k_1, \cdots, k_r + l_r; 1) : B$$

$$(2.1)$$

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Provided that

 $\min\{g,w,f_i,w_i,a_j,b_j,h_k,k_k\}>0 \text{ for } i=1,\cdots,v; j=1,\cdots,u; k=1,\cdots,r$

$$\begin{split} ℜ(\rho + \sum_{i=1}^{v} K_i f_i) + \\ &+ \sum_{i=1}^{r} h_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0 \qquad \text{and} \\ ℜ(\sigma + \sum_{i=1}^{v} K_i w_i) + \sum_{i=1}^{r} k_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0 \\ &\left| arg Z_i (1-x)^{h_i} (1+x)^{k_i} \right| < \frac{1}{2}, A_i^{(k)}, A_i^{(k)}. \text{ is defined by (1.8)} \quad |y| < 1 \end{split}$$

Proof

To prove (2.1), first expressing the generalized prolate spheroidal wave function, a class of multivariable polynomials defined by Srivastava [11] $S_{N_1,\dots,N_s}^{M_1,\dots,M_s}[.]$, with the help of (1.2)and (1.9) respectively, expressing the generalized hypergeometric in series and we interchange the order of summations and *x*-integral (which is permissible under the conditions stated). Expressing the multivariable Gimel-function of *r*-variables in Mellin-Barnes contour integral with the help of (1.5) and interchange the order of integrations which is justifiable due to absolute convergence of the integrals involved in the process. Now collecting the powers of (1 - x) and (1 + x) and evaluating the inner *x*-integral. Interpreting the Mellin-Barnes contour integral in multivariable Gimel-function, we obtain the desired result (2.1).

3. Solution of problem

The formal solution (of our problem) to be establish is :

Theorem 2.

$$U(x,t) = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} \sum_{p,q,l=0}^{\infty} \sum_{m=0}^{p} B' \frac{[(e_M)_q]y^q}{[(f_N)_q]q!} R_{p-l,l}^{\alpha,\beta}(s) \ y_1^{K_1} \cdots y_u^{K_u} 2^{\rho+\sigma+1+gq+wq+\sum_{i=1}^{u} (f_i+w_i)K_i} ds = 0$$

$$\frac{(-p)_m(\alpha+\beta+p+1)_m}{m!(\alpha+1)_m} \Big\{ N_{l,l}^{\alpha,\beta} \Big\}^{-1} e^{-(p/q)(p+\alpha+\beta+1)t} \phi_l^{\alpha,\beta}(s,x)$$

$$(-\sigma - wq - \sum_{i=1}^{u} w_i K_i : k_1, \cdots, k_r; 1), \mathbf{A}, A$$

$$\vdots$$

$$\mathbb{B}; \mathbf{B}, (-1-m-\sigma - \rho - (g+w)q - \sum_{i=1}^{u} (f_i + w_i)K_i : h_1 + k_1, \cdots, k_r + l_r; 1) : B$$
(3.1)

under the same conditions that (2.1).

Proof

ISSN: 2231 - 5373

http://www.ijmttjournal.org

Page 54

the solution of (1.3) can be written as ([4], page 107; eq.(2.3))

$$U(x,t) = \sum_{l=0}^{\infty} A_l e^{-B_l t} \phi_l^{(\alpha,\beta)}(s,x)$$
(3.2)

where $\phi_l^{(\alpha,\beta)}(s,x)$ is the generalized prolate spheroidal wave function given by (1.2). If t = 0 then by vertue of (1.4), we can be obtained

$$B_l = \frac{(l+p)(l+p+\alpha+\beta+1)}{q} \text{ and } A_l = \frac{G_l}{N_{l,l}^{\alpha,\beta}}$$

where

$$G_{l} = \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \phi_{l}^{(\alpha,\beta)}(s,x) F(x) \mathrm{d}x$$
(3.3)

This because the generalized prolate spheroidal wave function possesses the orthogonality property (1.5) Hence the formal solution of our problem may be expressed as

$$U(x,t) = \sum_{l=0}^{\infty} G_l \left[N_{l,l}^{\alpha,\beta} \right]^{-1} exp \left[-\frac{(l+p)(l+p+\alpha+\beta+1)}{q} t \right] \phi_l^{(\alpha,\beta)}(s,x)$$
(3.4)

Now if we take F(x) as given in (1.5) then by (3.2), we have

$$(1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}{}_{M}F_{N}\left[\begin{array}{c|c} (e_{M}) \\ \cdot \\ (f_{N}) \end{array}\right| y(1-x)^{g}(1+x)^{w} \\ \end{bmatrix} S^{M_{1},\cdots,M_{u}}_{N_{1},\cdots,N_{u}} \left(\begin{array}{c|c} y_{1}(1-x)^{f_{1}}(1+x)^{w_{1}} \\ \cdot \cdot \\ \cdot \\ y_{u}(1-x)^{f_{u}}(1+x)^{w_{u}} \end{array}\right)$$

$$\exists \begin{pmatrix} Z_1(1-x)^{h_1}(1+x)^{k_1} \\ & \ddots \\ & \ddots \\ Z_r(1-x)^{h_r}(1+x)^{k_r} \end{pmatrix} = \sum_{l=0}^{\infty} A_l \phi_l^{(\alpha,\beta)}(s,x)$$
(3.5)

The equation (3.5) is valide since F(x) is continuous and bounded variation in the open interval (-1, 1). Now multiplying both sides of (3.5) by $(1-x)^{\alpha}(1+x)^{\beta}\phi_n^{(\alpha,\beta)}(s,x), \alpha > -1, \beta > -1$ and integrating with respect to x from -1 to 1, change the order of integration and summation (which is permissible) on the right, we obtain

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} \phi_n^{(\alpha,\beta)}(s,x)_M F_N \begin{bmatrix} (\mathbf{e}_M) \\ \cdot \\ (\mathbf{f}_N) \end{bmatrix} \mathbf{y} (1-\mathbf{x})^g (1+x)^w \end{bmatrix}$$

$$S_{N_{1},\cdots,N_{u}}^{M_{1},\cdots,M_{u}} \begin{pmatrix} y_{1}(1-x)^{f_{1}}(1+x)^{w_{1}} \\ & \ddots \\ & & \\ & \ddots \\ & & \\ y_{u}(1-x)^{f_{u}}(1+x)^{w_{u}} \end{pmatrix}$$

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Using the orthogonality property of the generalized prolate spheroidal wave, see Lemma, on the right-hand side and result (2.1) on the left hand side of (3.6), we obtain

$$A_{l} = \sum_{K_{1}=0}^{[N_{1}/M_{1}]} \cdots \sum_{K_{u}=0}^{[N_{u}/M_{u}]} \sum_{p,q=0}^{\infty} \sum_{m_{1},\cdots,m_{v}=0}^{\infty} B' \frac{[(e_{M})_{q}]y^{q}}{[(f_{N})_{q}]q!} R_{p-l,l}^{\alpha,\beta}(s)$$

 $y_1^{K_1} \cdots y_u^{K_u} t_1^{m_1} \cdots t_v^{m_v} 2^{\rho + \sigma + 1 + gq + wq + \sum_{i=1}^u (f_i + w_i)K_i}$

$$\sum_{m=0}^{n+p} \frac{(-n-p)_m (\alpha+\beta+n+p+1)_m}{m! (\alpha+1)_m} \Big\{ N_{l,l}^{\alpha,\beta} \Big\}^{-1} e^{-(p/q)(p+\alpha+\beta+1)t}$$

On substituting the value A_l from (3.7) in (3.5) and use the lemma ([3], page 107, eq.(2)), we arrive at the formal solution of our problem as given in (3.1).

Remarks

We obtain the same formulae concerning the multivariable Aleph- function defined by Ayant [1], the multivariable I-function defined by Prasad [8] and the multivariable H-function defined by Srivastava and Panda [12,13], see Gupta [6], Gupta et al. [7] for more details concerning the multivariable H-function

4. Conclusion

Specializing the parameters of the multivariable Gimel-function, the generalized hypergeometric function and the multivariable polynomials, we can obtain a large number of results of problem of heat conduction involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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