The application of generalized prolate spheroidal wave function, class of multivariable polynomials, and the multivariable Gimel-function in heat conduction

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ABSTRACT
Gupta [5] and Gupta et al. [6] have studied the application of generalized prolate spheroidal wave function, generalized polynomials, and the multivariable H-function in heat conduction. In this paper we employ the generalized prolate spheroidal wave function, generalized hypergeometric function, class of multivariable polynomials and the multivariable Gimel-function in heat conduction in obtaining the formal solution of partial differential equation related to a problem of heat conduction in an anisotropic material. This problem occurs mainly in mechanics of solids, physics and applied mathematics.

Keywords : Modified multivariable Gimel-function, , classes of multivariable polynomials, generalized hypergeometric function, generalized prolate spheroidal wave function, expansion formula

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1. Introduction and preliminaries.

The generalized prolate spheroidal wave functions has been recently defined by Gupta [4] as the solution of the differential equation

$$(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+[\xi(s)-s^2x^2]y=0 \tag{1.1}$$

in the form of an infinite sum :

$$\phi_n^{(\alpha,\beta)}(s,x)=\sum_{p=0}^{\infty}R_{p,n}^{\alpha,\beta}(s)P_{n+p}^{(\alpha,\beta)}(x) \tag{1.2}$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomials, where $s=0$ and $\xi(0)=(n+p)(\alpha+\beta+n+p+1),p\geqslant 0$

In this paper we employ the generalized prolate spheroidal wave function, generalized hypergeometric , a class of multivariable polynomials and the multivariable Gimel-function in applied mathematics, we shall consider the problem of obtaining a solution of a problem of heat conduction. Consider the partial differential equation related to a problem of heat conduction in an anisotropic material which has been obtained by Saxena and Nageria [10] and given below :

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial U}{\partial x}\right]-\frac{pcv}{\lambda}\frac{\partial U}{\partial x}+\frac{Q(x)}{\lambda}=\frac{pc}{\lambda}\frac{\partial U}{\partial t} \tag{1.3}$$

With the law of conductivity $K=\lambda(1-x^2),Q(x)$ is the intensity of a continuous source of heat situated inside this solid . Let initial temperature of the rod be given by

$$U(x,o)=F(x) \tag{1.4}$$

Gupta([5], page 107) took $v=\dfrac{\alpha-\beta}{q},Q(x)=-(\alpha+\beta)\lambda x\dfrac{\partial U}{\partial x}-s^2x^2\lambda U,q=\dfrac{pc}{\lambda}$, in equation (1.3) and obtained the equation (1.3) to the form (1.1).

We consider a generalized transcendental function called Gimel function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{U;0,n_r:V}_{X;p_{i_r},q_{i_r},\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};\mathbf{A}:\ A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B}:\ B \end{array}\right)=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (1.1)$$

with $\omega=\sqrt{-1}$

The following quantities $\mathbb{A},\mathbf{A},A,\mathbb{B},\mathbf{B},B,X,Y,U,V,\ \psi(s_1,\cdots,s_r)$ and $\theta_k(s_k)(k=1,\cdots,r)$ are defined by Ayant [2].

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r)=0(\ |z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\ ),max(\ |z_1|,\cdots,|z_r|\ )\rightarrow 0$

$\aleph(z_1,\cdots,z_r)=0(\ |z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\ ),min(\ |z_1|,\cdots,|z_r|\ )\rightarrow\infty$ where $i=1,\cdots,r$:

$$\alpha_i=\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] \text{ and } \beta_i=\max_{1\leqslant j\leqslant n^{(i)}}Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$$

In this paper we shall assume that

$$F(x)={}_MF_N\left[\begin{array}{c|c} (\mathrm{e}_M) & \\ \cdot & \mathrm{y(1\text{-}x)}^g(1+x)^w \\ (\mathrm{f}_N) & \end{array}\right]S^{M_1,\cdots,M_u}_{N_1,\cdots,N_u}\left(\begin{array}{c} \mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u} \end{array}\right)$$

$$H\left(\begin{array}{c} \mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_r(1-x)^{h_r}(1+x)^{k_r} \end{array}\right) \quad (1.2)$$

The class of multivariable polynomials defined by Srivastava [11], is given in the following manner :

$$S^{M_1,\cdots,M_u}_{N_1,\cdots,N_u}[y_1,\cdots,y_u]\ =\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}\ A[N_1,K_1;\cdots;N_u,K_u]$$

$$A[N_1,K_1;\cdots;N_u,K_u]y_1^{K_1}\cdots y_u^{K_u} \quad (1.3)$$

We shall use the following notation

$$B'=\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_u)_{M_uK_u}}{K_u!}A[N_1,K_1;\cdots;N_u,K_u] \quad (1.4)$$

On suitably specializing the above coefficients , $S^{M_1,\cdots,M_u}_{N_1,\cdots,N_u}[y_1,\cdots,y_u]$ yields some of known polynomials, these include the Jacobi polynomials, Laguerre polynomials, and others polynomials ([14],p. 158-161)

The orthogonality property of the generalized prolate spheroidal wave function (Gupta [3], page 107, eq.(3.1)). We have the following integral.

Lemma

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\phi_n^{(\alpha,\beta)}(c,x)\phi_m^{(\alpha,\beta)}(c,x)\mathrm{d}x = N_{n,m}^{\alpha,\beta}\delta_{m,n} \quad \text{,where} \tag{1.5}$$

$$N_{n,n}^{\alpha,\beta} = 2^{\alpha+\beta+1}\sum_{p=0}^{\infty}\left[R_{p,n}^{\alpha,\beta}\right]^2 \frac{\Gamma(\alpha+p+1)\Gamma(\beta+p+1)}{(1+\alpha+\beta+2n+2p)\Gamma(n+p+1)\Gamma(1+\alpha+\beta+n+p)} \tag{1.6}$$

where $\delta_{m,n}$ is the Kronecker symbol.

2. Main integral

We have the following integral :

Theorem 1

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}\phi_n^{(\alpha,\beta)}(s,x)\,{}_MF_N\left[\begin{array}{c|c}(\mathrm{e}_M) & \\ . & \mathrm{y}(1\text{-x})^{g}(1+x)^{w} \\ (\mathrm{f}_N) & \end{array}\right]$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1}\\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u}\end{array}\right)$$

$$\daleth\left(\begin{array}{c}\mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{Z}_r(1-x)^{h_r}(1+x)^{k_r}\end{array}\right)\mathrm{d}x = 2^{\rho+\sigma+1}\sum_{p,q=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{m=0}^{n+p} B'\,\frac{[(e_M)_q]y^q}{[(f_N)_q]q!}$$

$$R_{p,n}^{\alpha,\beta}(s)\,\frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!(\alpha+1)_m}y_1^{K_1}\cdots y_u^{K_u}2^{gq+wq+\sum_{i=1}^{u}(f_i+w_i)K_i}$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c}\mathrm{Z}_1 2^{h_1+k_1} & \mathbb{A};(1\text{-m-p-gq-}\sum_{i=1}^{u}f_iK_i:h_1,\cdots,h_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_r 2^{h_r+k_r} & \cdot\cdot\cdot\end{array}\right.$$

$$\left.\begin{array}{c};(-\sigma-wq-\sum_{i=1}^{u}w_iK_i:k_1,\cdots,k_r;1):\mathbf{A},A\\ \cdot \\ \cdot \\ \mathbb{B};\mathbf{B},(-1\text{-m-}\sigma-\rho-(g+w)q-\sum_{i=1}^{u}(f_i+w_i)K_i:h_1+k_1,\cdots,k_r+l_r;1):B\end{array}\right) \tag{2.1}$$

Provided that

$min\{g,w,f_i,w_i,a_j,b_j,h_k,k_k\} > 0$ for $i=1,\cdots,v; j=1,\cdots,u; k=1,\cdots,r$

$$Re(\rho+\sum_{i=1}^{v}K_if_i)+\ +\sum_{i=1}^{r}h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0 \quad \text{and}$$

$$Re(\sigma+\sum_{i=1}^{v}K_iw_i)+\sum_{i=1}^{r}k_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$$

$$\left|argZ_i(1-x)^{h_i}(1+x)^{k_i}\right| < \frac{1}{2}, A_i^{(k)}, A_i^{(k)}. \text{ is defined by (1.8)} \quad |y|<1$$

Proof

To prove (2.1), first expressing the generalized prolate spheroidal wave function, a class of multivariable polynomials defined by Srivastava [11] $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[.]$, with the help of (1.2)and (1.9) respectively, expressing the generalized hypergeometric in series and we interchange the order of summations and x-integral (which is permissible under the conditions stated). Expressing the multivariable Gimel-function of r-variables in Mellin-Barnes contour integral with the help of (1.5) and interchange the order of integrations which is justifiable due to absolute convergence of the integrals involved in the process. Now collecting the powers of $(1-x)$ and $(1+x)$ and evaluating the inner x-integral. Interpreting the Mellin-Barnes contour integral in multivariable Gimel-function, we obtain the desired result (2.1).

3. Solution of problem

The formal solution (of our problem) to be establish is :

Theorem 2.

$$U(x,t)=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{p,q,l=0}^{\infty}\sum_{m=0}^{p} B'\frac{[(e_M)_q]y^q}{[(f_N)_q]q!}R_{p-l,l}^{\alpha,\beta}(s)\, y_1^{K_1}\cdots y_u^{K_u}2^{\rho+\sigma+1+gq+wq+\sum_{i=1}^{u}(f_i+w_i)K_i}$$

$$\frac{(-p)_m(\alpha+\beta+p+1)_m}{m!(\alpha+1)_m}\left\{N_{l,l}^{\alpha,\beta}\right\}^{-1}e^{-(p/q)(p+\alpha+\beta+1)t}\phi_l^{\alpha,\beta}(s,x)$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c}\mathrm{Z}_1 2^{h_1+k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{Z}_r 2^{h_r+k_r}\end{array}\middle| \begin{array}{c}\mathbb{A};\ (\text{1-m-p-gq-}\sum_{i=1}^{u}f_iK_i:h_1,\cdots,h_r;1),\\ \cdot\\ \cdot\\ \cdots\end{array}\right.$$

$$\left.\begin{array}{c}(\text{-}\sigma-wq-\sum_{i=1}^{u}w_iK_i:k_1,\cdots,k_r;1),\mathbf{A},A\\ \cdot\\ \cdot\\ \mathbb{B};\mathbf{B},\ (\text{-1-m-}\sigma-\rho-(g+w)q-\sum_{i=1}^{u}(f_i+w_i)K_i:h_1+k_1,\cdots,k_r+l_r;1):B\end{array}\right) \quad (3.1)$$

under the same conditions that (2.1).

Proof

$$\gimel\left(\begin{array}{c}\mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdots\\ \cdots\\ \mathrm{Z}_r(1-x)^{h_r}(1+x)^{k_r}\end{array}\right)\mathrm{d}x=\sum_{l=0}^{\infty}A_l\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\phi_n^{(\alpha,\beta)}(s,x)\phi_l^{(\alpha,\beta)}(s,x)\mathrm{d}x \tag{3.6}$$

Using the orthogonality property of the generalized prolate spheroidal wave, see Lemma, on the right-hand side and result (2.1) on the left hand side of (3.6), we obtain

$$A_l=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{p,q=0}^{\infty}\sum_{m_1,\cdots,m_v=0}^{\infty}B'\frac{[(e_M)_q]y^q}{[(f_N)_q]q!}R_{p-l,l}^{\alpha,\beta}(s)$$

$$y_1^{K_1}\cdots y_u^{K_u}t_1^{m_1}\cdots t_v^{m_v}2^{\rho+\sigma+1+gq+wq+\sum_{i=1}^{u}(f_i+w_i)K_i}$$

$$\sum_{m=0}^{n+p}\frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!(\alpha+1)_m}\left\{N_{l,l}^{\alpha,\beta}\right\}^{-1}e^{-(p/q)(p+\alpha+\beta+1)t}$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c}\mathrm{Z}_1 2^{h_1+k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{Z}_r 2^{h_r+k_r}\end{array}\right|\begin{array}{c}\mathbb{A};\ (1\text{-m-p-gq-}\sum_{i=1}^{u}f_iK_i:h_1,\cdots,h_r;1),\\ \cdot\\ \cdot\\ \cdots\end{array}$$

$$\left.\begin{array}{c}(-\sigma-wq-\sum_{i=1}^{u}w_iK_i:k_1,\cdots,k_r;1),\mathbf{A},A\\ \cdot\\ \cdot\\ \mathbb{B};\mathbf{B},\ (-1\text{-m-}\sigma-\rho-(g+w)q-\sum_{i=1}^{u}(f_i+w_i)K_i:h_1+k_1,\cdots,k_r+l_r;1):B\end{array}\right) \tag{3.7}$$

On substituting the value A_l from (3.7) in (3.5) and use the lemma ([3], page 107, eq.(2)), we arrive at the formal solution of our problem as given in (3.1).

Remarks

We obtain the same formulae concerning the multivariable Aleph- function defined by Ayant [1], the multivariable I-function defined by Prathima et al. [9], the multivariable I-function defined by Prasad [8] and the multivariable H-function defined by Srivastava and Panda [12,13], see Gupta [6], Gupta et al. [7] for more details concerning the multivariable H-function

4. Conclusion

Specializing the parameters of the multivariable Gimel-function, the generalized hypergeometric function and the multivariable polynomials, we can obtain a large number of results of problem of heat conduction involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

References

[1] F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

[2] F.Y .Ayant, Some transformations and idendities form multivariable Gimel-function, International Journal of Matematics and Technology (IJMTT), 59(4) (2018), 248-255.

[3] B.L.J. Braaksma, Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[4] Gupta R.K. Generalized prolate spheroidal wave functions. Proc. India. Acad. Sci 85A (2), (1977), page 104-114.

[5] Gupta V.G. The generalized prolate spheroidal wave function and its applications. Jnanabha. Vol 16, 1986, page 103-112.

[6] Gupta V.G. The application of generalized prolate spheroidal wave function, generalized polynomials and the multivariable H-function in heat conduction equation. Acta. Ciencia. Indica. Math. Vol 28, no 4, 2002, page 553-558.

[7] Gupta V.G., Kumari P. and Jain S. The application of generalized prolate spheroidal wave function, generalized polynomials and the multivariable H-function in linear flow in an anisotropic material. J. Rajasthan. Acad. Phy. Sci. Vol2, no 2, 2003, page 111-120.

[8] Y.N. Prasad, Multivariable I-function , Vijnana Parisha Anusandhan Patrika 29 (1986), 231-237.

[9] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 1-12.

[10] V.P. Saxena and Nageria. Linear flow of heat in an anisotropic finite solid moving in a conducting medium. Jnanabha. Sect A 4, 1974, page 1-6.

[11] H.M. Srivastava, A multilinear generating function for Konhauser set of biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math. 117(1985), 183-191.

[12] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[13] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.

[14] H.M. Srivastava and N.P. Singh, The integration of certains products of the multivariable H-function with a general class of polynomials, Rend. Circ. Mat. Palermo. 32(2)(1983), 157-187.