On Certain Generalized Families of Unified Elliptic- Type Integrals Pertaining to Euler Integrals, Generating Function and Multivariable A-Function

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ABSTRACT
Elliptic-type integrals have their importance and potential in certain problems in radiation physics and nuclear technology. A number of earlier works on the subject contains several interesting unifications and generalizations of some significant families of elliptic-type integrals. The present paper is intended to obtain certain new theorems on generating functions.The results obtained in this paper are of manifold generality and basic in nature. Beside deriving various known and new elliptic-type integrals and their generalizations these theorems can be used to evaluate various Euler-type integrals involving a number of generating functions.

Keywords Elliptic-type integrals, Euler-type integrals, generalized Lauricella function, multivariable A-function, generating functions.

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1. Introduction

The following integrals

$$\Omega_j(k) = \int_0^\pi (1-k^2\cos\theta)^{-j-\frac{1}{2}}\mathrm{d}\theta \tag{1.1}$$

where $j = 0, 1, 0, \cdots$ and $0 \leq k < 1$ was studied by Epstein-Hubbell [6], for the first time. Due to its occurrence in a number of physical problems [2, 3, 7, 9, 11, 13, 16, 17, 19, 28, 35], in the form of single and multiple integrals, several authors notably Kalla [12, 13] and Kalla and Al-Saqabi [20], Kalla et al. [16], Salman [26], Saxena et al. [27] and Srivastava and Bromberg [31], have investigated various interesting unifications (and generalizations) of the elliptic-type integrals (1). Some of the important generalizations of elliptic-type integral (1) are as follows:

Kalla [12, 13] introduced the generalization of the form:

$$R_\mu(k,\alpha,\gamma) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\gamma-2\alpha-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\mathrm{d}\theta \tag{1.2}$$

where $0 \leq k < 1, Re(\gamma) > Re(\alpha) > 0, Re(\mu) > -\frac{1}{2}$

Result for this generalization are also derived by Glasser and Kalla [10]. Al-Saqabi [20] defined and studied the generalization given by the integral

$$B_\mu(k,m,\upsilon) = \int_0^\pi \frac{\cos^{2m}\theta\sin^{2\upsilon}\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\mathrm{d}\theta \tag{1.3}$$

where $0 \leq k < 1; m \in \mathbb{N}_0, \mu \in \mathbb{C}, re(\mu) > -\frac{1}{2}.$

The integral

$$\Lambda_\upsilon(\alpha,k) = \int_0^\pi \frac{exp[\alpha sin^2(\theta/2)]}{(1-k^2\cos\theta)^{\upsilon+\frac{1}{2}}}\mathrm{d}\theta \tag{1.4}$$

where $0 \leq k < 1, \alpha, \upsilon \in \mathbb{R}$; presents another generalization of (1), given by Siddiqui [30]. Srivastava and Siddiqi [35] have given an interesting unification and extension of the families of elliptic integrals in the following form :

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho;k)=\int_0^{\pi}\frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\left[1-\rho\sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda}\mathrm{d}\theta \tag{1.5}$$

where $0\leq k<1, Re(\alpha)>0, Re(\beta)>0, \lambda,\mu\in\mathbb{C}, |\rho|<1$.

Kalla and Tuan [17] generalized the above equation by means of the following integral and also obtained its asymptotic expansions

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho;k)=\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)(1-k^2\cos\theta)^{-\mu-\frac{1}{2}}$$

$$\left(1-\rho\sin^2\left(\frac{\theta}{2}\right)\right)^{-\lambda}\left(1+\delta\cos^2\left(\frac{\theta}{2}\right)\right)\mathrm{d}\theta \tag{1.6}$$

where $0\leq k<1, Re(\alpha)>0, Re(\beta)>0, \lambda,\mu,\gamma\in\mathbb{C}, |\rho|<1.|\delta|<1$ or ρ (or δ) $\in\mathbb{C}$ whenever $\lambda=m$ or $\gamma=-m, m\in\mathbb{N}_0$, respectively.

Al-Zamel et al. [21] discussed a generalized family of elliptic- type integrals in the form :

$$Z^{(\alpha,\beta)}_{(\gamma)}(k)=Z^{(\alpha,\beta)}_{(\gamma_1,\cdots,\gamma_n)}(k)=\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n}(1-k_j^2\cos\theta)^{-\gamma_j}\mathrm{d}\theta=$$

$$B(\alpha,\beta)\prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j}F_D^{(n)}\left(\beta;\gamma_1,\cdots,n;\alpha+\beta;\frac{2k_1^2}{k_1^2-1},\cdots,\frac{2k_n^2}{k_n^2-1}\right) \tag{1.7}$$

where $Re(\alpha)>0, Re(\beta)>0, |k_j|<1; \gamma_j\in\mathbb{C}(j=1,\cdots,n), F_D^{(n)}(.)$ is the Lauricella hypergeometric function of n variables [32, p. 163].
Saxena and Kalla [28] have studied a family of elliptic-type integrals of the form

$$\Omega_{(\sigma_1,\cdots,\sigma_{n-2};\delta,\mu)}(\rho_1,\cdots,\rho_{n-2},\delta;k)=\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n-2}\left(1-\rho_j^2\sin\left(\frac{\theta}{2}\right)\right)^{-\sigma_j}$$

$$\left(1+\delta\cos^2\left(\frac{\theta}{2}\right)\right)^{-\gamma}(1-k^2\cos\theta)^{-\mu-\frac{1}{2}}\mathrm{d}\theta \tag{1.8}$$

where $0\leq k<1, Re(\alpha), Re(\beta)>0, \sigma_j(j=1,\cdots,n-2), \gamma,\mu\in\mathbb{C}; max\left\{|\rho_j|,\left|\frac{\delta}{1+\delta}\right|,\left|\frac{2k^2}{k^2-1}\right|\right\}<1$

In the recent paper, Saxena and Pathan [29] investigated an extension of the above equation in the fom

$$\Omega_{(\sigma_1,\cdots,\sigma_m;\gamma,\tau_1,\cdots,\tau_n)}(\rho_1,\cdots,\rho_m,\delta;\lambda_1,\cdots,\lambda_n)=\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{i=1}^{m}\left(1-\rho_j^2\sin\left(\frac{\theta}{2}\right)\right)^{-\sigma_j}$$

$$\left(1+\delta\cos^2\left(\frac{\theta}{2}\right)\right)^{-\gamma}\prod_{j=1}^{n}(1-\lambda_j^2\cos\theta)^{-\tau_j}\mathrm{d}\theta \tag{1.9}$$

where $Re(\alpha), Re(\beta)>0,\in\mathbb{C}; |\lambda_j|<1; \sigma_i,\gamma,\tau_j\in\mathbb{C}$,max$\left\{|\rho_i|,\left|\frac{\delta}{1+\delta}\right|,\left|\frac{2\lambda_j^2}{\lambda_j^2-1}\right|\right\}<1, (i=0,\cdots,m; j=1,\cdots,n)$

In a recent paper [5], the authors have proposed and investiged a new family of unified and generalized elliptic type integrals :

$$\mathbf{\Omega}^{(\alpha,\beta)}_{(\lambda_i,\tau_j)}((\rho_i),(\delta_i);k_j)=\Omega^{(\alpha,\beta)}_{\lambda_1,\cdots,\lambda_N,\tau_1,\cdots,\tau_M}(\rho_1,\cdots,\rho_N,\delta_1,\cdots,\delta_N;k_1,\cdots,K_M)=$$

$$\int_0^{\pi} \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{J=1}^{M}\left(1-k_j^2\cos\theta\right)^{-\tau_j}\prod_{i=1}^{N}\left[1+\rho_i\sin^2\left(\frac{\theta}{2}\right)+\delta_i\cos^2\left(\frac{\theta}{2}\right)\right]^{-\lambda_i}\mathrm{d}\theta \tag{1.10}$$

where

$$Re(\alpha), Re(\beta) > 0, \in \mathbb{C}; |\lambda_j| < 1; \lambda_i, \tau_j \in \mathbb{C}, \max\left\{|\rho_i|, |\delta_i|, \left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|, \left|\frac{2k_j^2}{k_j^2-1}\right|\right\} < 1, (i = 0, \cdots, m; j = 1, \cdots, n)$$

which include most of the known generalized and unified families of elliptic type integrals (includind the integrals (1.1) to (1.9).

Upon a closer examination of the above equation (1.10), it can be seen that the family of elliptic integral $\mathbf{\Omega}_{(\lambda_i,\tau_j)}^{(\alpha,\beta)}((\rho_i),(\delta_i);k_j)$ can be put in to the following Euler integral :

$$\mathbf{\Omega}_{(\lambda_i,\tau_j)}^{(\alpha,\beta)}((\rho_i),(\delta_i);k_j) = \Omega_{\lambda_1,\cdots,\lambda_N,\tau_1,\cdots,\tau_M}^{(\alpha,\beta)}(\rho_1,\cdots,\rho_N,\delta_1,\cdots,\delta_N;k_1,\cdots,K_M) = \prod_{j=1}^{M}(1-k_j^2)^{-\tau_j}\prod_{i=1}^{N}(1+\delta_i)^{-\lambda_i}$$

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}\prod_{j=1}^{M}\left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}\prod_{i=1}^{N}\left[1-\frac{(\delta_i-\rho_i)\omega}{1+\delta_i}\right]^{-\lambda_i}\mathrm{d}\omega \tag{1.11}$$

A two variables generating function $F(x,t)$ possess a formal power series representation, not necessarily convergent for $t = 0$ power series representation in t , can be written in the following form

$$F(x,t) = \sum_{n=0}^{\infty} C_n f_n(x) t^n \tag{1.12}$$

where each member of the generalized set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of x and t.

Now, we consider the multivariable A-function defined by Gautam et al. [8]. It's an extension of multivariable H-function, see [22,33,34].

$$A(z_1,\cdots,z_r) = A_{p,q:p_1,q_1;\cdots;p_r,q_r}^{m,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;A_j^{(1)},\cdots,A_j^{(r)})_{1,p} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;B_j^{(1)},\cdots,B_j^{(r)})_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},D_j^{(r)})_{1,q_r} \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1'}\cdots\int_{L_r'}\phi'(t_1,\cdots,t_r)\prod_{i=1}^{r}\theta_i'(t_i)Z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{1.13}$$

where $\phi'(t_1,\cdots,t_r), \theta_i'(t_i), i = 1,\cdots,r$ are given by :

$$\phi'(t_1,\cdots,t_r) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\sum_{i=1}^{s}B_j^{(i)}t_i)\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{s}A_j^{(i)}t_j)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{s}A_j^{(i)}t_j)\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{s}B_j^{(i)}t_j\right)} \tag{1.14}$$

$$\theta_i'(t_i) = \frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+C_j^{(i)}t_i)\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-D_j^{(i)}t_i)}{\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-C_j^{(i)}t_i)\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+D_j^{(i)}t_i)} \tag{1.15}$$

Here $M,N,P,Q,m_i,n_i,p_i,c_i \in \mathbb{N}^*; i=1,\cdots,r$; $a_j,b_j,c_j^{(i)},d_j^{(i)},A_j^{(i)},B_j^{(i)},C_j^{(i)},D_j^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if :

$$|arg(\Omega_i')Z_k| < \frac{1}{2}\eta_k'\pi, \xi'^* = 0, \eta_i' > 0 \tag{1.16}$$

$$\Omega_i' = \prod_{j=1}^{p}\{A_j^{(i)}\}^{A_j^{(i)}}\prod_{j=1}^{q}\{B_j^{(i)}\}^{-B_j^{(i)}}\prod_{j=1}^{q_i}\{D_j^{(i)}\}^{D_j^{(i)}}\prod_{j=1}^{p_i}\{C_j^{(i)}\}^{-C_j^{(i)}}; i=1,\cdots,r \tag{1.17}$$

$$\xi_i'^* = Im(\sum_{j=1}^{p}A_j^{(i)} - \sum_{j=1}^{q}B_j^{(i)} + \sum_{j=1}^{q_i}D_j^{(i)} - \sum_{j=1}^{p_i}C_j^{(i)}); i=1,\cdots,r \tag{1.18}$$

$$\eta_i' = Re\left(\sum_{j=1}^{n}A_j^{(i)} - \sum_{j=n+1}^{p}A_j^{(i)} + \sum_{j=1}^{m}B_j^{(i)} - \sum_{j=m+1}^{q}B_j^{(i)} + \sum_{j=1}^{m_i}D_j^{(i)} - \sum_{j=m_i+1}^{q_i}D_j^{(i)} + \sum_{j=1}^{n_i}C_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\right) \tag{1.19}$$

$i=1,\cdots,r$

In this paper, we shall note.

$$X = m_1,n_1,\cdots,m_r,n_r; Y = p_1,q_1,\cdots,p_r,q_r \tag{1.20}$$

$$\mathbf{A} = (a_j; A_j^{(1)},\cdots,A_j^{(r)})_{1,p} : \mathrm{C} = (\mathrm{c}_j^{(1)},C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \tag{1.21}$$

$$\mathbf{B} = (b_j; B_j^{(1)},\cdots,B_j^{(r)})_{1,q} : \ \mathrm{D} = (\mathrm{d}_j^{(1)},D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},D_j^{(r)})_{1,q_r} \tag{1.22}$$

2. Main result

In this section we see two general formulae

Theorem 1.

Let the generating function $F(x,t)$ and multivariable A-function defined respectively by (1.12) and (1.13). Then

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1}A((\omega z_1)^{\zeta_1},\cdots,(\omega z_r)^{\zeta_r})F(x,t\omega^{\eta}(1-\omega)^{\mu})\mathrm{d}\omega = \Gamma(\gamma-\alpha)\sum_{n=0}^{\infty}(\gamma-\alpha)_{\mu n}C_n f_n(x)t^n$$

$$A_{p+1,q+1:Y}^{m,n+1:X}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\alpha-\eta n;\zeta_1,\cdots,\zeta_r),\mathbf{A}:C \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \mathbf{B},(1\text{-}\gamma-\eta n-\mu n;\zeta_1,\cdots,\zeta_r):D \end{array}\right) \tag{2.1}$$

provided that

$\min[Re(\alpha),Re(\gamma-\alpha)]>0, Re(\eta)>0, Re(\mu)>0, \left|arg(\Omega_i')(\omega z_i)^{\zeta_i}\right| < \frac{1}{2}\eta_i\pi, \xi^*=0, \eta_i>0$

Ω_i, η_i and ξ^* are defined respectively by (1,17), (1.19) and (1.18).

Proof

Expressing $F(x,t)$ by the power serie form with the help of (1.12) in the integral (2.1) and the multivariable A-functions in terms of Mellin-Barnes type integrals contour with the help of (1.13), interchanging the order of summations and integrations which is justified under the conditions mentioned above. Evaluating the inner integral with the help of the beta-function and finally, reinterpreting the multiple Mellin-Barnes integrals contour in terms of multivariable A-function, we obtain the desired result after algebraic manipulations.

Let

$$\mathbf{A}_1 = (a_j; A_j^{(1)}, \cdots, A_j^{(r)}, \underbrace{0, \cdots, 0}_{M+N})_{1,p}; X_1 = m_1, n_1, \cdots, m_r, n_r, \underbrace{0, \cdots, 0}_{M+N}:$$

$$\mathbf{C}_1 = (c_j^{(1)}, C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, C_j^{(r)})_{1,p_r}; -; - \tag{2.3}$$

$$\mathbf{B}_1 = (b_j; B_j^{(1)}, \cdots, B_j^{(r)}, \underbrace{0, \cdots, 0}_{M+N})_{1,q}: \; Y_1 = p_1, q_1, \cdots, p_r, q_r, \underbrace{0, \cdots, 0}_{M+N}:$$

$$\mathbf{D}_1 = (d_j^{(1)}, D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(r)}, D_j^{(r)})_{1,q_r}; -; - \tag{2.4}$$

Theorem 2

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} A((\omega z_1)^{\zeta_1}, \cdots, (\omega z_r)^{\zeta_r}) F(x, t\omega^{\eta}(1-\omega)^{\mu}) \prod_{j=1}^{M}\left[1 - \frac{2\omega k_j^2}{k_j^2 - 1}\right]^{-\tau_j} \prod_{i=1}^{N}\left[1 - \frac{(\delta_i - \rho_i)\omega}{1+\delta_i}\right]^{-\lambda_i} \mathrm{d}\omega$$

$$= \Gamma(\alpha) \prod_{j=1}^{M}\left[1 - \frac{2k_j^2}{k_j^2 - 1}\right]^{\tau_i} \prod_{i=1}^{N}\left[1 - \frac{\delta_i - \rho_i}{1+\delta_i}\right]^{-\lambda_j} \sum_{n=0}^{\infty} C_n f_n(x) t^n (\alpha)_{\mu\eta}$$

$$A_{p+1,q+1:Y_1}^{m,n+1:X_1}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\beta - \eta n; \zeta_1, \cdots, \zeta_r, \underbrace{1, \cdots, 1}_{M+N}), \mathbf{A}_1 : \mathbf{C}_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \cdot \\ \frac{2k_1^2}{k_1^2-1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{2k_M^2}{k_M^2-1} & \cdot \\ \frac{\delta_1-\rho_1}{1+\delta_1} & \cdot \\ \cdot & \\ \cdot & \\ \frac{\delta_N-\rho_N}{1+\delta_N} & \mathbf{B}_1, (1-\beta-\alpha-\eta n-\mu n, \underbrace{1, \cdots, 1}_{M+N}) : \mathbf{D}_1 \end{array}\right) \tag{2.5}$$

We obtain the A-function of $(r+M+N)$-variables

Provided

$\min[Re(\alpha), Re(\beta)] > 0, Re(\eta) > 0, Re(\mu) > 0, \delta_i, \rho_i, \lambda_i, \tau_j \in \mathbb{C}, |k_j| < 1, i = 1, \cdots, N, j = 1, \cdots, M)$ and

$\max\left\{|\rho_i|, |\delta_i|, \left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|, \left|\frac{2k_j^2}{k_j^2-1}\right|\right\} < 1.$

The proof of (2.2) is similar that (2.1).

3. Applications.

In view of the importance of the theorems in the above section, we mention some interesting applications about the multivariable A-function.

1) We consider the generating function [32]

$$F(x,t)=(1-x)^{-\sigma}\sum_{n=0}^{\infty}(\sigma)_n\frac{x^nt^n}{n!} \tag{3.1}$$

and by the use the theorem 1, under the conditions mentionned in (2.1), we get the following interesting results :

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1}A((\omega z_1)^{\zeta_1},\cdots,(\omega z_r)^{\zeta_r})[1-xt\omega^{\eta}(1-\omega)^{\mu}]^{-\sigma}\mathrm{d}\omega=\Gamma(\gamma-\alpha)\sum_{n=0}^{\infty}(\gamma-\alpha)_{\mu n}(\sigma)_n\frac{x^nt^n}{n!}$$

$$A^{m,n+1:X}_{p+1,q+1:Y}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\alpha-\eta n;\zeta_1,\cdots,\zeta_r),\mathbf{A}:C \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \mathbf{B},(1\text{-}\gamma-\eta n-\mu n;\zeta_1,\cdots,\zeta_r):D \end{array}\right) \tag{3.2}$$

Let $\omega=\cos^2(\theta/2)$ and using the relation $\cos\theta=2\cos^2(\theta/2)-1$, the equation (3.2) gives the following generalization of the elliptic integral.

$$\int_0^{\pi}\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right)A\left(\left(z_1\cos^2\left(\frac{\theta}{2}\right)\right)^{\zeta_1},\cdots,\left(z_r\cos^2\left(\frac{\theta}{2}\right)\right)^{\zeta_r}\right)$$

$$\left[1-xt\cos^{2\eta}\left(\frac{\theta}{2}\right)\sin^{2\mu}\left(\frac{\theta}{2}\right)\right]^{-\sigma}\mathrm{d}\theta=\Gamma(\gamma-\alpha)\sum_{n=0}^{\infty}(\gamma-\alpha)_{\mu n}(\sigma)_n\frac{x^nt^n}{n!}$$

$$A^{m,n+1:X_1}_{p+1,q+1:Y_1}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\alpha-\eta n;\zeta_1,\cdots,\zeta_r),\mathbf{A}:C \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \mathbf{B},(1\text{-}\gamma-\eta n-\mu n;\zeta_1,\cdots,\zeta_r):D \end{array}\right) \tag{3.3}$$

Let $\sigma\to 0,\omega=\sin(\theta/2)$ and using the relation $\cos\theta=1-2\sin^2(\theta/2)$ in (3.2) we obtain the following generalized family of elliptic integrals :

$$\int_0^{\pi}\sin^{2\alpha-1}\left(\frac{\theta}{2}\right)\cos^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right)A\left(\left(z_1\sin^2\left(\frac{\theta}{2}\right)\right)^{\zeta_1},\cdots,\left(z_r\sin^2\left(\frac{\theta}{2}\right)\right)^{\zeta_r}\right)$$

$$\mathrm{d}\theta=\Gamma(\gamma-\alpha)\sum_{n=0}^{\infty}(\gamma-\alpha)_{\mu n}(\sigma)_n\frac{x^nt^n}{n!}A^{m,n+1:X}_{p+1,q+1:Y}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\alpha;\zeta_1,\cdots,\zeta_r),\mathbf{A}:C \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \mathbf{B},(1\text{-}\gamma;\zeta_1,\cdots,\zeta_r):D \end{array}\right) \tag{3.4}$$

No, we consider the theorem 2 and the generating function defined by (3.1), we have (under the same validity conditions that (2.2)),

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}A((\omega z_1)^{\zeta_1},\cdots,(\omega z_r)^{\zeta_r})[1-xt\omega^{\eta}(1-\sigma)^{\mu}]^{-\sigma}$$

$$\prod_{j=1}^{M}\left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}\prod_{i=1}^{N}\left[1-\frac{(\delta_i-\rho_i)\omega}{1+\delta_i}\right]^{-\lambda_i}\mathrm{d}\omega=\Gamma(\alpha)\prod_{j=1}^{M}\left[1-\frac{2k_j^2}{k_j^2-1}\right]^{\tau_i}\prod_{i=1}^{N}\left[1-\frac{\delta_i-\rho_i}{1+\delta_i}\right]^{-\lambda_j}$$

$$\sum_{n=0}^{\infty}(\alpha)_{\mu\eta}(\sigma)_n\frac{(xt)^n}{n!}A^{m,n+1:X_1}_{p+1,q+1:Y_1}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\beta-\eta n;\zeta_1,\cdots,\zeta_r,\underbrace{1,\cdots,1}_{M+N}),\mathbf{A}_1:\mathbf{C}_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \cdot \\ \frac{2k_1^2}{k_1^2-1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{2k_M^2}{k_M^2-1} & \cdot \\ \frac{\delta_1-\rho_1}{1+\delta_1} & \\ \cdot & \\ \cdot & \\ \frac{\delta_N-\rho_N}{1+\delta_N} & \mathbf{B}_1,(1-\beta-\alpha-\eta n-\mu n,\underbrace{1,\cdots,1}_{M+N}):\mathbf{D}_1 \end{array}\right) \tag{3.5}$$

2) Consider the generating relation [32,36]

$$F(x,t)=(1-X_1t)^{-\alpha_1}(1-X_2t)^{-\alpha_2}=\sum_{n=0}^{\infty}g_n^{\alpha_1,\alpha_2}(X_1,X_2)t^n \tag{3.6}$$

where

$$g_n^{\alpha_1,\alpha_2}(x,y)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_{n-r}}{r!(n-r)!}x^ry^{n-r} \tag{3.7}$$

By application of theorem 1, we have (under the same validity conditions that (2.1)),

$$\int_0^1\omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1}A((\omega z_1)^{\zeta_1},\cdots,(\omega z_r)^{\zeta_r})\prod_{j=1}^{2}[1-X_jtw^{\eta}(1-\omega)^{\mu}]^{-\sigma_j}\,\mathrm{d}\omega=$$

$$\Gamma(\gamma-\alpha)\sum_{n=0}^{\infty}(\gamma-\alpha)_{\mu n}\,g_n^{\sigma_1,\sigma_2}(X_1,X_2)t^nA^{m,n+1:X}_{p+1,q+1:Y}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\alpha-\eta n;\zeta_1,\cdots,\zeta_r),\mathbf{A}:C \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \mathbf{B},(1\text{-}\gamma-\eta n-\mu n;\zeta_1,\cdots,\zeta_r):D \end{array}\right) \tag{3.8}$$

Now, we use the theorem 2, we have (under the same validity conditions that (2.2)),

$$\int_0^1\omega^{\beta-1}(1-\omega)^{\alpha-1}A((\omega z_1)^{\zeta_1},\cdots,(\omega z_r)^{\zeta_r})\prod_{j=1}^{2}[1-X_jtw^{\eta}(1-\omega)^{\mu}]^{-\sigma_j}$$

$$\prod_{j=1}^{M}\left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}\prod_{i=1}^{N}\left[1-\frac{(\delta_i-\rho_i)\omega}{1+\delta_i}\right]^{-\lambda_i}\mathrm{d}\omega=\Gamma(\alpha)\prod_{j=1}^{M}\left[1-\frac{2k_j^2}{k_j^2-1}\right]^{\tau_i}\prod_{i=1}^{N}\left[1-\frac{\delta_i-\rho_i}{1+\delta_i}\right]^{-\lambda_j}$$

$$\sum_{n=0}^{\infty}g_n^{\sigma_1\sigma_2}(X_1,X_2)t^n(\alpha)_{\mu n}A^{m,n+1:X_1}_{p+1,q+1:Y_1}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\beta-\eta n;\zeta_1,\cdots,\zeta_r,\underbrace{1,\cdots,1}_{M+N}),\mathbf{A}_1:\mathbf{C}_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \cdot \\ \frac{2k_1^2}{k_1^2-1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{2k_M^2}{k_M^2-1} & \cdot \\ \frac{\delta_1-\rho_1}{1+\delta_1} & \\ \cdot & \\ \cdot & \\ \frac{\delta_N-\rho_N}{1+\delta_N} & \mathbf{B}_1,(1-\beta-\alpha-\eta n-\mu n,\underbrace{1,\cdots,1}_{M+N}):\mathbf{D}_1 \end{array}\right) \tag{3.9}$$

3) Consider the generating function

$$F(x,t) = e^{-xt} = \sum_{n=0}^{\infty} \frac{(-xt)^n}{n!} \tag{3.10}$$

and use the theorem 1. Under the same validity conditions that (2.1), we get

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1} A((\omega z_1)^{\zeta_1},\cdots,(\omega z_r)^{\zeta_r}) e^{-xt}[\omega^{\eta}(1-\omega)^{\mu}]\mathrm{d}\omega = \Gamma(\gamma-\alpha)\sum_{n=0}^{\infty}(\gamma-\alpha)_{\mu n}$$

$$\frac{(-xt)^n}{n!} A_{p+1,q+1:Y}^{m,n+1:X}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\alpha-\eta n;\zeta_1,\cdots,\zeta_r),\mathbf{A}:C \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \mathbf{B},(1\text{-}\gamma-\eta n-\mu n;\zeta_1,\cdots,\zeta_r):D \end{array}\right) \tag{3.11}$$

Now, we use the theorem 2 (under the same validity conditions that (2.2)), we get

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} A((\omega z_1)^{\zeta_1},\cdots,(\omega z_r)^{\zeta_r}) e^{-xt}[\omega^{\eta}(1-\omega)^{\mu}]\prod_{j=1}^{M}\left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}\prod_{i=1}^{N}\left[1-\frac{(\delta_i-\rho_i)\omega}{1+\delta_i}\right]^{-\lambda_i}\mathrm{d}\omega$$

$$\sum_{n=0}^{\infty}\frac{(\alpha)_{\mu\eta}(-xt)^n}{n!} A_{p+1,q+1:Y_1}^{m,n+1:X_1}\left(\begin{array}{c|c} \mathrm{z}_1^{\zeta_1} & (1\text{-}\beta-\eta n;\zeta_1,\cdots,\zeta_r,\underbrace{1,\cdots,1}_{M+N}),\mathbf{A}_1:\mathbf{C}_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r^{\zeta_r} & \cdot \\ \frac{2k_1^2}{k_1^2-1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{2k_M^2}{k_M^2-1} & \cdot \\ \frac{\delta_1-\rho_1}{1+\delta_1} & \cdot \\ \cdot & \mathbf{B}_1,(1-\beta-\alpha-\eta n-\mu n,\underbrace{1,\cdots,1}_{M+N}):\mathbf{D}_1 \\ \cdot & \\ \frac{\delta_N-\rho_N}{1+\delta_N} & \end{array}\right) \tag{3.12}$$

Remarks

By using the same method, We obtain the similar integrals about the multivariable I-function defined by Prasad [23], multivariable I-function defined by Prathima et al. [25], the modified multivariable H-function defined by Prasad and Singh [24], the multivariable Aleph-function defined by Ayant [1], the multivariable H-function defined by Srivastava and Panda [33,34], see Chaurasia and Megwal [4] concerning the multivariable H-function.

4. Conclusion.

We have established two theorems concerning the product of generating function and multivariable A-function, we have given several applications. Firstly, the elliptic integrals presented in this document are quite nature. Therefore, on specializing the parameters of these integrals involving here, we obtain various other results as its special cases. Secondly, by specializing the parameters of generating function, we can to obtain a large number of functions. Thirdly, by specializing the various parameters as well as variables in the multivariable A-function , we obtain a large number of formulae involving remarkably wide variety of useful functions or product of such functions) which are expressible in terms of E, F, G, H of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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