

A Study of Unified Fractional Integral Operator Involving Generalized Laguerre Function and (τ, β) -Generalized Associated Legendre Function

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Abstract

In this paper, we study a pair of a general class of fractional integral operators whose kernel involves the product of a generalized laguerre function, and (τ, β) -generalized associated Legendre function. We have given four images about the multivariable Gimel-function, the $\bar{H}$ -function and the Aleph function. At the end we shall see two applications.

Keywords : Unified fractional integral operators, generalized laguerre function, and (τ, β) -generalized associated Legendre function. multivariable Gimel function, Aleph-function, H -function.

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1. Introduction.

The paper deals with special function, generalized Laguerre function [17,p.532,eq(2.1)] defined for $x > 0, \rho \in \mathbb{R}, v \in \mathbb{C}$ and $\alpha > 1$ by serie and in terms of H-function representation of of generalized Laguerre function which is used in this paper is in the following form

$$\gamma L_v^\alpha(z) = \frac{\sin \pi v}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma n - v)}{\Gamma(\alpha + 1 + \gamma n)} \frac{z^n}{n!} = \frac{1}{2\pi\omega} \int_L \frac{\Gamma(\gamma s - v)}{\Gamma(1 + \alpha + \gamma s)} ds =$$

$$\frac{\Gamma(\alpha + v + 1)}{\Gamma(v)\Gamma(1 - v)} H_{1,2}^{1,1} \left[-z \left| \begin{matrix} (1+v; \gamma) \\ (0;1), (-\alpha, \gamma) \end{matrix} \right. \right] \quad (1.1)$$

provided : $Re(\alpha) \geq -1, Re(\alpha + v) > -\frac{1}{\alpha}, v$ is not integer, $(\gamma n - v)$ and $(\alpha + 1 + \gamma n)$ are such that $\Gamma(\gamma n - v), \Gamma(\alpha + 1 + \gamma n)$ are finite when $n = 0, 1, 2 \dots$.

The contour representation of (τ, β) -generalized associated Legendre function of first kind which is used in this paper is in the following form

$$\tau, \beta P_k^{m,n}(z) = \frac{(z+1)^{\frac{n}{2}}(z-1)^{-\frac{m}{2}}}{\Gamma(k - \frac{m-n}{2} + 1) \Gamma(-k - \frac{m-n}{2})} \int_C \frac{\Gamma(k - \frac{m-n}{2} + 1 + s) \Gamma(-k - \frac{m-n}{2} + \tau s) \Gamma(-s)}{\Gamma(1 - m + \beta s)} \left(\frac{z-1}{2}\right)^s ds =$$

$$\frac{(z+1)^{\frac{n}{2}}(z-1)^{-\frac{m}{2}}}{\Gamma(k - \frac{m-n}{2} + 1) \Gamma(-k - \frac{m-n}{2})} H_{2,2}^{1,2} \left[\frac{z-1}{2} \left| \begin{matrix} (-k + \frac{m-n}{2}, 1), (k + \frac{m-n}{2} + 1, \tau) \\ (0,1), (m, \beta) \end{matrix} \right. \right] \quad (1.2)$$

We consider a generalized transcendental function called Gimel function of several complex variables.

$$\mathfrak{J}(z_1, \dots, z_r) = \mathfrak{J}_{X;p_{i_r}, q_{i_r}, \tau_{i_r}; R_r; Y}^{U; 0, n_r; V} \left(\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} \mathbb{A}; \mathbf{A}: \mathbf{A} \\ \vdots \\ \mathbb{B}; \mathbf{B}: \mathbf{B} \end{matrix} \right. \right) = \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \theta_k(s_k) z_k^{s_k} ds_1 \dots ds_r \quad (1.3)$$

with $\omega = \sqrt{-1}$

The following quantities $\mathbb{A}, \mathbf{A}, A, \mathbb{B}, \mathbf{B}, B, X, Y, U, V, \psi(s_1, \dots, s_r)$ and $\theta_k(s_k) (k = 1, \dots, r)$ are defined by Ayant [2].

The Aleph- function , introduced by Südländ et al. [15,16], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph_{P_i, Q_i, c_i; r'}^{M, N} \left(z \left| \begin{matrix} (a_j, A_j)_{1, N}, [c_i(a_{ji}, A_{ji})]_{N+1, P_i; r'} \\ (b_j, B_j)_{1, M}, [c_i(b_{ji}, B_{ji})]_{M+1, Q_i; r'} \end{matrix} \right. \right) = \frac{1}{2\pi\omega} \int_L \Omega_{P_i, Q_i, c_i; r'}^{M, N}(s) z^s ds \quad (1.4)$$

for all z different to 0 and

$$\Omega_{P_i, Q_i, c_i; r'}^{M, N}(s) = \frac{\prod_{j=1}^M \Gamma(b_j - B_j s) \prod_{j=1}^N \Gamma(1 - a_j + A_j s)}{\sum_{i=1}^{r'} c_i \prod_{j=N+1}^{P_i} \Gamma(a_{ji} - A_{ji} s) \prod_{j=M+1}^{Q_i} \Gamma(1 - b_{ji} + B_{ji} s)} \quad (1.5)$$

$$\text{with } |\arg z| < \frac{1}{2}\pi\Omega \quad \text{where } \Omega = \sum_{j=1}^M B_j + \sum_{j=1}^N A_j - c_i \left(\sum_{j=N+1}^{P_i} A_{ji} + \sum_{j=M+1}^{Q_i} B_{ji} \right) > 0, i = 1, \dots, r'$$

For convergence conditions and other details of Aleph-function , see Südländ et al [15,16].

Inayat Hussain[6] has studied the $\bar{H}$ -function which is a new generalization of familial Fox-function [4,11]. The $\bar{H}$ -function contains polylogarithm, Riemann Zeta function. $\bar{H}$ -function will be represented as follows

$$\bar{H}_{p,q}^{m,n}(z) = \bar{H}_{p,q}^{m,n} \left(z \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1, n}, (a_j, \alpha_j)_{n+1, p} \\ (b_j, \beta_j)_{1, m}, (b_j, \beta_j; B_j)_{m+1, q} \end{matrix} \right. \right) = \frac{1}{2\pi\omega} \int_L \bar{\phi}(s) z^s ds \quad (1.6)$$

where

$$\bar{\phi}(s) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^n \Gamma^{A_j}(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^q \Gamma^{B_j}(1 - b_j + \beta_j s) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j s)} \quad (1.7)$$

The following sufficient condition for the absolute convergence of the defining integral for the $\bar{H}$ -function given by the equation (1.6) have been given by Buschman and Srivastava [3], Rathie [10] introduced the I-function of one variable, it's a generalization of $\bar{H}$ -function.

$$\Omega = \sum_{j=1}^m |\beta_j| + \sum_{j=1}^n |A_j \alpha_j| - \sum_{j=m+1}^q |B_j \beta_j| - \sum_{j=n+1}^p |\alpha_j| > 0 \quad (1.8)$$

and

$$|\arg(z)| < \frac{1}{2}\pi\Omega \quad (1.7)$$

2. Fractional integral operators

Throughout this paper Δ will denote the class of functions $f(t)$ for which

$$f(t) = \begin{cases} 0|t|^\zeta \}. \dots \max(|t|) \rightarrow 0 \\ 0[|t|^{w_1} e^{-w_2|t|}] \dots \min(|t|) \rightarrow \infty \end{cases}$$

In this paper we study the following two unified fractional integral operators involving the product of generalized Laguerre function and (τ, β) generalized associated Legendre function having general arguments.

$$I_{x:k;z_0,\gamma,\alpha}^{\mu,\lambda;\tau,\beta,m,n;\mu_0,\lambda_0,v}[f(t)] = x^{-\mu-\lambda-1} \int_0^x t^\mu (x-t)^\lambda {}_\gamma L_v^\alpha \left(z_0 \left(\frac{t}{x} \right)^{\mu_0} \left(1 - \frac{t}{x} \right)^{\lambda_0} \right)^{\tau,\beta} P_k^{m,n} \left(\frac{t}{x} \right) f(t) dt \quad (2.1)$$

where $f(t) \in \Delta$, provided that

$$Re(\mu + s + 1) > 0, Re(\lambda + 1) > 0, Re(\alpha) \geq -1, Re(\alpha + v) > -\frac{1}{\alpha}, \min Re(\mu_0, \lambda_0) > 0 \text{ (not all simultaneously zero)}$$

$$J_{x:k;z_0,\gamma,\alpha}^{\mu,\lambda;\tau,\beta,m,n;\mu_0,\lambda_0,v}[f(t)] = x^\lambda \int_x^\infty t^{-\mu-\lambda-1} (t-x)^\lambda {}_\gamma L_v^\alpha \left(z_0 \left(\frac{x}{t} \right)^{\mu_0} \left(1 - \frac{x}{t} \right)^{\lambda_0} \right)^{\tau,\beta} P_k^{m,n} \left(\frac{x}{t} \right) f(t) dt \quad (2.2)$$

provided

$$Re(w_2) > 0 \text{ or } Re(w_2) = 0 \text{ and } Re(\mu - w_1) > 0, Re(\alpha) > 0, Re(\alpha) \geq -1, Re(\alpha + v) > -\frac{1}{\alpha}$$

$$Re(\lambda) > -1, \min Re(\mu_0, \lambda_0) > 0, \text{ (not all simultaneously zero).}$$

3. Images formulae.

In this section , we shall use the following notations

$$\mathbf{A}_1 = [(a_{rj}; \alpha_{rj}^{(1)}, \dots, \alpha_{rj}^{(r)}, 0, 0, 0; A_{rj})_{1,n_r}], [\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \dots, \alpha_{rji_r}^{(r)}, 0, 0, 0; A_{rji_r})_{n+1,p_{i_r}}] \quad (3.1)$$

$$\mathbf{B}_1 = [\tau_{i_r}(b_{rji_r}; \beta_{rji_r}^{(1)}, \dots, \beta_{rji_r}^{(r)}, 0, 0, 0; B_{rji_r})_{1,q_{i_r}}] \quad (3.2)$$

We have

Theorem 1.

$$I_x^{\mu,\lambda} \left[t^\sigma \mathfrak{J} \left(\begin{matrix} z_1 t^{\mu_1} \left(1 - \frac{t}{x} \right)^{\lambda_1} \\ \cdot \\ \cdot \\ z_r t^{\mu_r} \left(1 - \frac{t}{x} \right)^{\lambda_r} \end{matrix} \right) \right] = \frac{(-)^{m/2} \Gamma(1 + \alpha + v)}{\Gamma(v) \Gamma(1 - v) \Gamma(k - \frac{m-n}{2} + 1) \Gamma(-k - \frac{m-n}{2})} x^\sigma$$

$$\mathfrak{J}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r};R_r;Y;2,2;1,2;1,1}^{U;0,n_r+2;V;2,2;1,1;1,1} \left(\begin{matrix} z_1 x^{v_1+\lambda_1} \\ \cdot \\ \cdot \\ z_r x^{v_r+\lambda_r} \\ -z_0 \\ -\frac{1}{2} \\ 1 \end{matrix} \middle| \begin{matrix} \mathbb{A}; \mathbf{A}_1; (1 + v, \gamma; 1); (-k + \frac{m-n}{2}, 1; 1), (k + \frac{m-n}{2} + 1, \tau; 1); (1 + \frac{n}{2}, 1; 1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mathbb{B}; \mathbf{B}_1; (0; 1; 1), (-\alpha, \gamma; 1); (0, 1; 1), (m, \beta; 1); (0, 1; 1) \end{matrix} \right) \quad (3.3)$$

$$A_1 = (-\mu - \sigma, \mu_1, \dots, \mu_r; \mu_0, 0, 1; 1), \left(-\lambda + \frac{m}{2}; \lambda_1, \dots, \lambda_r, \lambda_0, 1, 0; 1 \right), \mathbf{A}_1 \quad (3.4)$$

$$B_1 = \mathbf{B}_1, \left(-1 - \mu - \lambda - \sigma + \frac{m}{2}; \mu_1 + \lambda_1, \dots, \mu_r + \lambda_r, \mu_0 + \lambda_0, 1, 1; 1 \right) \quad (3.5)$$

Provided that

$f(t) \in \Delta$ and

$$Re(w_2) > 0 \text{ or } Re(w_2) = 0 \text{ and } Re(\mu - w_1) > 0, Re(\alpha) > 0, Re(\alpha) \geq -1, Re(\alpha + v) > -\frac{1}{\alpha}$$

$$Re(\lambda) > -1, \min Re(\mu_0, \lambda_0) > 0, \text{ (not all simultaneously zero).}$$

$$\left| \arg \left(z_i t^{\mu_i} \left(1 - \frac{t}{x} \right)^{\lambda_i} \right) \right| < \frac{1}{2} A_i^{(k)} \pi \text{ where } A_i^{(k)} \text{ is defined by Ayant [2].}$$

Proof

To prove the theorem 1, first we express the I-operator involved in its left hand side in the integral form with the help of (2.1). Then we express the generalized Laguerre function, (τ, β) -generalized associated Legendre function and the multivariable Gimel-function having general arguments of multiple Mellin-Barnes type integrals contour with the help of (1.1), (1.2) and (1.3) respectively. Then we interchange the order of $s_i (i = 1, 2, \dots, r + 3)$ -integral and t-integral, (which is permissible under the condition stated). Finally, on evaluating the t-integral and reinterpreting the result thus obtained in terms of multivariable Gimel-function, we arrive at the required result after algebraic manipulations.

Theorem 2.

$$J_x^{\mu, \lambda} \left[t^\sigma \mathfrak{J} \left(\begin{matrix} z_1 t^{-\mu_1} \left(1 - \frac{x}{t} \right)^{\lambda_1} \\ \cdot \\ z_r t^{-\mu_r} \left(1 - \frac{x}{t} \right)^{\lambda_r} \end{matrix} \right) \right] = \frac{(-)^{m/2} \Gamma(1 + \alpha + v)}{\Gamma(v) \Gamma(1 - v) \Gamma(k - \frac{m-n}{2} + 1) \Gamma(-k - \frac{m-n}{2})} x^\sigma$$

$$\mathfrak{J}_{X; p_{i_r} + 2, q_{i_r} + 1, \tau_{i_r}; R_r; Y; 2, 2; 1, 2; 1, 1}^{U; 0, n_r + 2; V; 2, 2; 1, 1; 1} \left(\begin{matrix} z_1 x^{v_1 + \lambda_1} \\ \cdot \\ \cdot \\ z_r x^{v_r + \lambda_r} \\ -z_0 \\ -\frac{1}{2} \\ 1 \end{matrix} \middle| \begin{matrix} \mathbb{A}; \mathbb{A}_2; (-k + \frac{m-n}{2}, 1; 1), (k + \frac{m-n}{2} + 1, \tau; 1); (1 + v, \gamma; 1); (1 + \frac{n}{2}, 1; 1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mathbb{B}; \mathbb{B}_2; (0; 1; 1), (m, \beta; 1); (0, 1; 1), (-\alpha, \gamma; 1); (0, 1; 1) \end{matrix} \right) \quad (3.6)$$

where

$$A_1 = (1 + \sigma - \mu, \mu_1, \dots, \mu_r; \mu_0, 0, 1; 1), \left(-\lambda + \frac{m}{2}; \lambda_1, \dots, \lambda_r, \lambda_0, 1, 0; 1 \right), \mathbf{A}_1 \quad (3.7)$$

$$B_1 = \mathbf{B}_1, \left(-\mu - \lambda + \sigma + \frac{m}{2}; \mu_1 + \lambda_1, \dots, \mu_r + \lambda_r, \mu_0 + \lambda_0, 1, 1; 1 \right) \quad (3.8)$$

The validity conditions are the same that (3.5) and

$$\left| \arg \left(z_i t^{\mu_i} \left(1 - \frac{x}{t} \right)^{\lambda_i} \right) \right| < \frac{1}{2} A_i^{(k)} \pi$$

The proof of (3.8) is similar that (3.5)

The multivariable Gimel function reduces to $\bar{H}$ -function defined in (1.6), we obtain

Corollary 1.

$$I_x^{\mu, \lambda} \left[t^\sigma \bar{H} \left(z t^\mu \left(1 - \frac{t}{x} \right)^\lambda \right) \right] = \frac{(-)^{m/2} \Gamma(1 + \alpha + v)}{\Gamma(v) \Gamma(1 - v) \Gamma(k - \frac{m-n}{2} + 1) \Gamma(-k - \frac{m-n}{2})} x^\sigma$$

$$\mathfrak{J}_{2, 1; p, q; 2, 2; 1, 2; 1, 1}^{0, 2; m, n; 2, 2; 1, 1; 1} \left(\begin{matrix} z x^{v + \lambda} \\ -z_0 \\ -\frac{1}{2} \\ 1 \end{matrix} \middle| \begin{matrix} A_1 : C; (1 + v, \gamma); (-k + \frac{m-n}{2}, 1), (k + \frac{m-n}{2} + 1, \tau; 1); (1 + \frac{n}{2}, 1) \\ \cdot \\ \cdot \\ B_1 : D; (0, 1), (-\alpha, \gamma); (0, 1), (m, \beta); (0, 1) \end{matrix} \right) \quad (3.9)$$

where

$$A_1 = (-\mu - \sigma, \mu_1, \dots, \mu_r; \mu_0, 0, 1), \left(-\lambda + \frac{m}{2}; \lambda_1, \dots, \lambda_r, \lambda_0, 1, 0 \right) : C = (a_j, \alpha_j; A_j)_{1, n}, (a_j, \alpha_j)_{n+1, p} \quad (3.10)$$

$$B_1 = \left(-1 - \mu - \lambda - \sigma + \frac{m}{2}; \mu_1 + \lambda_1, \dots, \mu_r + \lambda_r, \mu_0 + \lambda_0, 1, 1\right); D = (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \quad (3.11)$$

The validity conditions are the same that (3.5) and

$$\left| \arg \left(z t^\mu \left(1 - \frac{t}{x} \right)^\lambda \right) \right| < \frac{1}{2} \pi \Omega \text{ is defined by (1.8).}$$

The multivariable Gimel-function reduces in Aleph-function of one variable [16,17], we have

Corollary 2.

$$J_x^{\mu, \lambda} \left[t^\sigma \mathfrak{N} \left(z t^{-\mu} \left(1 - \frac{x}{t} \right)^\lambda \right) \right] = \frac{(-)^{m/2} \Gamma(1 + \alpha + v)}{\Gamma(v) \Gamma(1 - v) \Gamma(k - \frac{m-n}{2} + 1) \Gamma(-k - \frac{m-n}{2})} x^\sigma$$

$$\mathfrak{J}_{2,1:P,Q;2,2;1,1;1,1}^{0,2:M,N;2,2;1,1;1,1} \left(\begin{array}{c} z x^{\nu+\lambda} \\ -z_0 \\ -\frac{1}{2} \\ 1 \end{array} \middle| \begin{array}{c} A_2 : E; (-k + \frac{m-n}{2}, 1), (k + \frac{m-n}{2} + 1, \tau); (1 + v, \gamma); (1 + \frac{n}{2}, 1) \\ \vdots \\ B_2 : F; (0, 1), (m, \beta); (0, 1), (-\alpha, \gamma; 1); (0, 1) \end{array} \right) \quad (3.12)$$

where

$$A_2 = (-\mu - \sigma, \mu_1, \dots, \mu_r; \mu_0, 0, 1), \left(-\lambda + \frac{m}{2}; \lambda_1, \dots, \lambda_r, \lambda_0, 1, 0\right), E = (a_j, A_j)_{1,N}, [c_i(a_{ji}, A_{ji})]_{N+1, P_i; r'} \quad (3.13)$$

$$B_2 = \left(-1 - \mu - \lambda - \sigma + \frac{m}{2}; \mu + \lambda, \mu_0 + \lambda_0, 1, 1\right); F = (b_j, B_j)_{1,M}, [c_i(b_{ji}, B_{ji})]_{M+1, Q_i; r'} \quad (3.14)$$

The validity conditions are the same that (3.5) and

$$\left| \arg \left(z t^\mu \left(1 - \frac{x}{t} \right)^\lambda \right) \right| < \frac{1}{2} \pi \Omega \text{ is defined by (1.5).}$$

4. Application (see Kumawat [7])

In (3.5) let $\lambda_0, r = 1$ and reduce generalized Laguerre function to Laguerre function, (τ, β) -generalized associated function to τ -generalized associated Legendre function [18, p.241, eq.(5.4)] and multivariable Gimel-function to generalized Krätzel function [5 p. 610, eq.(420)], we obtain the following integral under the same validity conditions

$$\int_0^x \frac{1}{x} \left(\frac{t}{x} \right)^{\mu+\sigma} \tau P_k^{m,n} \left(\frac{t}{x} \right) L_v^\alpha \left(\left(\frac{t}{x} \right)^{\mu_0} \right) D_{\rho,k}^{\omega,\delta} \left(z_1 \left(\frac{t}{x} \right)^{\mu_1} \right) dt$$

$$= \frac{(-)^{m/2} [a(\delta - 1)]^{-\omega/\rho} \Gamma(1 + \alpha + v)}{\rho \Gamma(1/(\delta - 1)) \Gamma(v) \Gamma(1 - v) \Gamma(k - \frac{m-n}{2} + 1) \Gamma(-k - \frac{m-n}{2})} x^\sigma H_{1,1:1,2;1,2;3,2;1,1}^{0,1;2,1,1,1;2,3;1,1}$$

$$\left(\begin{array}{c} z_1 \\ -1 \\ -\frac{1}{2} \\ 1 \end{array} \middle| \begin{array}{c} A_3 : \left(1 - \frac{1}{\delta-1} - \frac{\omega}{\rho}, \frac{k}{\rho} \right); (1 + v, 1); (-k + \frac{m-n}{2}, 1; 1), (k + \frac{m-n}{2} + 1), (-\lambda + \frac{m}{2}, 1); (1 + \frac{n}{2}, 1) \\ \vdots \\ B_3 : (0; 1), \left(\frac{\omega}{\rho}, \frac{k}{\rho} \right); (0, 1), (-\alpha, 1); (0, 1), (m, \tau); (0, 1) \end{array} \right) \quad (4.1)$$

where

$$A_3 = (-\mu - \sigma; \mu_1, \mu_0, 0, 1); B_3 = \left(-1 - \mu - \lambda - \sigma + \frac{m}{2}; \mu_1, \mu_0, 1, 1\right) \quad (4.2)$$

On taking $\lambda_0 = \lambda_1 = 0$ and reduce (τ, β) -generalized associated function to associated Legendre function and generalized Laguerre function to Laguerre function, The multivariable Gimel function to generalized Hurwitz-Lerch zeta function [14, p.491, eq.(1.20)], we obtain the following integral

$$\begin{aligned} & \int_0^x \frac{1}{x} \left(\frac{t}{x}\right)^{\mu+\sigma} \left(1 - \frac{t}{x}\right)^{\lambda} P_k^{m,n} \left(\frac{t}{x}\right) L_v^{\alpha} \left(\left(\frac{t}{x}\right)^{\mu_0}\right) \phi_{\eta,\delta,\zeta}^{\omega,k',\gamma} \left(z_1 \left(\frac{t}{x}\right)^{\mu_1}; s, b\right) dt \\ &= \frac{\Gamma(1+\alpha+v)}{\Gamma(v)\Gamma(1-v)\Gamma\left(k - \frac{m-n}{2} + 1\right)\Gamma\left(-k - \frac{m-n}{2}\right)} \sum_{n=0}^{\infty} \frac{(\eta)_{\omega n}(\delta)_{k'n}}{(\zeta)_{\gamma n}} \frac{z_1^n}{(b+u)^{sn}} x^{\sigma+\mu_1 u} \\ & H_{1,1:1,2;3,2;1,1}^{0,1;1,1;2,3;1,1} \left(\begin{array}{c} z_1 \\ -\frac{1}{2} \\ 1 \end{array} \middle| \begin{array}{c} A_4 : (1+v, 1) : \left(-k + \frac{m-n}{2}, 1\right), \left(k + \frac{m-n}{2} + 1\right), \left(-\lambda + \frac{m}{2}, 1\right); \left(1 + \frac{n}{2}, 1\right) \\ \vdots \\ B_4 : (0, 1), (-\alpha, 1); (0, 1), (m, 1); (0, 1) \end{array} \right) \end{aligned} \quad (4.3)$$

where

$$A_4 = (-\mu - \sigma - \mu_1 u, 0, \mu_0, 1); B_4 = \left(-1 - \mu - \lambda - \sigma - \frac{m}{2} - \mu_1 u; 1, \mu_0, 1\right) \quad (4.4)$$

Remarks

We obtain the same formulae concerning the multivariable Aleph- function defined by Ayant [1], the multivariable I-function defined by Prathima et al. [9], the multivariable I-function defined by Prasad [8] and the multivariable H-function defined by Srivastava and Panda [12,13], see Kumawat [7] for more details concerning the multivariable H-function

5. Conclusion.

Firstly, the pair of fractional integral operators presented in this document are quite nature. Therefore, on specializing the parameters of these functions involving in this paper, we obtain various other results as its special cases. Secondly, by specializing the various parameters as well as variables in the generalized multivariable Gimel-function, we obtain a large number of formulae involving remarkably wide variety of useful functions or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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