Reduces Solution of Fredholm Integral Equation to a System of Linear Algebraic Equation

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Abstract

The objective of this research is to study some types of kernel of integral equations like iterated kernel, symmetric, different and Resolved kernel and to determine the Resolved kernel for Fredholm integral equation (FIE) and Volterra integral equation (VIE). It was shown that there is relation between iterated kernel and resolvent kernel as mentioned to some examples of these kernel ., also find the solution of fredholm complete neutralization for kernel and it reduces the solution of fredholm integration neutralization to a system of linear algebraic equation, give some problem of solving system.

Keywords: kernel, Volterra integral equation, Fredholm integral equation, linear algebraic equation.

I. INTRODUCTION

An integration neutralization is an neutralization in which the unidentified function show down an integration symbol. Accept to Bocher [1914], the name *integration* neutralization was proposes in 1888 by du Bois-Raymond. A generic structure of an integration neutralization in f(x) can be offered as:

$$h(x) f(x) = g(x) + \lambda \int_{a}^{b(x)} k(x, y) f(y) dy$$
 (1-1)

Where: "h, g" are given functions, K(x, y) function in two variables are names the kernel of the integration neutralization, λ is a scalar parameter, the given function K(x, y) which depends up on the current variables x as well as the variables y is known as the kernel or nucleus of the integration neutralization. The integration neutralization can be classify for two classes. The first, it is name of "Volterra integral equation" (VIE) where the Volterra's significant job in this domain was complete in 1884-1896 and the second, name "Fredholm integral equation" (FIE) where the Fredholm's significant "contribution was made in 1900-1903". Fredholm progressing the theory of this integration neutralization such as a limit to the linear system of neutralization[1]. Integrat equations play an important role in many branches of sciences such as mathematics, biology, chemistry, physics, mechanics and engineering. Therefore, many different techniques are used to solve these types of equations. Also Integral equations diverse evolved directly linked to the number of several branche of mathematics in the differentiation account, integration account, differential neutralization and rounding issues to addition to the very concepts and physical links issues^[2], ^[7]. There is equivalence relation between Integral equations and ordinary differential equations. There is a close relationship among differentials and integration neutralization, and several issues may be features either way. In example, Green's function, Fredholm theory, and Maxwell's neutralization [3]. The of the integral equations depends on the type of integral equation fredholm, volterra first kind, or second kind, linear or nonlinear, homogeneous, or non -hom. Also, the solution of integral equs. Depend on the kernel of integral equs., whether if it is symmetric or difference type, in some case, it is difficulty to solve integral equs.[4]. Therefore, there exist approximate and numerical method for solving integral equs. Integral equations are important in many applications. Problems in which integral equations are encountered include radiative transfer, and the oscillation of a string, membrane, or axle. Oscillation problems may also be solved as differential equations. Both Fredholm and Volterra equations are linear integral equations, due to the linear behavior of y(x) under the integral [5] ,[8].

II. CLASSIFICATION OF THE INTEGRAL EQUATIONS

Integration neutralization are categorized agree to three various dichotomies, creating eight various types: border of integration

- both fixed: "Fredholm neutralization $h(x) f(x) = g(x) + \lambda \int_{a}^{b} k(x, y) f(y) dy$, $a \le x \le b$ "
- one variable: "<u>Volterra neutralization</u> $h(x) f(x) = g(x) + \lambda \int_{a}^{b(x)} k(x, y) f(y) dy$, , when b(x) = x $a \le x < \infty$ "
- only inside integral: first kind, if h(x)=0: "<u>Fredholm neutralization</u> $g(x) = \int_{a}^{b} k(x, y) f(y) dy$ "
- "<u>Volterra neutralization</u> $g(x) = \lambda \int_{a}^{x} k(x, y) f(y) dy$,"
- both inside and outside integral: second kind, if h(x) = 1 <u>Fredholm neutralization</u> $f(x) = g(x) + \lambda \int_{a}^{b} k(x, y) f(y) dy$ "

"<u>Volterra equation</u>" $f(x) = g(x) + \lambda \int_{a}^{b} k(x, y) f(y) dy$ "

• identically zero: homogeneous : first kind , if g(x)=0 : <u>Fredholm neutralization</u> $\int k(x, y) f(y) dy = 0$

• Volterra neutralization
$$\lambda \int_{a}^{a} k(x, y) f(y) dy = 0$$
, "

• not identically zero: inhomogeneous

Examples :1- $\int_{0}^{x} (xy + x^{2}) f(y) dy = 0$ linear homo. Volterra integral eq. of the first type $\lambda = 1, k(x, y) = (xy + x^{2})$

2- $2\int_{0}^{1} (xy + x) f(y) dy = 0 \text{ linear homo. Fredholm integral eq. of the kind}$ $\lambda = 2, k(x, y) = (xy + x)$

III. TYPES KERNELS

There are several kinds of kernels *A- The Iterated Kernels :*

i) See the following non-homo-fredholm integration neutralization of the second kind

$$y(x) = f(x) + \lambda \int_{0}^{b} k(x,t) y(t) dt \dots (1)$$

The Iterated Kernels of equ. 1 defined as $:k_1(x,t)=k(x,t)$

$$k_{n}(x,t) = \int_{a}^{b} k(x,z)k_{n-1}(z,t)dz$$
"
...n=2,3,...
$$k_{n}(x,t) = \int_{a}^{b} k_{n-1}(x,z)k(z,t)dz$$
"

ii) Look the following non-homo-Volterra integration neutralization "of the 2nd type":

$$y(x) = f(x) + \lambda \int_{a} k(x,t) y(t) dt \dots (2)$$

The Iterated Kernels of equ. (2) defined as $:k_1(x,t)=k(x,t)$

$$k_{n}(x,t) = \int_{t}^{t} k(x,z)k_{n-1}(z,t)dz "$$

$$k_{n}(x,t) = \int_{t}^{t} k_{n-1}(x,z)k_{n-1}(z,t)dz "$$

B. Resolvent Kernel

i) Look the next non-homo-fredholm integration neutralization of the second type:

$$y(x) = f(x) + \lambda \int_{a}^{b} k(x,t) y(t) dt \dots (3)$$
, The Iterated Kernels is defined as $: R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t)$

Where: this series convergent absolutely and uniformly in this status of continuous k(x,t), $k_n(x,t)$ are iterated kernels as[1].

Then : the solution of equ. (3) is defined as: $y(x) = f(x) + \lambda \int_{a}^{b} R(x, t, \lambda) f(t) dt$ "

ii) Believe the next non-homo-volterra integration neutralization of the second type:

 $y(x) = f(x) + \lambda \int_{a}^{x} k(x,t) y(t) dt \dots (4)$, The Iterated Kernels is defined as $: R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t)$

Where: this series convergent absolutely and uniformly in this status of continuous k(x,t), $k_n(x,t)$ are iterated kernels as :

Then : the solution of equ. (4) is defined as: $y(x) = f(x) + \lambda \int R(x, t, \lambda) f(t) dt$ "

Then : the solution of equ. (4) is defined as: $y(x) = f(x) + \lambda \int_{a}^{b} R(x, t, \lambda) f(t) dt$ "

Theorem(1) : let R(x,t,) be the Resolvent kernel of a fredhlom integration neutralization:

$$y(x) = f(x) + \lambda \int_{a}^{b} k(x,t) y(t) dt$$

Thereafter, the Resolvent kernel satisfies the integration neutralization:

$$R(x,t,\lambda) = k(x,t) + \lambda \int k(x,z)R(z,t,\lambda)dz$$

Theorem(2) : let R(x,t,) be the Resolvent kernel of a volterra integration neutralization:

$$y(x) = f(x) + \lambda \int k(x,t) y(t) dt$$

Then, the Resolvent kernel satisfies the integration neutralization:""

$$R(x,t,\lambda) = k(x,t) + \lambda \int_{a} k(x,z) R(z,t,\lambda) dz$$

C. Symmetric kernel

A kernel is called symmetrical (or complex symmetrical) if it coincides with its own complex conjugate :

$$K\left(x,t\right)=K\left(x,t\right)$$

An integration neutralization with a real symmetrical kernel (cf. Kernel for an integration operation): K(x, s) = K(s, x)

Method of linear neutralizations together actual symmetrical kernel has primary structure uses drawing onto the notion of symmetrical square shapes "and going during from a finite to an infinite numeral of variables". Put forward a more primary way of substantiating Hilbert's results. To this cause, that method for integration neutralizations together symmetrical "kernel is also called the Hilbert–Schmidt way". A considerable sapping for limitation imposing that notion upon that information while unbeknown items has carried out. Look an integration neutralization of the second type together real symmetrical kernel:

$$y(x) - \lambda \int_{a}^{b} K(x,t) y(t) dt = f(x)$$
 " $x \in [a,b]$

The structure of this way at integration neutralizations together symmetrical kernel, it answer for assume this

symmetrical kernel **K** is measures upon the quadrate $[a, b] \times [a, b]$ and that $: \int_{a}^{b} \int_{a}^{b} |K(x, t)|^{2} |dxdt| < \infty$

D. Difference Kernel

Look integration eq. in which the kernel K(x, t) = K(x - t) where is a job of one variable, then the integration eq. "

$$y(x) = f(x) + \lambda \int k(x-t)y(t)dt$$
 and $y(x) = f(x) + \lambda \int k(x-t)y(t)dt$

Are called integral eq. of the convolution type .k(x-y) is called difference kernel . let $y_1(x), y_2(x)$ be pair continued jobs define for $x \ge 0$. Thereafter the gyration from y1 and y2 is denoted and defined by

$$y_1 * y_2 = \int_0^x y_1 (x - t) y_2 (t) dt = \int_0^x y_1 (x) y_2 (x - t) dt$$
"...(5)

The integrals (5) are called the coiling integrations.

E. Examples: Example 1: this example about the iterated kernel for " $y(x) = f(x) + \lambda \int_{a}^{b} k(x,t)y(t)dt$

Where k(x,t)=sin(x - 2t) $0 \le x \le 2\pi$, $0 \le t \le 2\pi$

Solution:
$$k1(x,t)=k(x,t)\sin(x-2t)$$
, $k_n(x,t) = \int_{-2\pi}^{2\pi} k(x,z)k_{n-1}(z,t)dz$ "

$$\therefore k_{2}(x,t) = \int_{a}^{2\pi} \sin(x-2z) \sin(z-2t) dz = 0$$

:.
$$k_{3}(x,t) = \int_{a}^{2\pi} \sin(x-2z).dz = 0$$
"
: $k_{n}(x,t) = 0$

The Iterated Kernels is $R(x, t, \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x, t) = 0$

Example 2: This example about the resolvent kernel for a volterra integration neutralization where :k(x,t)=1.

Solution: k1(x, t)=k(x, t) = 1 , $k_n(x, t) = \int_t^\infty k(x, z)k_{n-1}(z, t)dz$

$$k_{2}(x,t) = \int_{t}^{x} 1 dz = \frac{x-t}{1!} = x-t , \quad k_{3}(x,t) = \int_{t}^{x} 1 dx = \frac{(x-t)^{2}}{2!}$$

$$k_{n}(x,t) = \frac{(x-t)^{n-1}}{(n-1)!} , \quad R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_{n}(x,t) = \sum_{n=1}^{\infty} \frac{(\lambda(x-t))^{n-1}}{(n-1)!}$$

Example 3: Prove that k(x,t)=i(x-t) is symmetric and k(x,t)=i(x+t) is not symmetric, this example about symmetric kernel.

$$\frac{\mathbf{k}(\mathbf{x},t)=\mathbf{i}(\mathbf{x}-t)}{K(\mathbf{x},t)} = -\mathbf{i}\mathbf{k}(\mathbf{t}-\mathbf{x})=\mathbf{i}\mathbf{k}(\mathbf{x}-t)=\mathbf{k}(\mathbf{x}-t)$$
 is symmetric

 $K(x, t) = -ik(t+x) \neq k(x+t)$ is not symmetric

IV. FREDHOLM INTEGRATION NEUTRALIZATION DEGENERATION KERNEL

The major promoter of the integration neutralization are Fredholm (1903), There are many kinds of integration neutralization like Fredholm integration neutralization, Volterra integration neutralizations, Hammerstein integration neutralization, mixed integration neutralization and 2-D integration neutralization is [6]. when we have non –hom.

Fredholm integration neutralization. Then the way for solving it are depending on the degeneration case of the kernel.

A. Degenerate Kernel

A kernel k(x,t) can be so-called degenerate or separable kernel if it can be explicit as the summation of a finite digit of expression, every from which is the production from a task of x only and a task of t in the end.

$$k(x,t) = \sum_{k=1}^{n} a_{k}(x)b_{k}(t)$$
 ...

Here : n is positive integer . The task $a_k(x)$, $b_k(t)$, are continuous tasks at the requisite, $a \le x, t \le b$ and linearly not rely on [5],[6].

Example: Consider a Fredholm IE from that 2nd type, this example about a degenerate kernel [3].

$$x^{2} = \phi(x) - \int_{0}^{1} (x^{2} + y^{2})\phi(y) dy$$

Where: $K(x; y) = x^2 + y^2$ is a degenerate kernel $f(x) = x^2$ is a given task, and the parameter $\lambda = 1$.

B. Remark 1:

If put" k(x, y) = 0 for y > x" in volterra's equation. if could be consider as private status of fredholm's neutralization as given in the equation $\int_{a}^{b} k(x, y) f(y) dy = \int_{a}^{x} k(x, y) f(y) dy + \int_{x}^{b} k(x, y) f(y) dy$ $= \int_{a}^{x} k(x, y) f(y) dy + 0$

Remake 2: In several status a non Degenerate Kernel k(x,t) may be approximate uses Degenerate Kernel such a part of summation of the Taylor chain extension or other of k (x,t) and truncating the series. This method it is called convergent way. we can explain this by the following:

Fredholm IE of the 2nd type, with a degenerate kernel has the form:

b

$$``\phi(x) = f(x) + \lambda \int_{a}^{b} k(x,t)\phi(y) \, dy \dots (1`) , `` Where k(x,t) = \sum_{k=1}^{n} a_{k}(x)b_{k}(t)$$

$$\phi(x) = f(x) + \lambda \int_{a}^{b} (\sum_{k=1}^{n} a_{k}(x)b_{k}(t))\phi(t) \, dt \dots (2`)$$

$$``\phi(x) = f(x) + \lambda \sum_{k=1}^{n} a_{k}(x) \int_{a}^{b} b_{k}(t)\phi(t) \, dt \dots (3`)$$

$$\int_{a}^{b} b_{k}(t)\phi(t) \, dt = C_{k} , \quad k=1,2,3,\dots,n$$

Substituted C_k in (2[`]), we obtain $\phi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x) \dots (4^{`})$ " Where C_k are unknown constants (k = 1, 2, 3, ...,n)

Therefore the solution of Fredholm integration neutralization with Degenerate Kernel reduces to finding C_k are unknown fixeds (k = 1, 2, 3, ..., n)

Substituted (4°) in (2°) , we obtain :

$$\sum_{m=1}^{n} (C_m - \int_a^b b_m(t)(f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t)) dt a_m(x) = 0 "$$

By linear independent of $a_m(x)$, then

Let
$$a_{km} = \int_{a}^{b} a_{k}(t)b_{m}(t)dt$$
 ", $f_{m} = \int_{a}^{b} b_{m}(t)f(t)dt$ "
 $C_{m} - \lambda \sum_{k=1}^{n} C_{k}a_{km} = f_{m}$ ", $m = 1, 2, ..., n$
 $(1 - \lambda a_{11})C_{1} - \lambda a_{12}C_{2} - ... \lambda a_{1n}C_{n} = f_{1}$
 $- \lambda a_{21}C_{1} + (1 - \lambda a_{22})C_{2} - ... \lambda a_{2n}C_{n} = f_{2}$

 $-\lambda a_{n1}C_{1} - \lambda a_{n2}C_{2} - \dots + (1 - \lambda a_{nn})C_{n} = f_{n}$

The determinate of this system of linear algebraic equation:

$$D(\lambda) = \begin{vmatrix} 1 - \lambda a_{11} & -\lambda a_{12} & . & . & -\lambda a_{1n} \\ -\lambda a_{21} & 1 - \lambda a_{22} & . & . & -\lambda a_{2n} \\ . & . & . & . \\ . & . & . & . \\ -\lambda a_{n1} & -\lambda a_{n2} & +(1 - \lambda a_{nn}) \end{vmatrix}$$
 if $D(\lambda) \neq 0$ then:

The solution of equ. (1`) depend on finding C_k , (k=1,2,...,n), and $\phi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x)$(4`)

C. Example: Example 1 ;This is integration neutralization.

$$\phi(x) - \lambda \int_{-\pi}^{\pi} (x \cos t + t^2 \sin x + \cos x \sin x) \phi(t) dt = x \qquad \dots (1^*)$$

$$\phi(x) = x + \lambda x \int_{-\pi}^{\pi} (\cos t \phi(t) dt + \lambda \sin x) \int_{-\pi}^{\pi} t^2 \phi(t) dt + \lambda \cos x \int_{-\pi}^{\pi} \sin x) \phi(t) dt \dots (2^*)$$

Reduces to system of linear algebraic equ. Suppose C triple equal to the number of integrations :

$$C_{1} = \int_{-\pi}^{\pi} \cos t \phi(t) dt$$

$$C_{2} = \int_{-\pi}^{\pi} t^{2} \phi(t) dt \qquad \dots (3^{*}) \qquad C_{1}, C_{2}, C_{3} \text{ are unbeknown constants}$$

$$C_{3} = \int_{-\pi}^{\pi} \sin t \phi(t) dt$$
Substituted (3^{*}) in (2^{*})

Substituted (3^*) in (2^*)

$$\phi(x) = \lambda C_1 x + \lambda C_2 \sin x + \lambda C_3 \cos x \dots (4^*)$$

$$\phi(t) = \lambda C_1 t + \lambda C_2 \sin t + \lambda C_3 \cos t$$

Substituted $\phi(t)$ in (3*) we obtain:

$$C_{1} = \int_{-\pi}^{\pi} (t + \lambda C_{1}t + \lambda C_{2} \sin t + \lambda C_{3} \cos t) \cos t dt "$$

$$C_{2} = \int_{-\pi}^{\pi} (t + \lambda C_{1}t + \lambda C_{2} \sin t + \lambda C_{3} \cos t)t^{2} dt "$$

$$C_{3} = \int_{-\pi}^{\pi} (t + \lambda C_{1}t + \lambda C_{2} \sin t + \lambda C_{3} \cos t) \sin t dt "$$

To find integration we obtain the system of linear algebraic neutralization with unknown : C_1, C_2, C_3 . linear algebraic neutralization the system is :

$$C_{1} - \lambda \pi C_{3} = 0$$

$$C_{2} + 4 \lambda \pi C_{3} = 0$$

$$- 2 \pi C_{1} - \lambda \pi C_{2} + C_{3} = 2 \pi$$

....(5*)

The determinate of system (5^*) is :

$$\begin{vmatrix} 1 & 0 & -\lambda\pi \\ 0 & 1 & 4\lambda\pi \\ -2\lambda\pi & -\lambda\pi & 1 \end{vmatrix} = 1 + 2\lambda^2\pi^2 \neq 0 \quad \text{that is exist single resolution}$$

To find C_1, C_2, C_3 use crammer 's method , we get :

$$C_{1} = \frac{2\lambda\pi^{2}}{1+2\lambda^{2}\pi^{2}}$$

$$C_{2} = \frac{-8\lambda\pi^{2}}{1+2\lambda^{2}\pi^{2}} , \quad C_{3} = \frac{2\pi}{1+2\lambda^{2}\pi^{2}}$$

Substituted C_1, C_2, C_3 in (4*), we obtain

$$\phi(x) = x + \lambda \frac{2\lambda \pi^2}{1 + 2\lambda^2 \pi^2} x - \lambda \frac{8\lambda \pi^2}{1 + 2\lambda^2 \pi^2} \sin x + \lambda \frac{2\pi}{1 + 2\lambda^2 \pi^2}$$
$$\phi(x) = x + \lambda \frac{2\lambda \pi}{1 + 2\lambda^2 \pi^2} (\lambda \pi x - 4\lambda \pi \sin x + \cos x)....(6^*)$$

 (6^*) is solution for equation (1^*)

D. Remark :

Many problems of physics reduce to non- linear homo. Integration neutralization has the general forms:

$$\phi(x) = \int_{a}^{b} k(x,t) f(t,\phi(t)) dt \dots (1\#)$$

Where k(x,t), f are given function., $\phi(x)$ is the unknown function non-linear non-homo. Fredholm Integral neutralization has the form :

$$\phi(x) = g(x) + \int_{a}^{b} k(x,t) f(t,\phi(t)) dt$$

where : g(x) is the unbeknown job, k(x,t) be degenerate kernel.

$$k(x,t) = \sum_{i=1}^{m} C_{i}a_{i}(x)...(2\#) "$$

$$\phi(x) = \sum a_{i}(x)\int_{a}^{b} b_{i}(t) f(t,\phi(t)) dt(3\#) "$$

$$C_{i} = \int_{a}^{b} b_{i}(t) f(t,\phi(t)) dt(4\#) , \quad \therefore \phi(t) = \sum_{i=1}^{m} C_{i}a_{i}(x)...(5\#)$$

Substituted (5#) in (4#) ,we obtain the system of linear algebraic neutralization to m equ. If there is exist solution to integration equ. Is $\phi(x)$ after Substituted C_i, the number of solution to integration neutralization is equal to number to solution of system.

VI. CONCLUSIONS

- It appears the important of integral equation when we have difficulty to find a solution for O.D.E. we transform the last to integral equation which is equivalent to it
- The solution of the integral equation depend on the kernel of the integral equation and the Resolvent kernel $R(x, t, \lambda)$ satisfied the integral equation.
- The convolution definite is a particular case of the convolution :

$$y_1 * y_2 = \int_{-\infty}^{\infty} y_1(x-t) y_2(t) dt = \int_{-\infty}^{\infty} y_1(x) y_2(x-t) dt$$
, By setting $y_1(t) = y_2(t) = 0$ for t<0 & t>x.

- In many cases a non –degenerate kernel k(x,y) may be approximate by a degenerate kernel as a partial sum of the Taylor series expansion or other of k(x,y) and truncate the series .
- We obtained that the system of linear algebraic equation with C_1, C_2, C_3 are unknown constants after that solve the system (5*) by using determinates method after that find the solution of equation (6*).

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