Application of Special Functions in One Dimensional Advective Diffusion Problem

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ABSTRACT
In the present paper, we construct an one dimensional advective diffusion model problem to evaluate the substance concentration distribution along $x-$axis between $x=-1$ and $x=1$. Here, the diffusivity of the substance and the velocity of the solvent flow are both considered as a variable. Kumar and Satyarth [6] use the product of the class of multivariable polynomials and the multivariable H-function defined here to obtain the analytic solution. Then, we employ the product of the generalized hypergeometric function, class of multivariable polynomials and the multivariable Gimel-function to obtain an analytic formula of our problem. Finally, some particular cases will be given.

Keywords :Multivariable Aleph-function, , classes of multivariable polynomials, generalized hypergeometric function, advective-diffusion model problem, expansion formula, multivariable Gimel-function.

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1.Introduction and preliminaries.

. We consider a generalized transcendental function of several complex variables, called Gimel fnction.

$$\gimel(z_1,\cdots,z_r)=\gimel^{U;0,n_r:V}_{X;p_{i_r},q_{i_r},\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\mathbf{A}:\ A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B}:\ B \end{array}\right)=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

The following quantities $\mathbb{A},\mathbf{A},A,\mathbb{B},\mathbf{B},B,X,Y,U,V,\ \psi(s_1,\cdots,s_r)$ and $\theta_k(s_k)(k=1,\cdots,r)$ are defined by Ayant [2].

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r)=0(\ |z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\ )\ ,max(\ |z_1|,\cdots,|z_r|\ )\rightarrow 0$

$\aleph(z_1,\cdots,z_r)=0(\ |z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\ )\ ,min(\ |z_1|,\cdots,|z_r|\ )\rightarrow\infty$ where $i=1,\cdots,r$:

$\alpha_i=\min\limits_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]$ and $\beta_i=\max\limits_{1\leqslant j\leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.2}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.3}$$

$$A = [(c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,n^{(1)}}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}; C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]; \cdots ;$$

$$[(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}; C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.4}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2}; \beta_{2ji_2}^{(1)}, \beta_{2ji_2}^{(2)}; B_{2ji_2})]_{1,q_{i_2}}, [\tau_{i_3}(b_{3ji_3}; \beta_{3ji_3}^{(1)}, \beta_{3ji_3}^{(2)}, \beta_{3ji_3}^{(3)}; B_{3ji_3})]_{1,q_{i_3}}; \cdots ;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}}; \beta_{(r-1)ji_{r-1}}^{(1)}, \cdots, \beta_{(r-1)ji_{r-1}}^{(r-1)}; B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{B} = \quad [\tau_{i_r}(b_{rji_r}; \beta_{rji_r}^{(1)}, \cdots, \beta_{rji_r}^{(r)}; B_{rji_r})_{1,q_{i_r}}] \tag{1.6}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,m^{(1)}}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}; D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]; \cdots ;$$

$$[(\mathrm{d}_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}; D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.7}$$

An advective diffusion is a process where solute particles in a solvent are diffused and transported with the flow of solvent. If $C(x,t)$ represents be the concentration of the diffusing substance, D denotes the diffusivity of the substance, $\mathbf{u}$ is the velocity of the solvent flow and $F(x,t)$ is the externel source function, then the governing advective-diffusion equation is given by (see Harrison and Perry [5], Hanna et al. [4], Lyons et al [7])

$$\frac{\partial C}{\partial t} + div(C\mathbf{u}) = div(D\nabla C) + F \tag{1.8}$$

Here, we construct an one dimensional advective-diffusion model problem in which the diffusivity of the substance is considered as a variable of position x and the velocity of the solvent flow is supposed to be function of x. There is no generation or absorption of the solute in the solvent.

The class of multivariable polynomials defined by Srivastava [11], is given in the following manner :

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[y_1,\cdots,y_u] \quad = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_u=0}^{[N_u/M_u]} \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_u)_{M_uK_u}}{K_u!} \; A[N_1,K_1;\cdots;N_u,K_u]$$

$$A[N_1,K_1;\cdots;N_u,K_u] y_1^{K_1} \cdots y_u^{K_u} \tag{1.9}$$

On suitably specializing the above coefficients , $S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[y_1,\cdots,y_u]$ yields some of known polynomials, these include the Jacobi polynomials, Laguerre polynomials, and others polynomials ([14],p. 158-161)

We shall use the following notation

$$B' = \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_u)_{M_uK_u}}{K_u!} A[N_1,K_1;\cdots;N_u,K_u] \tag{1.10}$$

2. Statement of the problem and governing equation

Let us consider an advective-diffusion process in which the solute particles in a solvent are diffused and transported with the variable velocity $\mathbf{u}$ of the solvent flow in the direction of the $x-$axis between the limits $x=-1$ to $x=1$ and that is supposed to be a function of position and proportional to $[1+\alpha-\gamma-\beta+(\alpha+\beta-\gamma+1)x]$, $Re(\alpha-\gamma) > -2$ $Re(\beta) > -1$, the diffusivity D of the substance (solute) is a function of position and is proportional to $(1-x^2)$. The externel source function is absent in the process that is there is no generation or absorption of the solute in the solvent. The desired function $C(x,t)$ represents the concentration distribution of the diffusing substance at the position x and at the time t. Then due to (1.1), we get an one dimensional advective-diffusion equation in the form given below :

$$\frac{\partial C}{\partial t}+[\lambda_1(\alpha-\gamma-\beta+1)+(\alpha+\beta-\gamma+1)x]+2x\lambda_2]\frac{\partial C}{\partial x}+\lambda_1(\alpha+\beta-\gamma+1)C-\lambda_2(1-x^2)\frac{\partial^2 C}{\partial x^2}=0 \quad (2.1)$$

where λ_1 and λ_2 are arbitrary non zero proportionality constants, $Re(\alpha-\gamma)>-2$ and $Re(\beta)>-1, t>0$ and $-1\leqslant x\leqslant 1$.

Let us suppose that the initial concentration of the solute in the solvent is given by

$$C(x,0)=f(x), -1\leqslant x\leqslant 1 \quad (2.2)$$

In (2.1), for simplicity, letting λ_1 and λ_2 , we get

$$\frac{1}{\lambda_1}\frac{\partial C}{\partial t}=(1-x^2)\frac{\partial^2 C}{\partial x^2}+[\beta+\gamma-\alpha-1-(\alpha+\beta-\gamma+3)x]\frac{\partial C}{\partial x}-(\alpha+\beta-\gamma+1)C \quad (2.3)$$

where $\lambda_1\neq 0$, $Re(\alpha-\gamma)>-2, Re(\beta)>-1, t>0$ where $-1\leqslant x\leqslant 1$

3. Main integral

In our work, we also require the following integral

Theorem 1.

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_Q^{(\mu,\upsilon)}(x)\,{}_MF_N\left[\begin{array}{c|c}(\mathrm{e}_M) & \\ . & \mathrm{y(1\text{-}x)}^g(1+x)^w \\ (\mathrm{f}_N) & \end{array}\right]$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u}\end{array}\right)\gimel\left(\begin{array}{c}\mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{Z}_r(1-x)^{h_r}(1+x)^{k_r}\end{array}\right)\mathrm{d}x=$$

$$\frac{\Gamma(1+\mu+Q)2^{\rho+\sigma+1}}{\Gamma(1+\mu)\Gamma(1+Q)}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty}\sum_{k=0}^{Q}B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!}\frac{\Gamma(1+\mu+Q)2^{\rho+\sigma+1}}{\Gamma(1+\mu)\Gamma(1+Q)}$$

$$\frac{(-Q)_k(1+\mu+\upsilon+Q)_k}{(1+\mu)_k k!}y_1^{K_1}\cdots y_u^{K_u}2^{gq+wq+\sum_{i=1}^u(f_i+w_i)K_i}$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c}\mathrm{Z}_1 2^{h_1+k_1} & \mathbb{A};\ (\text{-}\rho-k-gq-\sum_{i=1}^u f_iK_i:h_1,\cdots,h_r;1),\ (\text{-}\sigma-wq-\sum_{i=1}^u w_iK_i:k_1,\cdots,k_r;1),\mathbf{A}:A\\ . & .\\ . & .\\ . & \cdots\\ \mathrm{Z}_r 2^{h_r+k_r} & \mathbb{B};\mathbf{B},(\text{-1-k-}\sigma-\rho-(g+w)q-\sum_{i=1}^u(f_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r;1):B\end{array}\right) \quad (3.1)$$

Provided that

$min\{g,w,f_i,w_i,h_l,k_l\} > 0$ for $i=1,\cdots,u; l=1,\cdots,r$

$$Re(\rho+\sum_{i=1}^{v}K_if_i)+\sum_{i=1}^{r}h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1 \quad \text{and}$$

$$Re(\sigma+\sum_{i=1}^{v}K_iw_i)+\sum_{i=1}^{r}k_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1; Re(\mu) > -1, Re(\upsilon) > -1$$

$$\left|argZ_i(1-x)^{h_i}(1+x)^{k_i}\right| < \frac{1}{2^{h_i+k_i-1}}, A_i^{(k)}, A_i^{(k)}. \;, \quad \text{where } A_i^{(k)} \text{ is defined by (1.8). } |y|<1$$

Proof

To prove (3.1), first expressing the Jacobi polynomial $P_Q^{(\mu,\upsilon)}(.)$ in series , a class of multivariable polynomials defined by Srivastava [11] $S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}[.]$ in series with the help of (1.14), expressing the generalized hypergeometric in series and we interchange the order of summations and x-integral (which is permissible under the conditions stated). Expressing the Gimel-function of r-variables in Mellin-Barnes contour integral with the help of (1.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integrals involved in the process. Now collecting the powers of $(1-x)$ and $(1+x)$ and evaluating the inner x-integral. Finally, interpreting the Mellin-Barnes contour integral in multivariable Gimel-function, we obtain the desired result (3.1).

4. Solution of the problem and analysis

To solve the differential equation (2.3), we suppose that $C(x,t)=X(x)T(t)$ in it, we get

$$\frac{1}{\lambda_1 T}\frac{dC}{dt}=\frac{1}{X}(1-x^2)\frac{d^2X}{dx^2}+[\beta+\gamma-\alpha-1-(\alpha+\beta-\gamma+3)x]\frac{dX}{dx}-(\alpha+\beta-\gamma+1)X$$

$$=-(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon) \tag{4.1}$$

and

$$\frac{dT}{dt}=-\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)T \tag{4.2}$$

The solution of the equation (4.1) is $P_\upsilon^{(\alpha+\gamma+1,\beta)}(x)$ (see 10 page 258). Thus by (4.1) and (4.2), we obtain

$$X(x)=A_1P_\upsilon^{(\alpha+\gamma+1,\beta)}(x), -1\leqslant x\leqslant 1 \tag{4.3}$$

$$T(t)=A_2exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big], t\geqslant 0 \tag{4.4}$$

The general solution of the equation (2.3) is :

$$C(x,t)=\sum_{\upsilon=0}^{\infty}A_\upsilon P_\upsilon^{(\alpha+\gamma+1,\beta)}(x)exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big], t\geqslant 0 \text{ and } -1\leqslant x\leqslant 1 \tag{4.5}$$

Now making an appeal to (2.2) and taking the expression of (4.5) at $t=0$, we have

$$f(x) = \sum_{\upsilon=0}^{\infty} A_\upsilon P_\upsilon^{(\alpha+\gamma+1,\beta)}(x) \tag{4.6}$$

Then to evaluate A_υ; multiply both sides of (4.6) by $(1-x)^{\alpha-\gamma+1}(1+x)^\beta P_\mu^{(\alpha-\gamma+1,\beta)}(x)$ and then integrate the two sides of (4.6) with respect x from -1 to 1 and use the orthogonality of Jacobi polynomials (see, [10, page 258]), we obtain

$$A_\upsilon = \frac{2^{\gamma-\alpha-\beta-2}\Gamma(\upsilon+1)\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha+\beta-\gamma+\upsilon+2)}{\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha+\gamma+\upsilon+2)\Gamma(\beta+\upsilon+1)}$$

$$\int_{-1}^{1}(1-x)^{\alpha-\gamma+1}(1+x)^\beta P_\upsilon^{(\alpha+\gamma+1,\beta)}(x) f(x)\mathrm{d}x \tag{4.7}$$

$Re(\alpha-\gamma) > -2, Re(\beta) > -1, \forall \upsilon \in \mathbb{N}.$

Finally, with the aid of (4.5) and (4.7), we obtain the concentration distribution of the solute in the form

$$C(x,t) = 2^{\gamma-\alpha-\beta-2}\sum_{\upsilon=0}^{\infty}\frac{\Gamma(\upsilon+1)\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha+\beta-\gamma+\upsilon+2)}{\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha+\gamma+\upsilon+2)\Gamma(\beta+\upsilon+1)}$$

$$P_\upsilon^{(\alpha+\gamma+1,\beta)}(x) exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big]\int_{-1}^{1}(1-x)^{\alpha-\gamma+1}(1+x)^\beta P_\upsilon^{(\alpha+\gamma+1,\beta)}(x) f(x)\mathrm{d}x \tag{4.8}$$

provided $Re(\alpha-\gamma) > -2, Re(\beta) > -1, t > 0$ and $-1 \leqslant x \leqslant 1$

5. Examples

Let the initial concentration of the solute in the solvent is given by (formal initial condition)

$$f(x) = (1-x)^\rho(1+x)^\sigma {}_M F_N \left[\begin{array}{c|c} (\mathrm{e}_M) & \\ . & \mathrm{y}(1\text{-}\mathrm{x})^g(1+x)^w \\ (\mathrm{f}_N) & \end{array} \right]$$

$$S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}\begin{pmatrix} \mathrm{y}_1(1-x)^{f_1}(1+x)^{w_1} \\ \cdots \\ \cdots \\ \mathrm{y}_u(1-x)^{f_u}(1+x)^{w_u}\end{pmatrix} \daleth \begin{pmatrix} \mathrm{Z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdots \\ \cdots \\ \mathrm{Z}_r(1-x)^{h_r}(1+x)^{k_r}\end{pmatrix} \tag{5.1}$$

Substituting the value of $f(x)$ in (4.8), we obtain the formal solution about this problem

$$C(x,t) = 2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{k=0}^{\upsilon}\frac{\Gamma(\alpha+\beta-\gamma+2\upsilon+3)\Gamma(\alpha+\beta-\gamma+\upsilon+2+k)}{\Gamma(\alpha+\beta-\gamma+2\upsilon+2)\Gamma(\alpha-\gamma+k+2)}\frac{(-\upsilon)_k}{\Gamma(\beta+\upsilon+1)k!}$$

$$P_\upsilon^{(\alpha+\gamma+1,\beta)}(x) exp\big[\lambda_1(\upsilon+1)(1+\alpha+\beta-\gamma+\upsilon)t\big] F^{\alpha,\beta,\gamma,\rho,\sigma}(k; N_1, M_1; \cdots; N_u, M_u) \tag{5.2}$$

where

$$F^{\alpha,\beta,\gamma,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty} B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!}\, y_1^{K_1}\cdots y_u^{K_u} 2^{gq+wq+\sum_{i=1}^u (f_i+w_i)K_i}$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c} Z_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ Z_r 2^{h_r+k_r} \end{array}\right| \begin{array}{c} \mathbb{A};\,(\gamma-\rho-\alpha-1-k-gq-\sum_{i=1}^u f_iK_i:h_1,\cdots,h_r:1), \\ \cdot \\ \cdot \\ \cdots \end{array}$$

$$\left.\begin{array}{c} (\text{-}\sigma-\beta-wq-\sum_{i=1}^u w_iK_i:k_1,\cdots,k_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ B;\mathbf{B},\,(\gamma-2-k-\sigma-\rho-\alpha-\beta-(g+w)q-\sum_{i=1}^u(f_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r;1):B \end{array}\right) \tag{5.3}$$

Provided that

$min\{g,w,f_i,w_i,h_l,k_l\} > 0$ for $i=1,\cdots,u;l=1,\cdots,r$

$$Re(\rho+\alpha-\gamma+\sum_{i=1}^v K_if_i)+\sum_{i=1}^r h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1 \text{ and}$$

$$Re(\sigma+\beta+\sum_{i=1}^v K_iw_i)+\sum_{i=1}^r k_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1, Re(\beta) > -1$$

$$\left|argZ_i(1-x)^{h_i}(1+x)^{k_i}\right| < \frac{1}{2^{h_i+k_i+1}}, A_i^{(k)}.$$

6. Particular cases

In this section, we shall study two particular cases and we obtain the formal solution concerning the problems.

(i) If $\gamma=\alpha+\beta+1$ in (2.3), then the solvent moves with the constant velocity in the direction of $\mathbf{u}$ and the the equation (2.3) reduces to

$$\frac{1}{\lambda_1}\frac{\partial C}{\partial t} = (1-x^2)\frac{\partial^2 C}{\partial x^2}+2(\beta-x)\frac{\partial C}{\partial x} \tag{6.1}$$

and the solution is

Corollary 1.

$$C(x,t) = 2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{k=0}^{\upsilon}\frac{\Gamma(2\upsilon+2)\Gamma(\upsilon+1+k)}{\Gamma(2\upsilon+1)\Gamma(1-\beta+k)}\frac{(-\upsilon)_k}{\Gamma(\beta+\upsilon+1)k!}$$

$$P_\upsilon^{(-\beta,\beta)}(x)exp\left[\lambda_1(\upsilon+1)\upsilon t\right]F^{\alpha,\beta,\alpha+1,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) \tag{6.2}$$

where

$$F^{\alpha,\beta,\alpha+1,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty} B' \frac{[(e_M)]_q y^q}{[(f_N)]_q q!} \; y_1^{K_1}\cdots y_u^{K_u} 2^{gq+wq+\sum_{i=1}^{u}(f_i+w_i)K_i}$$

$$\daleth^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|l} Z_1 2^{h_1+k_1} & \mathbb{A};\ (\beta-\rho-k-gq-\sum_{i=1}^{u} f_iK_i : h_1,\cdots,h_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_r 2^{h_r+k_r} & \cdots \end{array}\right.$$

$$\left.\begin{array}{l} (\text{-}\sigma-\beta-wq-\sum_{i=1}^{u} w_iK_i : k_1,\cdots,k_r;1), \mathbf{A} : A \\ \cdot \\ \cdot \\ B;\mathbf{B},\ (\text{-k-1-}\sigma-\rho-(g+w)q-\sum_{i=1}^{u}(f_i+w_i)K_i : h_1+k_1,\cdots,h_r+k_r;1) : B \end{array}\right) \tag{6.3}$$

Provided that

$min\{g,w,f_i,w_i,h_l,k_l\} > 0$ for $i=1,\cdots,u; l=1,\cdots,r$

$$Re(\rho-\beta+\sum_{i=1}^{v} K_if_i)+\sum_{i=1}^{r} h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1 \text{ and}$$

$$Re(\sigma+\beta+\sum_{i=1}^{v} K_iw_i)+ \quad \sum_{i=1}^{r} k_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1,$$

$$\left|argZ_i(1-x)^{h_i}(1+x)^{k_i}\right| < \frac{1}{2^{h_i+k_i+1}}, A_i^{(k)}.$$

(ii) If we put $\gamma=\alpha+1$ and $\beta=0$ in (2.3), the medium is stationary that is $\mathbf{u}=0$ and the equation (2.3) reduces to the following equation

$$\frac{1}{\lambda_1}\frac{\partial C}{\partial t} = (1-x^2)\frac{\partial^2 C}{\partial x^2} - 2x\frac{\partial C}{\partial x} \tag{6.4}$$

and the solution is

Corollary 2

$$C(x,t) = 2^{\rho+\sigma}\sum_{\upsilon=0}^{\infty}\sum_{k=0}^{\upsilon}\frac{\Gamma(2\upsilon+2)\Gamma(\upsilon+1+k)}{\Gamma(2\upsilon+1)\Gamma(1-\beta+k)}\frac{(-\upsilon)_k}{\Gamma(\upsilon+1)k!}$$

$$P_\upsilon(x)exp\left[\lambda_1(\upsilon+1)\upsilon t\right]F^{\alpha,0,\alpha+1,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) \tag{6.5}$$

where

$$F^{\alpha,0,\alpha+1,\rho,\sigma}(k;N_1,M_1;\cdots;N_u,M_u) = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_u=0}^{[N_u/M_u]}\sum_{q=0}^{\infty} B'\frac{[(e_M)]_q y^q}{[(f_N)]_q q!} y_1^{K_1}\cdots y_u^{K_u} 2^{gq+wq+\sum_{i=1}^u (f_i+w_i)K_i}$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c} \mathrm{Z}_1 2^{h_1+k_1} & \mathbb{A};\ (\text{-}\rho-k-gq-\sum_{i=1}^u f_iK_i:h_1,\cdots,h_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdots \\ \mathrm{Z}_r 2^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\sigma-wq-\sum_{i=1}^u w_iK_i:k_1,\cdots,k_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ B;\mathbf{B},\ (\text{-k-1-}\sigma-\rho-(g+w)q-\sum_{i=1}^u (f_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r;1):B \end{array}\right) \tag{6.6}$$

Provided that

$min\{g,w,f_i,w_i,h_l,k_l\}>0$ for $i=1,\cdots,u;l=1,\cdots,r$

$$Re(\rho+\sum_{i=1}^v K_if_i)+\sum_{i=1}^r h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1 \text{ and}$$

$$Re(\sigma+\sum_{i=1}^v K_iw_i)+ \sum_{i=1}^r k_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1,$$

$$\left|argZ_i(1-x)^{h_i}(1+x)^{k_i}\right|<\frac{1}{2^{h_i+k_i+1}},A_i^{(k)}.$$

Remarks

We obtain the same formal solutions concerning the multivariable Aleph- function defined by Ayant [1], the multivariable I-function defined by Prathima et al. [9], the multivariable I-function defined by Prasad [8] and the multivariable H-function defined by Srivastava and Panda [12,13], see Kumar and P.S.Y. Satyarth [6] for more details concerning the multivariable H-function

7. Conclusion

Specializing the parameters of the multivariable Gimel-function, the generalized hypergeometric function and the multivariable polynomials, we can obtain a large number of results of problem of advective-diffusion process involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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