On Vector Sequence Spaces and Representation of Compact Operators on BK Spaces

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Abstract

The study gives some properties of compact operators between Banach-Koordinat (BK) spaces. Further the study looks at the vector sequence spaces associated with these spaces. Finally, we characterize compact operators through a BK space is studied and also characterizes the spaces of compact linear maps from locally convex spaces into BK spaces in terms of certain subspaces of the generalized sequence spaces. These characterizations are vital in the proofs of representations of the BK spaces used in the generalizations of the classical results on Spaces of Compact operators and their Dual spaces.

Keywords: *BK space, compact operator, sequence space.*

I. INTRODUCTION AND NOTATION

The theory of operator ideals is playing an increasingly important role in the theory of locally convex spaces. At present, a good deal of research in Functional Analysis, a branch of mathematical analysis concerned with the study of spaces of functions and operators acting on them, is devoted to a classification problem: what is the relation between a different types of locally convex spaces which occur in Functional Analysis? The answers to such and related questions usually lead to a clear understanding of the structures of the spaces under considerations, and in this paper operator ideal are important tools.

The theory of operator ideals has been developed in [8, 17]. In [2] the importance of this theory to nuclear spaces was pointed out by the author in a statement: ‘I like to consider the theory of nuclear spaces to be to a large extent of a beautiful application of the theory of ideals of operators between Banach spaces’. Nuclear spaces can be characterized as dense subspaces of projective limits of Banach spaces with nuclear linking mapping. In a similar manner, Schwartz spaces can be characterized as dense subspaces of projective limits of Banach spaces with compact linking mapping.

In the present paper vector sequence spaces and representation of operators which factor compactly through BK spaces are studied and characterized. The results are applied to the study of locally convex spaces which are subspaces of projective limits of BK spaces with compact linking mapping; and to the study of locally convex spaces which are subspaces of projective limits of subspaces of BK spaces with compact linking mapping.

Important objects of study in this paper are the continuous linear operators defined on Banach and Hilbert spaces. These lead naturally to the definition of C*-algebras and other operators algebras.

Let Ω be an ideal of operators on the family of all Banach spaces. The concept of locally convex Ω-space is introduced in section 1. We refer to [1] and [2] for more details concerning these spaces. In these references it is clearly shown how the permanence properties of these spaces depend on the properties of Ω. Let N^1 and K denote the ideals of all nuclear and compact operators between Banach spaces, respectively. The family of all operators which factor compactly through a given BK space lies between these two ideals. It is known that the N^1 spaces are the nuclear spaces and the K spaces are the Schwartz spaces. The motivation for our study is guided by the need to seek to know which spaces are defined by the family of all operators which factor compactly through a given BK space, and what is the relation between these spaces and other known locally convex spaces. To answer this, a study of the operators themselves is necessary, and as these operators turn out to be very interesting and important, this family of operators finally became the main object this paper.

Unless otherwise stated, we shall use E, F, G, etc, to denote locally convex spaces and X, Y, Z to denote Banach spaces (B-spaces). As far as locally convex spaces and duality theory are concerned, we shall adhere to the notations of [2] and [13].

All vector spaces under consideration will be vector spaces with respect to $K \in \{\mathbb{R}, \mathbb{C}\}$, where $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively.
All locally convex spaces will tacitly be assumed to be separated. The family of all these spaces will be denoted by **LCS.** $\left(X, \|\cdot\|_X\right)$ and $\left(Y, \|\cdot\|_Y\right)$ to denote the B-spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively.
For any $(E,\pi) \in LCS$ the base of neighbourhoods of the origin (0-neighbourhood base) consisting of all absolutely convex closed neighbourhoods of the origin is denoted by $\bigcup(E,\pi)$. Consider a B-space $\left(X, \|\cdot\|_X\right)$. The closed unit ball in X will be denoted by B_X. Let E be a vector pace. By ΓA we denote the absolutely convex hull of a set $A \subseteq E$. Consider absolutely convex subsets A and B in E. If A is absorbed by B, we write $A \prec B$. By q_A we denote the seminorm on the vector subspace $E_A = \bigcup_{n=1}^{\infty} nA$ of E, defined by $q_A(X) := \inf\{\lambda > 0 \mid x \in \lambda A\}$ for all $x \in E_A$.

II. VECTOR SEQUENCE SPACES AND MATRIX MAPS OF BK SPACES

We define the following vector sequence spaces where Λ is a normal BK space with AK:

$$\Lambda(E) := \left\{(x_n) \in E^N \mid \left(\langle a, x_n\rangle\right)_n \in \Lambda \quad \text{for all } a \in E'\right\} \tag{2.1}$$

$$\Lambda(E') := \left\{(a_n) \in E'^N \mid \left(\langle a_n, x\rangle\right)_n \in \Lambda \quad \text{for all } x \in E\right\} \tag{2.2}$$

Here E^N *and* E'^N denote the set of all sequences in E and E' respectively. These vector sequence spaces are of fundamental importance.

Theorem 2.1. *Let* $(a_n) \in \Lambda(E')$ *.Then*

$$W(a_n, B_\Lambda x) := \left\{x \in E \mid \sum_{i=1}^{\infty} \left|\lambda(i)\langle a_i, x\rangle\right|\right\} \leq 1 \ \text{ for all } \lambda \in B_\Lambda x \tag{2.3}$$

is a barrel in E.

Proof. Clearly $W(a_n, B_\Lambda x)$ is absolutely convex. Let $x \in \overline{W(a_n, B_\Lambda x)}^{\sigma}$, the $\sigma(E,E')$-closure of $W(a_n, B_\Lambda x)$. If $\left(x^{(v)}\right)_v$ is a net in $W(a_n, B_\Lambda x)$, which converges to x, then $\sum_{i=1}^{k} \left|\lambda(i)\langle a_i, x\rangle\right| = \sum_{i=1}^{k} \left|\lambda(i)\left(\lim\langle a_i, x^v\rangle\right)\right| \leq 1$ for every $\lambda \in B_\Lambda x, k \in \mathbb{N}$. Hence $W(a_n, B_\Lambda x)$ is $\sigma(E,E')$-closed. Further, let $y \in E$. For any $\lambda \in B_\Lambda x$ holds $\sum_{i=1}^{\infty} \left|\lambda(i)\langle a_i, y\rangle\right| \leq \left\|\left(\langle a_n, y\rangle\right)_n\right\|_\Lambda (= M,\ say)$. Hence $y \in M\ W(a_n, B_\Lambda x)$, which proves that $W(a_n, B_\Lambda x)$ is absorbing.

□

Theorem 2.2. *Let E be a barreled space. If B is a bounded set in E and* $(a_n) \in \Lambda(E^{'})$ *is given, there exists a* $\rho \geq 0$ *such that* $\sum_{i=1}^{\infty} |\lambda(i)\langle a_i, x\rangle| \leq \rho$ *for all* $x \in B$ *and* $\lambda \in B_{\Lambda} x$. □

Proof: Consider $W(a_n, B_{\Lambda} x)$ as defined in (3.1) .Clearly $W(a_n, B_{\Lambda} x)$ is 0-neighborhood in E.

□

In connection with the AK-property for the vector sequence spaces $\Lambda(E)$ and $\Lambda(E^{'})$, it is natural to consider the following subspaces of these two spaces, respectively:

$$\Lambda_c(E) = \{(x_n) \in \Lambda(E) \mid \varepsilon_{\Lambda,U}((x_i)_{i \geq n}) \to 0 \text{ if } n \to \infty, \text{ for all } U \in U(E)\} \tag{2.4}$$

$$\Lambda_c(E^{'}) = \{(a_n) \in \Lambda(E^{'}) \mid \varepsilon_{\Lambda,B}((a_i)_{i \geq n}) \to 0 \text{ if } n \to \infty, \text{ for all } B \in B(E)\} \tag{2.5}$$

$\Lambda(E)$ will have AK if and only if $\Lambda(E) = \Lambda_c(E^{'})$.

Similarly, if $\varepsilon_{\Lambda,B}$ defines a seminorm on $\Lambda(E^{'})$ for all $B \in B(E)$ when E is a barreled space, for instance then $\Lambda(E^{'})$ has AK if and only if $\Lambda(E^{'}) = \Lambda_c(E^{'})$.
Extending this principles to BK space, we recall that a Banach Space E is a BK space if it is a vector subspace ω and $p_n \in E^{'}$ *for all* $n \in \mathbb{N}$. c_0 and l_{∞}, with their usual meaning, are BK spaces. Let $x, y \in \omega$ and $A = (a_{n,k})_{n,k=1}^{\infty}$ be an infinite array of scalars. We write $y = A(x)$ if $y_n = \sum_{k=1}^{\infty} a_{n,k} x_k$ for every $n \in \mathbb{N}$. If E and F are BK spaces and $A(x) \in E$ for every $x \in F$, then A defines a continuous linear map from F to E and we call A a matrix map from F into E. If $A(B_F)$ is relatively compact in E, we say that A is a compact map.
If $A = (a_{n,k})_{n,k=1}^{\infty}$ defines a matrix map from F into E, we let $a^k = A(e^k)$ denote the k^{th} column of A. Noting that E and F denote BK spaces and A denote the array $(a_{n,k})_{n,k=1}^{\infty}$ we state an important lemma.

Lemma 2.3. *Suppose A defines a matrix map from* l_{∞} *into E. if A is weakely compact, then* $\sum_{k=1}^{\infty} a^k$ *is unconditionally convergent in E.*

Proof. We show that $\sum_{k=1}^{\infty} a^k$ is weakly subseries convergent and hence, via the Orlicz-Pettis theorem, unconditionally convergent.

Let $\theta = (\theta_k)$ be an arbitrary sequence of 0's and 1's, and consider $\sum_{k=1}^{\infty} \theta_k a^k$.

For observe that

$$\begin{aligned} C &= \left\{ A\left(\sum_{k=1}^{n} \theta_k e^k \right) : n \in \mathbb{N} \right\} \\ &= \left\{ \sum_{k=1}^{n} \theta_k a^k : n \in \mathbb{N} \right\} \end{aligned} \tag{2.6}$$

is a relatively weakly compact set in E since $\left\{ \sum_{k=1}^{n} \theta_k e^k : n \in \mathbb{N} \right\}$ is bounded in l_∞. Since E is a BK space, we have that E's weak topology and the topology of coordinatewise convergence inherited from ω are equivalent on the weak closure of C: this follows from the fact that comparable Hausdorff topologies on a compact set are equal. We claim that the coordinatewise limit point, hence the limit point, of C is $A(\theta)$. Observe that $x = \sum_{k=1}^{\infty} \theta_k a^k$ coordinatewise if and only if $x_n = \lim_\eta \sum_{k=1}^{\eta} \theta_k a_{n,k}$ for all $n \in \mathbb{N}$ since $p_n\left(a^k\right) = a_{n,k}$.

Now since $\theta \in l_\infty$, $A(\theta) = \left(\sum \theta_k a_{n,k} \right)_{n=1}^{\infty}$ is an element of E and $A(\theta)_n = \lim_\eta \sum_{k=1}^{\eta} \theta_k a_{n,k} = \sum_{k=1}^{\infty} \theta_k a_{n,k}$ it follow that $A(\theta)$ is the coordinatewise, hence weak, limit point of C. Observe that C can only have one coordinatewise limit point. Since θ was arbitrary, we now have that $\sum_{k=1}^{\infty} a^k$ is weakly subseries convergent and hence unconditionally convergent. □

Lemma 2.4. *If* $\sum_{k=1}^{\infty} a^k$ *is unconditionally convergent in E, then A defines a matrix map from* l_∞ *into E and* $A(\beta) = \sum_{k=1}^{\infty} \beta_k a^k$ *for every* $\beta = (\beta_k)_{k=1}^{\infty} \in l_\infty$.

Proof. Let $\beta = (\beta_k) \in l_\infty$. First we observe that $\sum \beta_k a^k$ converges since $\sum a^k$ is unconditionally convergent, and note that

$$p_n\left(\sum_{k=1}^{\infty} \beta_k a^k \right) = \sum_{k=1}^{\infty} a_{n,k} \beta_k = p_n\left(A(\beta) \right) \textit{ for every } n \in \mathbb{N}. \tag{2.7}$$

Now, since $\{p_n\}_{n=1}^{\infty}$ is total over E , it follows that $A(\beta) = \sum \beta_k a^k$.

Since β was arbitrary, we have that $A(\beta) \in E$ for every $\beta \in l_\infty$, hence A defines a matrix map from l_∞ into E. □

Theorem 2.5. *If* $A : l_\infty \to E$ *is a weakly compact matrix map, then A is compact.*

Proof. Here, it suffices to show that, for every $\varepsilon > 0$ there is a compact set K_ε in E such that $A\left(B_{l_\infty}\right) \subseteq K_\varepsilon + \varepsilon B_E$. Let $\beta = (\beta_k) \in l_\infty$ and recall that, for every $\delta > 0$, there is an N_δ such that

$$\left\| \sum_{k=n}^{\infty} \beta_k a^k \right\|_E < \delta \sup_k |\beta_k| \qquad (2.8)$$

For all $n \geq N_\delta$, since $\sum_{k=1}^{\infty} a^k$ I unconditionally convergent.

Now suppose that $A\left(B_{l_\infty}\right) \subseteq MB_E$, $M > 0$, and $\delta = \varepsilon M^{-1}$. Select N such that (3.8) is satisfied and let F_N be the linear span of $\left\{a^k\right\}_{k-1}^{N}$. Note that $K_\varepsilon = F_N \bigcap MB_E$ is compact since F_N is finite dimensional and that

$$\begin{aligned} A\left(B_{l_\infty}\right) &= \left\{ \sum_{k=1}^{\infty} \beta_k a^k : (\beta_k) \in B_{l_\infty} \right\} \\ &\subseteq K_\varepsilon + \delta M^{-1} B_E \\ &= K_\varepsilon + \varepsilon B_E \end{aligned}$$

which completes the proof since ε was arbitrary. □

Rosenthal [18] has shown that any continuous linear operator from l_∞ into a Banach space that does not contin an isomorphic copy of l_∞ must be weakly compact. This in tandem with theorem 3.1.5 yields the following corollary.

Corollary 2.6. *Let E be a BK space that does not contain an isomorphic copy of l_∞. If A is a matrix map from l_∞ into E, then A is compact.*

III. CHARACTERIZATION OF COMPACT OPERATORS THROUGH A BK SPACE

In this chapter we characterize some classes of compact operators on the space $c_0(\Lambda), c(\Lambda)$ *and* $c_\infty(\Lambda)$. This is most efficiently achieved by applying the Hausdorff measure of noncompactness. We also give a view of the results concerning general bounded linear operators and their compactness as introduced by Eberhard Malkowsky . We embrace notations and results as used by [1, 5, 15]. If X and Y are infinite-dimensional complex Banach spaces, then a linear operator $L : X \to Y$ is said to be compact if the domain of L is all of X and Y, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence. We write $C(X,Y)$ for the class of all compact operators form X into Y. we recall that a set in a topological space is said to be *precompact* or *relatively compact* if its closure is compact. The first measure of of noncompactness, the function α, was defined and studied by Kuratowski in 1930. In 1955, Darbo was the first who continued to use the function α. He proved that if T is continuous self-mapping of a non-empty, bounded, closed and convex subset C of a Banach space X such that there exists a constant $K \in (0,1)$ such that $\alpha(T(Q)) \leq K\alpha(Q)$ for all subsets Q of C, then T has at least one fixed point in the set *C*. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and included the existence part of Banach's fixed point theorem.

Other measures of noncompactness were introduced and studied by Gohberg, Goldenstein and Markus in 1957 , the ball or Hausdorff measure of noncomapactness, and by Istratesku in 1972.

The following result due to Gohberg, Goldenstein and Markus gives an explicit estimate for the Hausdorff measure of noncompactness of any bounded set in a Banach space with a Schauder basis.

Theorem 3.1. *Let X be a Banach space with a Schauder basis* $(b_n)_{n=1}^{\infty}$, $P_n : X \to X$ *be the projector onto the linear span,* $span(\{b_1, b_2, ..., b_n\})$ *of* $b_1, b_2, ..., b_n$, $R_n = I - P_n$, *where I is the identity on X, and* $Q \in M_X$. *Then we have, writing* $\mu_n(Q) = \sup_{x \in Q} \|R_n(x)\|$ *for* $n = 1, 2, ...,$

$$\frac{1}{a}.\limsup_{n\to\infty} \mu_n(Q) \le \chi(Q) \le \limsup_{n\to\infty} \mu_n(Q) \text{ where } a = \limsup_{n\to\infty} \|R_n\|. \tag{3.1}$$

We obtain an explicit formula for the Hausdorff measure of noncompactness of bounded subsets of special BK spaces, and of *c*. We say that a norm $\|.\|$ on a sequence space X is *monotonous*, if $x, x' \in X$ with $|x_k| \le |x_k'|$ for all k implies $\|x\| \le \|x'\|$, and call such a space X monotonous.

Theorem 3.2. *Let X be a monotonous BK space with AK and* $P_n : X \to X$ *be the projectors onto* $span(\{e^{(1)}, e^{(2)}, ..., e^{(n)}\})$ *for* $n \in \mathbb{N}$ *.Then we have*

$$\chi(Q) = \lim_{n\to\infty} \mu_n(Q) \text{ for all } Q \in M_X \tag{3.2}$$

Corollary 3.3. $P_n : c \to c$ *be the projector onto the linear span of* $\{e^{(1)}, e^{(2)}, ..., e^{(n)}\}$. *Then the limit of (4.2) exists for all* $Q \in M_c$, and

$$a = \lim_{n\to\infty} \|R_n\| = 2. \tag{3.3}$$

In a general sense, we state some properties of $\|.\|_X$.

Theorem 3.4. *Let X and Y be Banach spaces and* $L \in B(X, Y)$. *Then:*

a. $\|L\|_X = \chi(L(B_X))$

b. $\|.\|_X$ *is a seminorm of* $B(X, Y)$, *and* $\|L\|_X = 0$ *if and only if* $L \in C(X, Y)$

If X is a BK space with AK and $L \in B(X, Y)$, let $A \in (X, Y)$ denote the matrix with $Ax = L(x)$ for all $x \in X$ also let $L \in B(X, c)$ and $\tilde{A}$ be the matrix with the row $\tilde{A}_n = A_n - (\alpha_k)_{k=1}^{\infty} (n = 1, 2, ...)$, where $\alpha_k = \lim_{n\to\infty} a_{nk}$ for every $k \in \mathbb{N}$ and $(\alpha_k)_{k=1}^{\infty} \in X^{\beta}$.

Then we have

$$\frac{1}{2}.\lim_{r\to\infty}\left(\sup_{n\ge r} \|\tilde{A}_n\|_X^*\right) \le \|L\|_X \le \lim_{r\to\infty}\left(\sup_{n\ge r} \|\tilde{A}_n\|_X^*\right).$$

Similarly, if $L \in B(X, c_0)$, then we have

$$\|L\|_X = \lim_{r\to\infty}\left(\sup_{n\ge r} \|\tilde{A}_n\|_X^*\right).$$

One of the most important results in the theory of sequence spaces states that matrix transformations between BK spaces are continuous.

The spaces $c_0(\Lambda), c(\Lambda)$ and $c_\infty(\Lambda)$ were defined a studied explicitly by [17] for exponentially bounded sequences Λ.

A nondecreasing sequence $\Lambda = (\lambda)_{n=1}^{\infty}$ of positive reals to be exponentially bounded sequences if there exists an integer $m \geq 2$ such that for each $v \in \mathbb{N}_0$ there is at least one λ_n in the interval $\left[m^v, m^{v+1}\right)$. The following result is a useful characterization of exponentially bounded sequences.

Theorem 3.5. *Let* $\Lambda = (\lambda)_{n=1}^{\infty}$ *be an exponentially bounded sequence and* $\Lambda = \left(\lambda_{n(v)}\right)_{v=0}^{\infty}$ *be an associated subsequence.*

a. *Then* $c_0(\Lambda) = \tilde{c}_0(\Lambda), c(\Lambda) = \tilde{c}(\Lambda)$ and $c_\infty(\Lambda) = \tilde{c}_\infty(\Lambda)$. *Where the sets are defined as*

$$\tilde{c}_0(\Lambda) = \left\{x \in \omega : \lim_{n\to\infty}\left(\frac{1}{\Lambda_n}\sum_{k=1}^{n}\left|\Delta_k(\Lambda.x)\right|\right) = 0\right\},\ c_0(\Lambda) = \left\{x \in \omega : \lim_{v\to\infty}\left(\frac{1}{\lambda_{n(v+1)}}\sum\nolimits_v\left|\Delta_k(\Lambda.x)\right|\right) = 0\right\},$$

$$\tilde{c}_\infty(\Lambda) = \left\{x \in \omega : \sup_n\left(\frac{1}{\Lambda_n}\sum_{k=1}^{n}\left|\Delta_k(\Lambda.x)\right|\right) < \infty\right\},\ c_\infty(\Lambda) = \left\{x \in \omega : \sup_v\left(\frac{1}{\lambda_{n(v+1)}}\sum\nolimits_v\left|\Delta_k(\Lambda.x)\right|\right) < \infty\right\},$$

$$\tilde{c}(\Lambda) = \left\{x \in \omega : x - \xi.e \in \tilde{c}_0(\Lambda) \text{ for some } \xi \in \mathbb{C}\right\}, \text{ and } c(\Lambda) = \left\{x \in \omega : x - \xi.e \in c_0(\Lambda) \text{ for some } \xi \in \mathbb{C}\right\}$$

b. *The block and sectional norms* $\|.\|_b$ and $\|.\|_s$ *defined by*

$\|x\|_b = \sup_v \frac{1}{\lambda_{n(v+1)}}\sum_v\left|\Delta_k(\Lambda.x)\right|$ and $\|x\|_s = \sup_n \frac{1}{\Lambda_n}\sum_{k=1}^{n}\left|\Delta_k(\Lambda.x)\right|$ *are equivalent on* $c_0(\Lambda), c(\Lambda)$ and $c_\infty(\Lambda)$, *more precisely* $\|x\|_b \leq \|x\|_s \leq K(s,t).\|x\|_b$.

c. *Each of the spaces* $c_0(\Lambda), c(\Lambda)$ and $c_\infty(\Lambda)$ *is a BK space,* $c_0(\Lambda)$ *is a closed subspace of* $c(\Lambda)$, $c(\Lambda)$ *is a closed subspace of* $c_\infty(\Lambda)$, $c_0(\Lambda)$ *has AK, and every sequence* $x = (x_k)_{k=1}^{\infty} \in c(\Lambda)$ *has a unique representation* $x = \xi.e + \sum_{k=1}^{\infty}(x_k - \xi)e^{(k)}$.

d. *The space* $c_\infty(\Lambda)$ *has no Schauder basis.*

If *X* is a normed sequence space and $a \in \omega$ then we write $\|a\| = \|a\|_X^* = \sup\left\{\left|\sum_{k=1}^{\infty} a_k x_k\right| : x \in S_X\right\}$ provided the last term exist and is finite which is the case whether $X \supset \phi$ is a BK space and $a \in X^\beta$.

Theorem 3.6. We have $\left(c_0(\Lambda)\right)^\alpha = \left(c(\Lambda)\right)^\alpha = \left(c_\infty(\Lambda)\right)^\alpha = \ell_1$ *and* $\left(c_0(\Lambda)\right)^{\alpha\alpha} = \left(c_\infty(\Lambda)\right)^{\alpha\alpha} = \ell_\infty$. We put $C(\Lambda) = \left\{a \in \omega : \sum_{v=0}^{\infty}\lambda_{n(v+1)}\max_v\left|\sum_{j=k}^{\infty}\frac{a_j}{\lambda_j}\right| < \infty\right\}$ and write $\|a\|_{C(\Lambda)} = \sum_{v=0}^{\infty}\lambda_{n(v+1)}\max_v\left|\sum_{j=k}^{\infty}\frac{a_j}{\lambda_j}\right|$.

Theorem 3.7. *Let* $A \in \left(c\left(\Lambda\right), c\right)$. *Then we have*

$$\frac{1}{2}\lim_{n\to\infty}\left(\left|\sum_{k=1}^{\infty}\alpha_k - \alpha\right| + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right) \le \|A\|_\chi \le \lim_{n\to\infty}\left(\left|\sum_{k=1}^{\infty}\alpha_k - \alpha\right| + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right),$$ *where* $\alpha_k = \lim_{n\to\infty} a_{nk}$ *for each* $k \in \mathbb{N}$, $\alpha = \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}$ *and* $\tilde{A}_n = A_n - \left(\alpha_k\right)_{k=1}^{\infty}$ *for all* $n \in \mathbb{N}$.

Proof. This is an immediate consequence to theorem 6.0 of [15] with $b_n = 0$ for $n = 1, 2, \ldots$ and the fact that for $A \in \left(c\left(\Lambda\right), c\right) \subset \left(c_0\left(\Lambda\right), c\right)$, we have $\left\|\left(L_A\right)_n\right\| = \|A_n\|_{C(\Lambda)}$ for all n, so we get $K\left(s,t\right) = 1$ and equality in [22].□

Corollary 3.8. *Let* $L \in B\left(c\left(\Lambda\right), c_0\right)$. *Then we have*

$$\lim_{n\to\infty}\left(|b_n| + \|A_n\|_{C(\Lambda)}\right) \le \|L\|_\chi \le K\left(s,t\right).\lim_{n\to\infty}\left(|b_n| + \|A_n\|_{C(\Lambda)}\right).$$

Proof. This is an immediate consequence of theorem 4.1.11 with $\beta = 0$ *and* $\alpha_k = 0$ for all *k*. □

It turns out that a strongly regular matrix cannot be compact; that is analogous to the classical result by Aywa and Fourie which strongly indicates that a regular matrix cannot be compact. Importantly we point out that the spaces $c_0\left(\Lambda\right), c\left(\Lambda\right)$ and $c_\infty\left(\Lambda\right)$ have the block norm.

IV. CONCLUSION

In this paper we have established some new results on vector sequence spaces and matrix maps of BK spaces. The approach we have envisaged is via the spaces of convergent and bounded sequences and characterization of compact operators through a BK space. The result helps us to see how vital the compact bounded operators and vector sequence spaces are to the BK spaces.

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