Glorious Era of Glorious Glitterness in Golden Graphs

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Abstract:
A relentless endeavour to traverse deeply into the details of definition of golden graphs are acquiring shape as concrete projects, which serve as a building block for solid structures of co-operation of pure golden graphs. It makes us mesmerized to see the seeds of transforming the characterized for a cyclic groups as pure golden graphs. Embarking on the wonderous journey generated, infinite class of p_n, a path on 'n' nodes as golden graphs. The destinies of this are embedded in more than one way.

Keywords—*Trees, Paths, adjacency matrix, characteristic polynomial of a graph **H** and Golden ratio.*

I. INTRODUCTION

In the pursuit of weaving the threads into a rich mathematical diaspora it is inevitable to rediscover and reinvent its historic evidence of the appearance of Golden ratio is graph theory; This paper binds together and there by brings so many diverse conditions emerging of different stages is connection with chromatic polynomials. W.T Tutte(1970), Michel O Alberston(1973),Saeid Alikhani and Yee-hock(2009) all immersed with GR in profound connection with chromatic polynomials. Pavel Chebotarev(2008) ,deal with GR in connection with spanning forest . We are giving an account of GR in graph which we came across while studying the spectral properties of graphs. While studying the spectra of P_4 ,path on 4 vertices ,we find that its Eigen values are $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$ which are nothing but Golden ratio (Divine ratio). Mesmerisingly, we put forward the question which graphs have Eigen values as Golden ratio. In this paper we have proved logically that, there are infinite class P_n , a path on n nodes which have GR as Eigen value and path P_4 is the only pure golden tree.

II. PRELIMINARIES

Let H be a graph devoid of loops or multiple links having n nodes. Then the adjacency matrix of H , A(H) = A, is a square matrix, symmetric matrix of order n ,whose elements A_{ij} are ones or zeros if the corresponding nodes are adjacent or not, respectively. This matrix has (not necessarily distinct) real-valued Eigen values ,which are indeed denoted by $\lambda_1, \lambda_2, ..., \lambda_n$. The set of Eigen values of A together with their multiplicities form the diaspora spectrum of H ,which will be represented here as Spec (G) $=\{[\lambda_1]^{m}{}_{1}, [\lambda_2]^{m}{}_{2} ...... \lambda_n]^{m}{}_{n}\}$ where λ_i is the i^{th} Eigen value with m_i multiplicity. Here the Eigen values are assumed to be labelled in a non-increasing manner .
$\lambda_1 \geq \lambda_2, \geq ...... \geq \lambda_n$.
Part of its appeal lay in the way we combined ancient wisdom coupled with contemporary insights. Let P_n, C_n ,K_n be the path graph .the cycle graph and the complete graph on n nodes, respectively .The path P_n is a tree with two nodes of degree 1and the other two nodes with degree 2. A tree is a acyclic graph(with out cycles). A cycle C_n is a graph on n nodes containing a single cycle through all nodes'

Theorem2.1 [1]: Let $\lambda_1 \geq \lambda_2, \geq ...... \geq \lambda_n$.
be the Eigen values of the adjacency matrix of a simple graph H .Then the following are equivalent .

a) H is a bipartite graph.
b) For all $1 \leq i \leq n$, $\lambda_{n+1-i} = -\lambda_i$

Theorem2.2 [1]: Let ϕ (H, x) $C_0 x^n + C_1 x^{n-1} C_2 x^{n-2} + ... C_n$
be the characteristic polynomial of a graph G ,then the co-efficient of ϕ (H, x) satisfy:

a) $C_0 = 1$

b) $C_1 = o$

c) $-C_2$ is the number of edges in G.

d) $-C_3$ is twice the triangles in H.

***Theorem2.3*[1]**: Let H be a graph of order n ,size m and k be the number of components of H,then m ≥ n - k holds. Further ,the equality holds if and only if each component of H is a tree.

III. MAIN RESULTS.

We defined pure golden graph and Golden graph as follows,

Definition.3.1: A graph H is said to be Pure golden graph, if all the Eigen values of H are Golden ratios (i.e $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$)

Definition3.2: A graph H is said to be golden graph, if at least one of the Eigen values of H are Golden ratios.

Lemma.3.3: A graph H is a pure golden tree if and only if H = P_4 ,a path on four nodes.

Proof: Let H be a pure golden tree of order n ,size m and only Eigen values of H are

$$\lambda_1 = \frac{1+\sqrt{5}}{2},\ \lambda_2 = \frac{-1+\sqrt{5}}{2},\ \lambda_3 = \frac{1-\sqrt{5}}{2},\ \lambda_4 = \frac{-1-\sqrt{5}}{2}$$

Cleary $\lambda_1 = -\lambda_4$ & $\lambda_2 = -\lambda_3$.

By the property of a bipartite graph [Theorm:2.1] and hence for a tree, the multiplicities of λ_1, λ_4 be l and of λ_2, λ_3 be

k. Thus , the characteristic polynomial of H can be expressed as

$$\phi(G,x) = (x^2-\lambda_1{}^2)^l (x^2-\lambda_2{}^2)^k \ldots\ldots\ldots\ldots (1)$$

By expanding the equation (1) ,we have

$$\begin{aligned}\phi(\mathrm{H},x) &= \left[(x^2)^l - {}^lC_1(x^2)^{l-1}\lambda_1{}^2 + {}^lC_2(x^2)^{l-2}\lambda_1{}^4 - \ldots\right]\times \\ &\quad \left[(x^2)^k - {}^kC_1(x^2)^{k-1}\lambda_2{}^2 + {}^kC_2(x^2)^{k-2}\lambda_2{}^4 - \ldots\right] \\ &= x^{2l+2k} - \left[{}^kC_1\lambda_2{}^2 + {}^lC_1\lambda_1{}^2\right]x^{2l+2k-2} + \ldots\ldots\end{aligned}$$

By the properties of characteristic polynomial of graph [Theorm:2.2],

we have $n = 2l + 2k$, $m = {}^kC_1\lambda_2{}^2 + {}^lC_1\lambda_1{}^2$

But $\lambda_1{}^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$ & $\lambda_2{}^2 = \left(\frac{-1+\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$

And thus , $m = l\left(\frac{3+\sqrt{5}}{2}\right) + k\left(\frac{3-\sqrt{5}}{2}\right)$

$$= \frac{3l+3k}{2} + \frac{\sqrt{5}(l-k)}{2}$$

The equation , 2m = (3l + 3k) + $\sqrt{5}$(l - k) holds ,only when l - k = 0 .Since l , k and n are non-negative integers.

Thus , l = k must hold .This proves that all roots are of same multiplicities and hence the equation(1) becomes

$$\phi(\mathrm{H},x) = \left[(x^2-\lambda_1{}^2)(x^2-\lambda_2{}^2)\right]^l \ldots\ldots\ldots\ldots (2)$$

Thus, we have n = 4l & m = 3l .As H is a tree ,and therefore 3l = m = n -1 = 4l -1 ⇒l = 1.Thus , H is tree of order 4.

But only trees of order 4 are $K_{1,3}$ & P_4 .Hence, H must be P_4 ,as none of the Eigen values of $K_{1,3}$ are golden ratio.

Converse is obvious trivial.

Theorem.3.4: An acyclic graph H is pure golden if and only if every component of H is P_4 .
Proof: Suppose H is a pure golden graph of order n and size m .Now we assert that, each component of H is P_4 . On the similar argument arguments as in the first half of the proof of lemma 3.3, we have the characteristic polynomial
of G as

$$\phi(\mathrm{H},x)=\left[(x^2-\lambda_1{}^2)(x^2-\lambda_2{}^2)\right]^l$$

By the properties of the degree and co-efficient of ϕ (H, x) ,we have n = 4l & m= 3l .

Thus , m = n - l holds . Hcontains exactly l components and each of them is a tree[Theorem2.3] .But by the lemma.3.3 ,each component of H is P 4 and hence the assertion holds.
Converse is obvious trivial.

Theorem. 3.5: Pn is golden graph if and only if n = 5k -1.
Proof: Let Pn be a golden graph .We know that spectrum of

P_n is $2\cos\left(\frac{\pi k}{n+1}\right)$, where $k=1,2......n$.

Therefore $2\cos\left(\frac{\pi k}{n+1}\right)=\frac{1+\sqrt{5}}{2}$, *for* some k

$$\Rightarrow \quad \cos\left(\frac{\pi k}{n+1}\right)=\frac{1+\sqrt{5}}{4}$$

We know that, $\cos 36^\circ=\frac{1+\sqrt{5}}{4}$ = sin 54^0

$$\therefore \quad \frac{\pi k}{n+1}=2l\pi\pm\alpha \text{, where } \alpha\in I$$

$$\Rightarrow \quad \frac{\pi k}{n+1}=2l\pi\pm 36^\circ$$

$$\Rightarrow \quad \frac{k}{n+1}=2l\pm\frac{1}{5}$$

$$\Rightarrow \quad 5k=10l(n+1)\pm(n+1)$$

$$\Rightarrow \quad 5k=(n+1)(10l\pm 1)$$

∴ l = 0, because l being the number of full rotation and π representing only half of the rotation must be zero, the above equation becomes

$$\Rightarrow \quad 5k=\pm(n+1)$$

$$\Rightarrow \quad 5k=(n+1)$$

$$\Rightarrow \quad n=5k-1.$$

∴ If P_n is golden graph, then $n=5k-1$.

For the converse, it is enough to claim that ϕ (P_{5k-1}) is divisible by x^2 + x- 1.
We prove this by induction on k. If k - 1 , we get

$$\phi(P_{5k-1})=\phi(P_4)=(x^2+x-1)(x^2+x-1).$$

Hence true for k = 1.

Next, we assume that the result is true for k -1.This implies that

$$x^2+x-1\big/\phi(P_{5k-6}).$$

Now, $\phi(P_{5k-1})=\phi(P_4)\times\phi(P_{5k-5})-\phi(P_3)\times\phi(P_{5k-6})$.

Since by induction hypothesis, $x^2+x-1\big/\phi(P_{5k-6})$.

We see that $x^2+x-1\big/\phi(P_{5k-6})$.

Thus by induction $\phi(P_{5k-1})$ is divisible by x^2+x-1.

IV. CONCLUSION

The narrative of this paper is to in still and infuse the sizzling relationship between the characteristics for cyclic groups as pure graphs.Further it unfolds an eclectic range of conditions for establishing infinite class of P_n a path on 'n' nodes as golden graphs, through exemplorary proofs coupled with blended theorems in the above incredible paper

V. ACKNOWLEDGEMENT

This journal paper could not have been written without the help of numerous people who have been kind enough to comment on various transformation aspects of the paper.My former guide Mrs Radha rugmini and aide has guided me with affection in the pursuit by which she asked me always to point out the areas that she agrees with my depiction of concepts , but not with my rejection of mathematical faiths. Dr Ramdass, the dean and HOD of our department undoubtedly India's most popular interpreter of mathematical diaspora came up with a number of pertinent suggestions, on an earlier draft which have helped, influenced ,inspired and motivated me immensely in my final research version paper. Mr Paneerselvam sir , the everbrightening star have been a superb trend setter for me and it is his intelligence, vision and humanity that have accompanied me through many of my papers in the process of upbringing my mathematical skills. Mrs Rajakumari.N, the professor an intellectual moonlighting as guide raised a number of questions I have summarised in the note and offered other insights into the faith. While many minds have therefore contributed to the contents of the volume, the final responsibility for the arguments and interpretations in this rests with me .If after reading this paper , mathematicians and non mathematicians come away with a new appreciation of the faith I cherish and the challenges it currently dealing with in the contemporary planet, why I am a researcher would have served its purpose.

REFERENCES

[1] D.M. Cvetkovic, M. Doob,and H. Sachs, Spectra of graphs, Acadmec press , New York , 1979.
[2] Diestel R. Graph Theory. New York: Springer-Verlag; 2000
[3] M. Doob, Graphs with a small number of distinct eigen values, Ann. N.Y. Acad.Sci. 175:104-110(1970).
[4] A. J. Hoffman , Some results on spectral properties of graphs, Beitrage Zur Graphentheorie (Kolloquium, Manebach, 1967) Leipzig, !968, 75-80.
[5] Livio M. The golden ratio: The story ofPhi, the world most astonishing number. New York: Broadway Books; 2002.
[6] L. Lovasz and J.Pelikan ,On the eigen values of trees, Periodica Mathmatica Hungarica Vol. 3 (1-2), 1973, pp. 175-182