Some Finite Integrals Involving the Multivariable Beth-Function

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ABSTRACT
The aim of this paper is to establish some finite integrals involving the products of multivariable Beth-function.

KEYWORDS : Multivariable Beth-function, multiple integral contours, finite integrals.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

We define a generalized transcendental function of several complex variables noted $\beth$. This function is a modification of the multivariable Aleph-function recently defined by Ayant [1].

$$\beth(z_1,\cdots,z_r)=\beth^{m_2,0;m_3,0;\cdots;m_r,0;m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right.$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,m_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{m_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,m_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{m_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,m_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})]_{m_r+1,p_r}:[(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)]_{1,m^{(1)}},[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})]_{m^{(1)}+1,p^{(1)}_i}$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})]_{1,q_r}:\quad[(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)]_{1,n^{(1)}},[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})]_{n^{(1)}+1,q^{(1)}_i}$$

$$;\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)]_{1,m^{(r)}},[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})]_{m^{(r)}+1,p^{(r)}_i}$$
$$\left.;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)]_{1,n^{(r)}},[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}}]_{n^{(r)}+1,q^{(r)}_i}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{A_{2j}}(a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=m_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(1-a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{A_{3j}}(a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=m_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(1-a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{A_{rj}}(a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=m_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(1-a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{n^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{m^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=n^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=m^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,\cdots,m_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant n_2,0\leqslant m_2\leqslant p_{i_2},\cdots,0\leqslant n_r,0\leqslant m_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant p_{i^{(r)}}$ and

$0\leqslant n^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant q_{i^{(r)}}$

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)},C_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\delta_{ji^{(k)}}^{(k)},D_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, m_2), \Gamma^{A_{3j}}\left(a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, m_3)$, $\cdots$, $\Gamma^{A_{rj}}\left(a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, m_r)$, $\Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$to the left of the contour L_k and the poles of $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$ lie to the right of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{n^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{m^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=n^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=m^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$\sum_{j=1}^{m_2} A_{2j} \beta^{(k)}_{2j} - \tau_{i_2} \left(\sum_{j=m_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) + \cdots +$$

$$\sum_{j=1}^{m_r} A_{rj} \beta^{(k)}_{rj} - \tau_{i_r} \left(\sum_{j=m_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}) , max(|z_1|, \cdots, |z_r|) \to 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}) , min(|z_1|, \cdots, |z_r|) \to \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant n^{(i)}} Re \left[D^{(i)}_j \left(\frac{d^{(i)}_j}{\delta^{(i)}_j} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant m^{(i)}} Re \left[C^{(i)}_j \left(\frac{c^{(i)}_j - 1}{\gamma^{(i)}_j} \right) \right]$

In your investigation, we shall use the following notations.

$\mathbb{A} = [(\mathrm{a}_{2j}; \alpha^{(1)}_{2j}, \alpha^{(2)}_{2j}; A_{2j})]_{1,m_2}, [\tau_{i_2}(a_{2ji_2}; \alpha^{(1)}_{2ji_2}, \alpha^{(2)}_{2ji_2}; A_{2ji_2})]_{m_2+1,p_{i_2}}, [(a_{3j}; \alpha^{(1)}_{3j}, \alpha^{(2)}_{3j}, \alpha^{(3)}_{3j}; A_{3j})]_{1,m_3},$

$[\tau_{i_3}(a_{3ji_3}; \alpha^{(1)}_{3ji_3}, \alpha^{(2)}_{3ji_3}, \alpha^{(3)}_{3ji_3}; A_{3ji_3})]_{m_3+1,p_{i_3}}; \cdots ; [(\mathrm{a}_{(r-1)j}; \alpha^{(1)}_{(r-1)j}, \cdots, \alpha^{(r-1)}_{(r-1)j}; A_{(r-1)j})]_{1,m_{r-1}},$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}}; \alpha^{(1)}_{(r-1)ji_{r-1}}, \cdots, \alpha^{(r-1)}_{(r-1)ji_{r-1}}; A_{(r-1)ji_{r-1}})]_{m_{r-1}+1,p_{i_{r-1}}} \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})]_{1,m_r},[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{m+1,p_{i_r}} \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,p_i^{(1)}};\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}} \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}}]_{1,q_{i_{r-1}}} \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_{i_r}} \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,q_i^{(1)}};\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}} \tag{1.10}$$

$$U = m_2,0;m_3,0;\cdots;m_{r-1},0;V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

In this section, we give several required results. These results will be utilized in the present investigations :

Lemma 1. (Erdelyi [5],1953)

$$(1-x^2)^v 2P_k^v(x)P_l^v(x) = \frac{(k+v)!(l+v)!}{2^v(k-v)!(l-v)!}\sum_{u=0}^{l+v}A_{u,v}A_{k-u}^vA_{l-u}^vA_{k+l+v-2u}^{-v}$$

$$\left[\frac{(2k+2l+2v-4u+1)}{(2k+2l+2v-2u+1)}\right]P_{k+l+v-2u}(x) \tag{2.1}$$

provided

$k,l,v\in\mathbb{N}$ such that $l+2v\leqslant k, v\leqslant l$ and $A_s^\delta = \frac{(\frac{1}{2},s)}{(s+\delta)!}: A_{\delta,s} = \frac{(\frac{1}{2}-s,\delta)}{\delta!}$, Bailey (1935).

Lemma 2. (Brafman [4], (1951))

$$P_k^{(\alpha,\beta)}(t+\rho)P_k^{(\alpha,\beta)}(t-\rho) = \frac{(-)^k(1+\alpha)_k(1+\beta)_k}{(k!)^2}\sum_{R=0}^{k}\frac{(-k)_R(1+\alpha+\beta+k)_R}{(1+\alpha)_R(1+\beta)_R}P_R^{(\alpha,\beta)}(x)t^R \tag{2.2}$$

Lemma 3.(Brafman [4], (1951))

$$\rho^kP_k^{(\alpha,\beta)}\left(\frac{1-xt}{\rho}\right) = \frac{(1+\alpha)_k}{k!}\sum_{R=0}^{k}\frac{(-k)_R}{(1+\alpha)_R}P_R^{(\alpha,\alpha)}(x)t^R \tag{2.3}$$

Lemma 4. (Erdelyi [5], (1953))

$$\frac{1}{\rho}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}=2^{-\alpha-\beta}\sum_{R=0}^{\infty}P_R^{(\alpha,\beta)}(x)t^R \tag{2.4}$$

In each of the formulae (2.2), (2.3) and (2.4) and throughout the paper $\rho=(1-2xt+t^2)^{-\frac{1}{2}}$.

Lemma 5. (Erdelyi [6], (1954))

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\beta+\rho+1}\Gamma(\rho+1)\Gamma(\beta+n+1)\Gamma(\alpha-\rho+n)}{n!\Gamma(\alpha-\rho)\Gamma(\beta+\rho+n+2)} \tag{2.5}$$

provided $Re(\rho), Re(\beta)>-1$.

Lemma 6. (Erdelyi [5], (1953))

$$\int_{-1}^{1}x^{\rho}(1-x^2)^{\frac{m}{2}}P_v^m(x)\mathrm{d}x=\frac{(-)^m2^{-m-1}\Gamma\left(\frac{\sigma+1}{2}\right)\left(1+\frac{\sigma}{2}\right)\Gamma(1+v+m)}{\Gamma(1-m+v)\Gamma\left(1+\frac{\sigma+m-v}{2}\right)\Gamma\left(\frac{3+\sigma+m+v}{2}\right)} \tag{2.6}$$

provided $Re(\sigma)>-1$.

Lemma 7. (Gradshteyn and Ryzhik [7], (1965))

$$\int_{a}^{b}(x-a)^{\alpha-1}(b-x)^{\beta-1}(x-c)^{-\alpha-\beta}\mathrm{d}x=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(b-a)^{\alpha+\beta-1}(b-c)^{-\alpha}(a-c)^{-\beta} \tag{2.7}$$

Provided $Re(\alpha),\Re(\beta)>0<a<b$.

3. Main results.

In this section, we evaluate several finite integrals .

Theorem 1.

$$\int_{-1}^{1}(1-x)^{\eta}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\gimel(z_1(1-x)^{a_1},\cdots,z_r(1-x)^{a_r})\mathrm{d}x=\frac{(-)^n\Gamma(1+\beta+n)2^{\beta+\eta+1}}{n!}$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,0:V}\left(\begin{array}{c|c}2^{a_1}z_1 & \mathbb{A};(1+\eta;a_1,\cdots,a_r;1),(1-\alpha+\eta;a_1,\ldots,a_r;1),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{a_r}z_r & \mathbb{B};\mathbf{B},(1-\alpha+\eta-n;a_1,\cdots,a_r;1),(2+\beta+\eta+n;a_1,\cdots,a_r;1):B\end{array}\right) \tag{3.1}$$

provided

$$a_i,\sigma>0(i=1,\cdots,r),Re(\eta)>-1,\,Re(\beta)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1$$

and $|arg\,(z_i(1-x)^{a_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 2.

$$\int_{-1}^{1}(1-x)^{\eta}(1-x^2)^{\frac{v}{2}}P_n^{v}(x)\gimel(z_1(1-x)^{a_1},\cdots,z_r(1-x)^{a_r}\mathrm{d}x=\frac{(-)^v\Gamma(1+v+n)}{2^{v+1}\Gamma(1-v+n)}\;\gimel_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,0:V}$$

$$\left(\begin{array}{c|c}2^{a_1}z_1 & \mathbb{A};\left(\frac{1+\eta}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\left(1+\frac{\eta}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{a_r}z_r & \mathbb{B};\mathbf{B},\left(1+\frac{\eta+v+u}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\left(\frac{3+\eta+v+u}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right):B\end{array}\right) \tag{3.2}$$

provided

$$a_i,\sigma>0(i=1,\cdots,r), Re(\eta)>-1\ Re(\eta)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1$$

and $|arg\,(z_i(1-x)^{a_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove the theorems 1 and 2, expressing the multivariable Beth-function with the help of (1.1), interchanging the order of integrations which is justified under the conditions stated with the integrals, evaluating the inner integral with the help of lemmae 5 and 6 respectively and Interpreting the resulting expression with the help of (1.1), we obtain the desired theorems 1 and 2.

Theorem 3.

$$\int_0^1 x^{\eta}(1-x^2)^v P_k^v(x)P_l^v(x)\beth(z_1x^{a_1},\cdots,z_rx^{a_r})\mathrm{d}x=\frac{(-)^v(k+v)!(l+v)!}{(k-v)!(l-v)!}$$

$$\sum_{u=0}^{l+v}\frac{A_{r,v}A_{k-u}^{v}A_{l-u}^{v}}{A_{k+l+v+n}^{-v}}\left(\frac{2k+2l+2v-4n-1}{2k+2l+2v-2n-1}\right)\frac{\Gamma(k+l+2v-2n+1)}{2^{2v+1}\Gamma(k+l-2n+1)}\beth_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,0:V}$$

$$\left(\begin{array}{c|c}2^{a_1}z_1 & \mathbb{A};\left(\frac{1+\eta}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\left(1+\frac{\eta}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{a_r}z_r & \mathbb{B};\mathbf{B},\left(\frac{1+\eta-k-l+2n}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\left(\frac{k-2n+l+2v+\eta+3}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right):B\end{array}\right) \quad (3.3)$$

provided

$k,l,a_i,\sigma>0(i=1,\cdots,r), Re(\eta)>-1,$ such that $l+2v\leqslant k, v\leqslant l$ and $A_s^{\delta}=\frac{(\frac{1}{2},s)}{(s+\delta)!}:A_{\delta,s}=\frac{(\frac{1}{2}-s,\delta)}{\delta!}$,

$$Re(\eta)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1$$

and $|arg\,(z_ix^{a_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 4.

$$\int_{-1}^{1}(1-x)^{\eta}(1+x)^{\beta}P_k^{(\alpha,\beta)}(t+\rho)P_k^{(\alpha,\beta)}(t-\rho)S_N^M[(1+x)^{\theta}]\beth\left(z_1(1-x)^{a_1},\cdots,z_r(1-x)^{a_r}\right)\mathrm{d}x=$$

$$\frac{2^{\beta+\gamma+1}(-)^k\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}{(k!)^2}\sum_{u=0}^{k}\frac{(-k)_u(1+\alpha+\beta+k)_u(-)^u}{(1+\alpha)_u u!}t^u\,\beth_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,0:V}$$

$$\left(\begin{array}{c|c}2^{a_1}z_1 & \mathbb{A};(1+\eta;a_1,\cdots,a_r;1),(1-\alpha+\eta;;a_1,\cdots,a_r;1),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{a_r}z_r & \mathbb{B};\mathbf{B},(1-\alpha+\eta+u;a_1,\cdots,a_r;1),(2+\beta+\eta+u;a_1,\cdots,a_r;1):B\end{array}\right) \quad (3.4)$$

provided $a_i>0(i=1,\cdots,r), Re(\beta)>-1\ Re(\eta)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1$ and

$|arg\,(z_i(1+x)^{a_i})| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 5.

$$\int_{-1}^{1}(1-x)^{\eta}(1+x)^{\alpha}\rho^{k}P_k^{(\alpha,\alpha)}\left(\frac{1-xt}{\rho}\right)(t+\rho)S_N^M[(1+x)^{\theta}]\daleth\,(z_1(1-x)^{a_1},\cdots,z_r(1-x)^{a_r})\,\mathrm{d}x$$

$$= \frac{2^{\alpha+\eta+1}\Gamma(1+\alpha+k)}{k!}\sum_{u=0}^{k}\frac{(-)^u(-k)_u}{u!}\daleth^{U;m_r+2,0:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} 2^{a_1}z_1 & \mathbb{A};\,(1{+}\eta;a_1,\cdots,a_r;1),(1-\alpha+\eta;;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r}z_r & \mathbb{B};\mathbf{B},\,(2{+}\alpha+\eta+u;a_1,\cdots,a_r;1),(1-\alpha+\eta-u;a_1,\cdots,a_r;1):B \end{array}\right) \quad (3.5)$$

provided

$, a_i > 0 (i=1,\cdots,r), Re(\beta) > -1\; Re(\eta) + \sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1$ and

$|arg\,(z_i(1-x)^{a_i})| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 6.

$$\int_{-1}^{1}(1-x)^{\eta}(1+x)^{\beta}\rho^{-1}\frac{(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}}{\rho}\daleth\,(z_1(1-x)^{a_1},\cdots,z_r(1-x)^{a_r})\,\mathrm{d}x =$$

$$2^{-\alpha+\eta+1}\sum_{u=0}^{\infty}\frac{(-)^u\Gamma(\beta+u+1)}{u!}\daleth^{U;m_r+2,0:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} 2^{a_1}z_1 & \mathbb{A};\,(1{+}\eta;a_1,\cdots,a_r;1),(1-\alpha+\eta;;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{a_r}z_r & \mathbb{B};\mathbf{B},(2{+}\beta+\eta+u;a_1,\cdots,a_r;1),(1-\alpha+\eta+u;a_1,\cdots,a_r;1):B \end{array}\right) \quad (3.6)$$

under the same validity conditions that theorem 2.

provided

$a_i > 0 (i=1,\cdots,r), Re(\eta) > -1, Re(\beta) > -1\; Re(\eta) + \sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1$ and

and $|arg\,(z_i(1-x)^{a_i})| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 7.

$$\int_{a}^{b}(x-a)^{\alpha-1}(b-x)^{\beta-1}(x-c)^{-\alpha-\beta}\daleth\left(z_1\left(\frac{x-a}{x-c}\right)^{a_r},\cdots,z_r\left(\frac{x-a}{x-c}\right)^{a_r}\right)\mathrm{d}x = \frac{b-a)^{\alpha+\beta-1}}{(b-a)^{\alpha}(a-c)^{\beta}}$$

$$\beth_{X;p_{i_r},q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;m_r+2,0:V}\left(\begin{array}{c|c} \left(\frac{b-a}{b-c}\right)^{a_1} z_1 & \mathbb{A};\, (\alpha; a_1,\cdots,a_r;1),(\beta;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \left(\frac{b-a}{b-c}\right)^{a_r} z_r & \mathbb{B};\,(\alpha+\beta;a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right) \tag{3.7}$$

provided

$$a_i>0(i=1,\cdots,r), Re(\alpha)>-0, Re(\beta)>0,\ Re(\alpha)+\sum_{i=1}^r a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,$$

$$Re(\beta)+\sum_{i=1}^r a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

and $\left|arg\left(z_i\left(\frac{x-a}{x-c}\right)^{a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

To prove the theorem 3, Multiplying both side of the equation (2.1) by $x^\eta(1-x^2)^{\frac{v}{2}}\beth(x^{a_1}z_1,\cdots,x^{a_r}z_r)$and integrating with respect to x between the limits 0 to 1, interchanging the the order of integration of summation which is permissible under the conditions mentioned about the theorem 3, evaluating the inner integral with the help of theorem 1, simplifying the right hand side of (3.3) after algebraic manipulations, we get the desired result.

To prove the theorems 4 to 7, we use the similar manner that theorem 3, but we use the lemmae 2, 3, a and 7 respectively 7.

Remark 1.
If $m_2=\cdots=m_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$ $A_{rj}=A_{rji_r}=B_{rji_r}=1$, then the multivariable beth-function reduces in the modified multivariable Aleph- function. This function is a modification of the multivariable Aleph-function defined by Ayant [1].

Remark 2.
If $m_2=\cdots=m_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Beth-function reduces in a modified multivariable I-function. This function is a modification of the multivariable I-function defined by Prathima et al. [9].

Remark 3.
If$A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}$ $=\cdots=R^{(r)}=1$, then the multivariable Beth-function reduces in modified multivariable I-function. This function is a modification of the multivariable I-function defined by Prasad [8].

Remark 4.
If the three above conditions are satisfied at the same time, then the multivariable Beth-function reduces in the modified multivariable H-function. This function is a modification of the multivariable H-function defined by Srivastava and Panda [10,11].

Remark 5.
We obtain easily the same integrals about the above functions.

4. Conclusion.

The importance of our all the results lies in their manifold generality. By specialising the various parameters as well as variables in the multivariable Beth-function, we get several integrals formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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