# Integrals Involving The Multivariable Beth-Function

Frédéric Ayant

Teacher in High School , France

ABSTRACT
Certain integrals involving multivariable Beth-function have been obtained in terms of multivariable Beth-function. The given integrals include a large number of integrals involving special functions as particular cases.

KEYWORDS : Multivariable Beth-function, multiple integral contours, finite integrals.

**2010 Mathematics Subject Classification. 33C99, 33C60, 44A20**

## 1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables noted $\beth$. This function is a modification of the multivariable Aleph-function recently defined by Ayant [1].

$$\beth(z_1,\cdots,z_r) = \beth^{m_2,0;m_3,0;\cdots;m_r,0;m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right|$$

$$\begin{array}{c}[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,m_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{m_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,m_3}, \\ [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};\end{array}$$

$$\begin{array}{l}[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{m_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})]_{1,m_r}, \\ [\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;\qquad\qquad \cdots\end{array}$$

$$\begin{array}{l}[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{m_r+1,p_r}:[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,p_i^{(1)}} \\ [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_r}:[(\mathrm{d}_j^{(1)}),\delta_j^{(1)};D_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,q_i^{(1)}}\end{array}$$

$$\left.\begin{array}{l};\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}} \\ ;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,n^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)}]_{n^{(r)}+1,q_i^{(r)}}\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{m_2}\Gamma^{A_{2j}}(a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=m_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(1-a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{A_{3j}}(a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=m_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(1-a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\frac{\prod_{j=1}^{m_r}\Gamma^{A_{rj}}(a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=m_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(1-a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{n^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{m^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=n^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=m^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $\triangle(m,a)$ represents the sequence $\frac{a}{m},\frac{a+1}{m},\cdots,\frac{a+m+1}{m}$

2) $m_2,\cdots,m_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant n_2, 0\leqslant m_2\leqslant p_{i_2},\cdots,0\leqslant n_r,0\leqslant m_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant p_{i^{(r)}}$ and

$0\leqslant n^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant q_{i^{(r)}}$

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)},C_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\delta_{ji^{(k)}}^{(k)},D_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour $L_k$ is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where $\sigma$ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, m_2), \Gamma^{A_{3j}}\left(a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, m_3)$ , $\cdots$ , $\Gamma^{A_{rj}}\left(a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, m_r)$, $\Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$to the left of the contour $L_k$ and the poles of $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$ lie to the right of the contour $L_k$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{n^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{m^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left( \sum_{j=n^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=m^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$\sum_{j=1}^{m_2} A_{2j} \beta^{(k)}_{2j} - \tau_{i_2} \left( \sum_{j=m_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) + \cdots +$$

$$\sum_{j=1}^{m_r} A_{rj} \beta^{(k)}_{rj} - \tau_{i_r} \left( \sum_{j=m_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0( |z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r} ), max( |z_1|, \cdots, |z_r| ) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0( |z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r} ), min( |z_1|, \cdots, |z_r| ) \rightarrow \infty$ where $i = 1, \cdots, r$ :

$\alpha_i = \min_{1 \leqslant j \leqslant n^{(i)}} Re \left[ D^{(i)}_j \left( \frac{d^{(i)}_j}{\delta^{(i)}_j} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant m^{(i)}} Re \left[ C^{(i)}_j \left( \frac{c^{(i)}_j - 1}{\gamma^{(i)}_j} \right) \right]$

In your investigation, we shall use the following notations.

$\mathbb{A} = [(a_{2j}; \alpha^{(1)}_{2j}, \alpha^{(2)}_{2j}; A_{2j})]_{1,m_2}, [\tau_{i_2}(a_{2ji_2}; \alpha^{(1)}_{2ji_2}, \alpha^{(2)}_{2ji_2}; A_{2ji_2})]_{m_2+1,p_{i_2}}, [(a_{3j}; \alpha^{(1)}_{3j}, \alpha^{(2)}_{3j}, \alpha^{(3)}_{3j}; A_{3j})]_{1,m_3},$

$[\tau_{i_3}(a_{3ji_3}; \alpha^{(1)}_{3ji_3}, \alpha^{(2)}_{3ji_3}, \alpha^{(3)}_{3ji_3}; A_{3ji_3})]_{m_3+1,p_{i_3}}; \cdots ; [(a_{(r-1)j}; \alpha^{(1)}_{(r-1)j}, \cdots, \alpha^{(r-1)}_{(r-1)j}; A_{(r-1)j})]_{1,m_{r-1}},$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}}; \alpha^{(1)}_{(r-1)ji_{r-1}}, \cdots, \alpha^{(r-1)}_{(r-1)ji_{r-1}}; A_{(r-1)ji_{r-1}})]_{m_{r-1}+1,p_{i_{r-1}}} \tag{1.5}$$

$$\mathbf{A} = [(a_{rj}; \alpha^{(1)}_{rj}, \cdots, \alpha^{(r)}_{rj}; A_{rj})]_{1,m_r}, [\tau_{i_r}(a_{rji_r}; \alpha^{(1)}_{rji_r}, \cdots, \alpha^{(r)}_{rji_r}; A_{rji_r})]_{m+1,p_{i_r}} \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,p_i^{(1)}};\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}} \tag{1.7}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}}]_{1,q_{i_{r-1}}} \tag{1.8}$$

$$\mathbf{B}=\ [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_{i_r}} \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,q_i^{(1)}};\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}} \tag{1.10}$$

$$U=m_2,0;m_3,0;\cdots;m_{r-1},0;V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

## 2. Required results.

In this section, we give several integrals. They will utilized in your investigation.

**Lemma 1.** (McRobert, 1961) )

$$\int_0^1\frac{x^{a-1}(1-x)^{b-1}}{[1+cx+d(1-x)]^{a+b}}\mathrm{d}x=(1+c)^{-a}(1+d)^{-b}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{2.1}$$

provided $Re(a),Re(b)>0$.

**Lemma 2.** (Erdelyi, 1954).

$$\int_0^1 c^{c-1}(1-x)^{d-1}{}_2F_1(a,b;c;x)\mathrm{d}x=\frac{\Gamma(c)\Gamma(d)\Gamma(c+d-a-b)}{\Gamma(c+d-a)\Gamma(c+d-b)} \tag{2.2}$$

provided $Re(c),Re(c)Re(c+d-a-b)>0$.

**Lemme 3.** (MacRobert, 1939).

$$\int_0^\pi(\sin\theta)^{a-1}T_c^{-b}(\cos\theta)\mathrm{d}\theta=\frac{\Gamma\left(\frac{a\pm b}{2}\right)}{2^b\Gamma\left(\frac{1+b\pm c}{2}\right)\Gamma\left(\frac{1+a+c}{2}\right)\Gamma\left(\frac{a-c}{2}\right)} \tag{2.3}$$

provided $Re(a\pm b)>0$

**Lemma 5.** (MacRobert, 1962).

$$\int_0^\infty e^{-ax}x^{b-1}E(c,d;ax)\mathrm{d}x=\frac{\Gamma(c)\Gamma(d)\Gamma(c+b)\Gamma(c+d)}{a^b\Gamma(b+c+d)} \tag{2.4}$$

provided $Re(a), Re(b+d), Re(b+c) > 0$.

**Lemma 6**. (Mathai and Saxena, p.78, Eq. (3.1.45))

$$\int_0^\infty x^{a-1}K_c(px)\mathrm{d}x = p^{-a}2^{a-2}\Gamma\left(\frac{a\pm c}{2}\right) \tag{2.5}$$

provided $Re(a) > Re(c)$.

**Lemma 6**. (Erdelyi, 1954)

$$\int_0^\infty x^{a-1}e^{-px}K_v(px)\mathrm{d}x = \frac{\sqrt{\pi}\Gamma(a\pm v)}{2^a p^a \Gamma\left(a+\frac{1}{2}\right)} \tag{2.6}$$

provided $Re(a) > Re(v)$

**Lemma 7.** (Erdelyi, 1954)

$$\int_0^\infty x^{a-1}J_v(px)\mathrm{d}x = \frac{2^{a-1}\Gamma\left(\frac{a+v}{2}\right)}{p^a\Gamma\left(\frac{v-a}{2}+1\right)} \tag{2.7}$$

**Lemma 8.** (Srivastava and Manocha, p.23, Eq.27))

If $m \in \mathbb{N}$, then

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}} \prod_{i=1}^{m-1}\Gamma\left(z+\frac{i}{m}\right) \tag{2.8}$$

## 3. Main integrals.

**Theorem 1.**

$$\int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{[1+cx+d(1-x)]^{a+b}} \daleth\left(z_1\frac{x^{ma_1}(1-x)^{na_1}}{[1+cx+d(1-x)]^{(m+n)a_1}}, \cdots, z_r\frac{x^{ma_r}(1-x)^{na_r}}{[1+cx+d(1-x)]^{(m+n)a_r}}\right)\mathrm{d}x =$$

$$\sqrt{\pi}(1+c)^{-a}1+d)^{-b}m^{a-\frac{1}{2}}n^{b-\frac{1}{2}}(m+n)^{-a-b-\frac{1}{2}}\daleth^{U;m_r+m+n,0:V}_{X;p_{i_r},q_{i_r}+m+n,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1(m+n)^{a_1(m+n)}\left(\frac{m}{1+c}\right)^{ma_1}\left(\frac{n}{1+c}\right)^{na_1} & \mathbb{A};(\triangle(m,a);a_1,\cdots,a_r;1),(\triangle(n,b);a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r(m+n)^{a_r(m+n)}\left(\frac{m}{1+c}\right)^{ma_r}\left(\frac{n}{1+c}\right)^{na_r} & \mathbb{B};(\triangle(m+n,a+b);a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right) \tag{3.1}$$

provided that

$m,n,a_i(i=1,\cdots,r) > 0.1+c, 1+d, 1+cx+d(1-x) \neq 0\text{in}0 < x < 1.$

$$Re(a) + m\sum_{i=1}^r a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0,\ Re(b) + n\sum_{i=1}^r a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$$

and $\left|arg\left(z_i\frac{x^{ma_i}(1-x)^{na_i}}{[1+cx+d(1-x)]^{(m+n)a_i}}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof
To prove the theorem 1, expressing the multivariable Beth-function with the help of (1.1), interchanging the order of integrations which is justified under the conditions stated with the integrals, evaluating the inner integral with the help

of lemma 1, using the lemma 8 and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

**Theorem 2.**

$$\int_0^1 c^{c-1}(1-x)^{d-1}{}_2F_1(a,b;c;x)\beth\left(z_1(1-x)^{ma_1}\cdots,z_r(1-x)^{ma_r}\right)\mathrm{d}x=\Gamma(c)m^{-c}$$

$$\beth^{U;m_r+2m,0:V}_{X;p_{i_r},q_{i_r}+2m,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(\triangle(m,d);a_1,\cdots,a_r;1),(\triangle(m,c+d-a-b);a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};(\triangle(m,c+d-a);a_1,\cdots,a_r;1),(\triangle(m,c+d-b);a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right) \tag{3.2}$$

provided that

$$m,a_i(i=1,\cdots,r)>0.Re(d)+m\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,$$

$$Re(c+d)+m\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

and $|arg\,(z_i(1-x)^{ma_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

**Theorem 3.**

$$\int_0^{\pi}(\sin\theta)^{a-1}T_c^{-b}(\cos\theta)\beth(z_1(\sin\theta)^{2ma_1},\cdots,z_r(\sin\theta)^{2ma_r})\mathrm{d}x=\frac{1}{2^b\sqrt{m}\Gamma\left(\frac{1+b\pm c}{2}\right)}$$

$$\beth^{U;m_r+2m,0:V}_{X;p_{i_r},q_{i_r}+2m,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};\left(\triangle\left(m,\frac{a\pm b}{2}\right);a_1,\cdots,a_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\left(\triangle\left(m,\frac{a+c+1}{2}\right);a_1,\cdots,a_r;1\right),\left(\triangle\left(m,\frac{a-c}{2}\right);a_1,\cdots,a_r;1\right),\mathbf{B}:B \end{array}\right) \tag{3.3}$$

provided that

$$m,a_i(i=1,\cdots,r)>0.Re(a\pm b)+2m\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,$$

and $\left|arg\left(z_i\sin(\theta)^{2ma_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

**Theorem 4.**

$$\int_0^{\infty}e^{-ax}x^{b-1}E(c,d;ax)\,\beth(z_1x^{2ma_1},\cdots,z_rx^{2ma_r})\mathrm{d}x=\frac{\Gamma(c)\Gamma(d)}{a^b}(2\pi)^{\frac{1-m}{2}}m^{b-\frac{1}{2}}$$

$$\beth^{U;m_r+2m,0:V}_{X;p_{i_r}+2m,q_{i_r}+m,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\left(\frac{x}{a}\right)^{ma_1} & \mathbb{A};(\triangle(m,b+c);a_1,\cdots,a_r;1),(\triangle(m,b+d);a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\left(\frac{x}{a}\right)^{ma_r} & \mathbb{B};(\triangle(m,c+b-d);a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right) \tag{3.4}$$

provided that

$$Re(a), m, a_i(i=1,\cdots,r) > 0. Re(b+c) + m\sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0,$$

$$Re(b+d) + m\sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$$

and $\left|arg\left(z_i x^{2ma_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

**Theorem 5.**

$$\int_0^\infty x^{a-1}K_c(px)\daleth(z_1x^{2ma_1},\cdots,z_rx^{2ma_r})\mathrm{d}x = \quad p^{-a}2^{a-2}(2\pi)^{1-m}m^{a-1}$$

$$\daleth^{U;m_r,0:V}_{X;p_{i_r}+2m,q_{i_r},\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1\left(\frac{m}{p}\right)^{ma_1} & \mathbb{A};\triangle\left(m,\frac{a\pm c}{2}\right);a_1,\cdots,a_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r\left(\frac{m}{p}\right)^{ma_r} & \mathbb{B};\mathbf{B}:\mathrm{B} \end{array}\right) \tag{3.5}$$

$$Re(a), m, a_i(i=1,\cdots,r) > 0. Re(a) + m\sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > Re(v),$$

and $\left|arg\left(z_i x^{2ma_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

**Theorem 6.**

$$\int_0^\infty x^{a-1}e^{-px}K_v(px)\daleth(z_1x^{ma_1},\cdots,z_rx^{ma_r})\mathrm{d}x = \pi^{1-\frac{m}{2}}2^{\frac{1-m}{2}-a}m^{a-\frac{1}{2}}p^m$$

$$\daleth^{U;m_r+2m,0:V}_{X;p_{i_r}+,q_{i_r}+m,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1\left(\frac{m}{2p}\right)^{ma_1} & \mathbb{A};(\triangle(m+\pm v);a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r\left(\frac{m}{2p}\right)^{ma_r} & \mathbb{B};\triangle\left(m+\frac{1}{2}\right);a_1,\cdots,a_r;1\right),\mathbf{B}:B \end{array}\right) \tag{3.6}$$

$$Re(a), m, a_i(i=1,\cdots,r) > 0. -Re(v) < Re(a) + m\sum_{i=1}^{r} a_i \min_{1\leqslant j\leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] < \frac{3}{2},$$

and $|arg\left(z_i x^{ma_i}\right)| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

**Theorem 7.**

$$\int_0^\infty x^{a-1}J_v(px)\daleth(z_1x^{ma_1},\cdots,z_rx^{ma_r})\mathrm{d}x = (2m)^{a-1}p^{-a}$$

$$\gimel^{U;m_r+m,0:V}_{X;p_{i_r}+,q_{i_r}+m,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\left(\frac{2}{p}\right)^{ma_1} & \mathbb{A};\left(\triangle\left(m,\frac{a+v}{2}\right);a_1,\cdots,a_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\left(\frac{2}{p}\right)^{ma_r} & \mathbb{B};\left(\triangle\left(m,\frac{v-a+2}{2}\right);a_1,\cdots,a_r;1\right),\mathbf{B}:B \end{array}\right) \tag{3.6}$$

$$Re(a),m,a_i(i=1,\cdots,r)>0.-Re(v)<Re(a)+m\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]<\frac{3}{2},$$

and $|arg\left(z_i x^{ma_i}\right)|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

To prove the theorems 2, 3, 4 ,5 ,6 and 7, we use the similar methods that theorem 1 bu we use the lemmae 2 ,3, 4, 5 ,6 and 7 respectively.

**Remark 1.**
If $m_2=\cdots=m_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$ $A_{rj}=A_{rji_r}=B_{rji_r}=1$, then the multivariable beth-function reduces in the modified multivariable Aleph- function. This function is a modification of the multivariable Aleph-function defined by Ayant [1].

**Remark 2.**
If $m_2=\cdots=m_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Beth-function reduces in a modified multivariable I-function. This function is a modification of the multivariable I-function defined by Prathima et al. [9].

**Remark 3.**
If$A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Beth-function reduces in modified multivariable I-function. This function is a modification of the multivariable I-function defined by Prasad [8].

**Remark 4.**
If the three above conditions are satisfied at the same time, then the multivariable Beth-function reduces in the modified multivariable H-function. This function is a modification of the multivariable H-function defined by Srivastava and Panda [11,12].

**Remark 5.**
We obtain easily the same integrals about the above functions.

## 4. Conclusion.

The importance of our all the results lies in their manifold generality. By specialising the various parameters as well as variables in the multivariable Beth-function, we get several integrals formulae involving remarkably wide variety of useful functions ( or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

## REFERENCES.

[1] F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

[2] B.L.J. Braaksma, Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[3] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Tables of integrals transforms, Vol I, McGraw-Hill, New York (1954),

[4] A.M. Mathai and R.K. Saxena, Generalized hypergeometric functions with applications in Statistics and physical sciences.

[5] T.M. MacRobert, Expressions for generalized hypergeometric function in multiple series, Proc. Royal. Edinburg 59( 1939), 141-144.

[6] T.M. MacRobert, Beta function formulae and integrals involving E-functions , Math. Annalen 142, 450452, 1961

[7]T.M. MacRobert, Barnes integral as a sum of E-function, Math. Ann 147(1962), 240-243.

[8] Y.N. Prasad, Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , 231-237.

[9] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 1-12.

[10] H.M. Srivastava and H.L. Manocha, A Treatise of generating functions, Ellis Horwood Ltd, Chichester, U.K. 1984.

[11] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment certain of. Math. Univ. St. Paul. 24 (1975),119-137.

[12] H.M. Srivastava and R.Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.