Hopf Bifurcation and Stability Analysis in a Price Model with Time-Delayed Feedback

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Abstract *- A Rayleigh price model with time-delayed feedback is investigated in this paper. First, a time-delayed feedback controller is introduced to the Rayleigh price model and we discussed the effect of the delay on the system. Second, the linear stability of the model and the local Hopf bifurcation are studied and we derived the conditions for the stability and the existence of Hopf bifurcation at the equilibrium of the system. Besides, the direction of Hopf bifurcation and the stability of bifurcation periodic solutions are studied by adopting the center manifold theorem and the normal form theory. At last, some numerical simulation results are confirmed that the feasibility of the theoretical* analysis.

Keywords—*Rayleigh price model,Time-delayed, Hopf bifurcation, Stability, Numerical simulation.*

I. INTRODUCTION

In recent years, with the differential equations have been widely applied to biology, economics and other fields, scholars have established some models that can reflect the characteristics of the dynamical systems of differential equations. Price is a dynamic economic phenomenon, which is closely related to people's life and it is also affected by the supply and demand. Recently, because the price model has important application background and extremely rich dynamic behavior, it has attracted the attention of scholars at home and abroad[1-3]. Among them, the Rayleigh price model is a classical economic model. In[4], the author ignorced the effect of time delay and studied the price differential equation model which gived a dynamic system to research the dynamics properties of a price model. Reference[5] further studied on price model with delay and provided a qualitative analysis for different kinds of economic phenomenon by qualitative theory of differential equations. In[6], by using the method of $\tau - D$ partitioning approach of exponential polynomial, Lv and Liu investigated the Rayleigh price model with time delay, they draw a conclusion that supply depends only on the price of the past. However, there have few studies are related to price model with time-delayed feedback control. In this paper, our study based on the Rayleigh price model and introduced a time-delayed feedback controller to the model.

In[4], the original Rayleigh price model can be described by the following nonlinear differential equations:

$$\ddot{x}(t) - l(ax^2(t) + bx(t) + c)\dot{x}(t) + x(t) = 0 \tag{1}$$

where $x(t)$ represents the price at time t, $y(t)$ denotes the amount of supply at time t, $l > 0$ and a, b, c are the constants.

The system (1) is equivalent to the following two-dimensional system:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = l\left(\frac{1}{3}ax^3(t) + \frac{1}{2}bx^2(t) + c\,x(t)\right)y(t) - x(t). \end{cases} \tag{2}$$

In this paper, we add a time-delayed feedback controller $k(x(t) - x(t-\tau))$ to the system (2). The classical Rayleigh price model can be modified by the following delay differential equations:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = l\left(\frac{1}{3}ax^3(t) + \frac{1}{2}bx^2(t) + c\,x(t)\right)y(t) - x(t) + k(x(t) - x(t-\tau)). \end{cases} \tag{3}$$

where $\tau > 0$ is the time delay and the coefficient k is the feedback gain.

The rest of the paper is arranged as follows, the linear stability of the model and the local Hopf bifurcation are studied and the conditions for the stability and the existence of Hopf bifurcation at the equilibrium are derived in section 2. In section 3, according to the method of theory and applications of Hopf bifurcation by Hassard et al.[7], the direction and stability of bifurcating periodic solutions are investigated. In section 4, the correctness of theoretical analysis are confirmed by some numerical simulation results. At last, some conclusions are obtained in section 5.

II. STABILITY AND LOCAL HOPF BIFURCATION ANALYSIS

In this section, we only discuss the problems of the Hopf bifurcation and stability for the unique equilibrium point $(0,0)$. The linearation of system (3) at $(0,0)$ is

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = (k-1)x(t) - kx(t-\tau) + lcy(t). \end{cases} \tag{4}$$

The correspoding characteristic equation of system (3) at the equilibrium point is as follows.

$$\lambda^2 - lc\lambda + ke^{-\lambda\tau} - (k-1) = 0 \tag{5}$$

Lemma 1. When $\tau = 0$ and $c < 0$ are satisfied, the equilibrium point $(0,0)$ of price model (3) is locally asymptotically stable.

Proof. When $\tau = 0$ is met, Eq. (5) becomes

$$\lambda^2 - lc\lambda + 1 = 0 \tag{6}$$

further, if $c < 0$ is met, we have the following conditions:

$$D_1 = -lc > 0, \quad D_2 = 1 > 0 \tag{7}$$

According to the Routh-Hurwotz criteria [8-9], all roots of characteristic equation (6) have negative real parts. Hence, when $\tau = 0$ and $c < 0$ hold, the equilibrium point $(0,0)$ of system (3) is locally asymptotically stable.

Lemma 2. For the system (3), assume that $k > 1$ is met.Then Eq.(5) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_0$, where

$$\omega_0 = \sqrt{\frac{-(l^2c^2 + 2(k-1)) + \sqrt{(l^2c^2 + 2(k-1))^2 + 4(2k-1)}}{2}},$$

$$\tau_0 = \frac{1}{\omega_0}\arctan\left(-\frac{lc\omega}{\omega_0^2 + (k-1)}\right).$$

Proof. Let $\lambda = i\omega$ $(\omega > 0)$ is a solution of the characteristic equation (5), then

$$-\omega^2 - ilc\omega + k(\cos\omega\tau - i\sin\omega\tau) - (k-1) = 0. \tag{8}$$

The separation of the real and imaginary parts, it follows

$$\begin{cases} -\omega^2 + k\cos\omega\tau - (k-1) = 0 \\ -lc\omega - k\sin\omega\tau = 0. \end{cases} \tag{9}$$

From (9) we obtain

$$\omega = \sqrt{\frac{-(l^2c^2 + 2(k-1)) + \sqrt{(l^2c^2 + 2(k-1))^2 + 4(2k-1)}}{2}}$$

$$\tau_j = \frac{1}{\omega}\left[\arctan\left(-\frac{lc\omega}{\omega^2 + (k-1)}\right) + j\pi\right], \quad j = 0,1,2,\cdots.$$

Obviously, set $j = 0$, then

$$\omega_0 = \sqrt{\frac{-(l^2c^2 + 2(k-1)) + \sqrt{(l^2c^2 + 2(k-1))^2 + 4(2k-1)}}{2}} \tag{10}$$

$$\tau_0 = \frac{1}{\omega_0}\arctan\left(-\frac{lc\omega}{\omega_0^2 + (k-1)}\right) \tag{11}$$

As a result, when $\tau = \tau_0$, the equation (5) have a pair of purely imaginary roots.

Lemma 3. Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of (5) with $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$ then we have the following transversality condition $\mathrm{Re}\left(\frac{d\lambda}{d\tau_0}\right)^{-1}\Big|_{\tau=\tau_0} > 0$ is satisfied.

Proof. By differentiating both sides of Eq. (5) with regard to τ and applying the implicit function theorem, we have

$$\begin{aligned} \frac{d\lambda}{d\tau}\Big|_{\tau=\tau_0} &= \frac{k\lambda e^{-\lambda\tau_0}}{2\lambda - lc - k\tau e^{-\lambda\tau_0}} \\ &= \frac{k\omega_0\sin\omega_0\tau_0 - ik\omega_0\cos\omega_0\tau_0}{(-lc - k\tau_0\cos\omega_0\tau_0) + i(2\omega_0 + k\tau_0\sin\omega_0\tau_0)} \end{aligned}$$

then

$$\mathrm{Re}\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_0} = \frac{l^2c^2\omega^2 + 2\omega^4 + 2\omega^2(k-1)}{(-lc - k\tau_0\cos\omega_0\tau_0)^2 + (2\omega_0 + k\tau_0\sin\omega_0\tau_0)^2} \tag{12}$$

Since $k > 1$, thus $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_0} > 0$. The proof is completed.

Lemma 4. For Eq. (5), when $\tau < \tau_0$, all of his roots have negative real parts. The equilibrium $(0,0)$ is locally asymptotically stable, and system (3) produces a Hopf bifurcation at the equilibrium $(0,0)$ when $\tau = \tau_0$.

By applying the Hopf bifurcation theorem for time-delayed differential equation and the above four lemmas[10], we have the following results.

Theorem 1. For system (3), when $c < 0$ and $k > 1$ hold, we have the following conclusions:

a) If $\tau \in [0, \tau_0)$, the equilibrium point $(0,0)$ is asymptotically stable .

b) If $\tau = \tau_0$, model (3) exhibits a Hopf bifurcation at the equilibrium point $(0,0)$.

c) If $\tau > \tau_0$, then the equilibrium point of system (3) is unstable.

III. DIRECTION AND STABILITY OF THE HOPF BIFURCATION

In this section, by using the normal form theory and the center manifold theorem introduced in [11-12], we discuss the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions when $\tau = \tau_0$.

For notational convenience, let $\tau = \tau_0 + \mu$, $u(t) = (x_1(t), x_2(t))^T$ and $u_t(\theta) = u(t+\theta)$ for $\theta \in [-\tau, 0]$, clearly, $\mu = 0$ is Hopf bifurcation value for system (3). For initial condition $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))^T \in C[-\tau, 0]$, Then the system (3) is equivalent to the following Functional Differential Equation (FDE) system

$$\dot{u}(t) = L_\mu u + F(u_t, \mu). \tag{13}$$

with

$$L_\mu(\varphi) = B_1\varphi(0) + B_2\varphi(-\tau) \tag{14}$$

and

$$F(\mu, \varphi) = \begin{pmatrix} 0 \\ la\varphi_1^2(t)\varphi_2(t) + lb\varphi_1(t)\varphi_2(t) \end{pmatrix} \tag{15}$$

where L_μ is the one family of bounded linear operator in $C[-\tau, 0]$ and

$$B_1 = \begin{pmatrix} 0 & 1 \\ k-1 & lc \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 \\ -k & 0 \end{pmatrix}.$$

By the Riesz representation theorem [13] , there exists a bounded variation function $\eta(\theta, \mu)$ for $\theta \in [-\tau, 0]$, such that

$$L_\mu \varphi = \int_{-\tau}^{0} d\eta(\theta, \mu)\varphi(\theta), \varphi \in C. \tag{16}$$

In fact, we can choose

$$\eta(\theta, \mu) = B_1\delta(\theta) + B_2\delta(\theta + R). \tag{17}$$

where $\delta(\theta)$ is a Delta function. For $\varphi \in C([-\tau, 0])$, the operators A and R are defined as follow

$$A(\mu)\varphi(\theta) = \begin{cases} \dfrac{d(\varphi(\theta))}{d\theta}, & \theta \in [-\tau, 0), \\ \displaystyle\int_{-\tau}^{0} d(\eta(\theta, \mu)\varphi(\theta)), & \theta = 0. \end{cases} \tag{18}$$

$$R(\mu)\varphi(\theta)=\begin{cases}0, & \theta\in[-\tau,0),\\ F(\mu,\varphi), & \theta=0.\end{cases} \tag{19}$$

Hence the Eq. (13) can be written as the following form:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t \tag{20}$$

Since $\frac{du_t}{d\theta}=\frac{du_t}{dt}$, then Eq.(20) can be written as

$$\frac{du_t}{dt}=\begin{cases}\frac{du_t}{dt}+0, & \theta\in[-\tau,0),\\ L_\mu u_t + F(u_t,\mu), & \theta=0.\end{cases} \tag{21}$$

For $\psi\in C[0,\tau]$, we define the adjoint operator $A^*(\mu)$ of $A(\mu)$ as

$$A^*(\mu)\psi(s)=\begin{cases}-\frac{d\psi(s)}{ds}, & s\in(0,\tau]\\ \int_{-\tau}^{0} d(\eta^T(s,0)\psi(-s)), & s=0\end{cases} \tag{22}$$

For $\varphi(\theta)\in C[-\tau,0)$ and $\psi\in C[0,\tau]$, define a bilinear inner product

$$<\psi,\varphi>=\overline{\psi}(0)^T\varphi(0)-\int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta}\overline{\psi}^T(\xi-\theta)[d\eta(\theta)]\varphi(\xi)d\xi. \tag{23}$$

where $\eta(\theta)=\eta(\theta,0)$.

Let $\mu=0$; To determine the normal form of operator A , we need to calculate the eigenvectors $q(\theta)$ and $q^*(s)$ of A and A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively. we can obtain

$$\begin{cases}A(0)q(\theta)=i\omega_0 q(\theta)\\ A^*(0)q^*(s)=-i\omega_0 q^*(s)\end{cases} \tag{24}$$

Assume that $q(\theta)=Ve^{i\omega_0\theta}$ is eigenvector of $A(0)$ corresponding to $i\omega_0$ and $q^*(s)=DV^*e^{-i\omega_0 s}$ is eigenvector of $A^*(0)$ corresponding to $-i\omega_0$. By direct calculate, we get

$$q(\theta)=Ve^{i\omega_0\theta}=(v_1,v_2)^T e^{i\omega_0\theta}=(1,\rho_1)^T e^{i\omega_0\theta}=(1,i\omega_0)^T e^{i\omega_0\theta} \tag{25}$$

$$q^*(s)=DV^*e^{-i\omega_0 s}=D(v_1^*,v_2^*)^T e^{-i\omega_0 s}=D(\rho_2,1)^T e^{-i\omega_0 s}=D(-lc-i\omega_0,1)^T e^{-i\omega_0 s} \tag{26}$$

Now, we verify that $<q^*,q>=1$ and $<q^*,\overline{q}>=0$. From(23), we obtain

$$\begin{aligned}\langle q^*,q\rangle &= \overline{q}^{*T}q(0)-\int_{\theta=-\tau_0}^{0}\int_{\xi=0}^{\theta}\overline{q}^{*T}(\xi-\theta)d\eta(\theta)q(\xi)d\xi.\\ &= \overline{D}[\overline{V}^{*T}V-\int_{\theta=-\tau_0}^{0}\int_{\xi=0}^{\theta}\overline{V}^{*T}e^{-i\omega_0(\xi-\theta)}d\eta(\theta)Ve^{i\omega_0\xi}d\xi\}\\ &= \overline{D}[\overline{V}^{*T}V-\int_{\theta=-\tau_0}^{0}\overline{V}^{*T}[d\eta(\theta)]\theta e^{-i\omega_0\theta}V]\end{aligned}$$

$$= \bar{D}[\bar{V}^{*T}V - \tau_0 e^{i\omega_0\theta}\bar{V}^{*T}B_2V]. \tag{27}$$

Let $\bar{D} = [\bar{V}^{*T}V - \tau_0 e^{i\omega_0\theta}\bar{V}^{*T}B_2V]^{-1}$, we can get $< q^*, q >= 1$.

By $\langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle$, we obtain

$$-i\omega_0 \langle q^*, \bar{q} \rangle = \langle q^*, A\bar{q} \rangle = \langle A^*q^*, \bar{q} \rangle = \langle -i\omega_0 q^*, \bar{q} \rangle = i\omega_0 \langle q^*, \bar{q} \rangle. \tag{28}$$

Therfore $< q^*, \bar{q} >= 0$. The proof is completed.

Using the same notations as in Hassard et al.[7], we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Define

$$z(t) =< q^*, u_t >, \qquad W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \tag{29}$$

On the center manifold C_0, we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta) \tag{30}$$

Where $W(z,\bar{z},\theta) = W_{20}(\theta)\dfrac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\dfrac{\bar{z}^2}{2} + \cdots$.

z and $\bar{z}$ are local coordinates for center manifold C_0 in C in the direction of q and $\bar{q}$, respectively. Note that W is real if u_t is real, therefore we only real solutions. Since $\mu = 0$, it is easy to see that

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{\mu}_t \rangle \\ &= \langle q^*, (A(0) + R(0))\mu_t \rangle \\ &=< q^*, A\mu_t > + < q^*, R\mu_t > \\ &= i\omega_0 z(t) + \bar{q}^* f_0(z,\bar{z}). \end{aligned} \tag{31}$$

Let

$$\dot{z}(t) = i\omega_0 z + g(z,\bar{z}), \tag{32}$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{33}$$

from (20) and (32), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\dot{q} = \begin{cases} AW - 2\,\mathrm{Re}\{\bar{q}^*(0)f_0(z,\bar{z})q(\theta)\}, & \theta \in [-\tau_0, 0], \\ AW - 2\,\mathrm{Re}\{\bar{q}^*(0)f_0(z,\bar{z})q(\theta)\} + f_0(z,\bar{z}), & \theta = 0. \end{cases} \tag{34}$$

Which can be rewritten as

$$\dot{W} = AW + H(z,\bar{z},\theta) \tag{35}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{36}$$

On the other hand, on C_0,

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}} \tag{37}$$

Using (30) and (32) to replace W_z and $\dot{z}$ and their conjugates by their power series expansions, we obtain

$$\dot{W} = i\omega_0 W_{20}(\theta)z^2 - i\omega_0 W_{02}(\theta)\bar{z}^2 + \cdots . \tag{38}$$

Comparing the coefficients of the above equation with those of (35) and (38), we get

$$\begin{cases} (A - 2i\omega_0)W_{20}(\theta) = -H_{20}(\theta), \\ AW_{11}(\theta) = -H_{11}(\theta), \\ (A + 2i\omega_0)W_{02}(\theta) = -H_{02}(\theta). \end{cases} \tag{39}$$

Notice that $u_t = u(t+\theta) = W(z(t),\bar{z}(t),\theta) + zq + \overline{zq}$ and $q(\theta) = (1,\rho_1)^T e^{i\omega_0\theta}$, we get

$$u_t = \begin{pmatrix} x_1(t+\theta) \\ x_2(t+\theta) \end{pmatrix} = \begin{pmatrix} W^{(1)}(z,\bar{z},\theta) \\ W^{(2)}(z,\bar{z},\theta) \end{pmatrix} + z\begin{pmatrix} 1 \\ \rho_1 \end{pmatrix} e^{i\omega_0\theta} + \bar{z}\begin{pmatrix} 1 \\ \bar{\rho}_1 \end{pmatrix} e^{-i\omega_0\theta}.$$

$$\varphi_1(0) = z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\varphi_2(0) = z\rho_1 + \overline{z\rho_1} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\varphi_1^2(0) = z^2 + 2z\bar{z} + \bar{z}^2 + [2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)]z^2\bar{z} + \cdots$$

$$\varphi_1(0)\varphi_2(0) = \rho_1 z^2 + (\bar{\rho}_1 + \rho_1)z\bar{z} + \bar{\rho}_1\bar{z}^2 + [W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\bar{\rho}_1 + W_{11}^{(1)}(0)\bar{\rho}_1]z^2\bar{z}$$

$$\varphi_1^2(0)\varphi_2(0) = (\bar{\rho}_1 + 2\rho_1)z^2\bar{z} + \cdots$$

From the (32) and (33), we obtain

$$f(z,\bar{z}) = \begin{pmatrix} 0 \\ K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z} \end{pmatrix}$$

where $K_1 = lb\rho_1$, $K_2 = lb(\rho_1 + \bar{\rho}_1)$, $K_3 = lb\bar{\rho}_1$,

$$K_4 = la(\bar{\rho}_1 + 2\rho_1) + lb[W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\bar{\rho}_1 + W_{11}^{(1)}(0)\bar{\rho}_1].$$

In order to get the values of g_{20}, g_{11}, g_{02} and g_{21}. Comparing the cofficients of the above equation with those in (33), we get

$$g_{20} = 2\bar{D}\bar{\rho}_2 K_1, g_{11} = \bar{D}\bar{\rho}_2 K_2, g_{02} = 2\bar{D}\bar{\rho}_2 K_3, g_{20} = 2\bar{D}\bar{\rho}_2 K_4. \tag{40}$$

In order to determine the value of g_{21}, we also need to compute the values of $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain

$$\begin{aligned} H(z,\bar{z},\theta) &= -2\,\mathrm{Re}[\bar{q}^{*T}(0)\,f_0(z,\bar{z})q(\theta)] \\ &= -(g_{20}(\theta)\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots)q(\theta) \\ &\quad -(\bar{g}_{20}(\theta)\frac{z^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(\theta). \end{aligned} \tag{41}$$

Comparing the coefficients with (36), we gives that

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned} \tag{42}$$

When $\theta = 0$, we have

$$\begin{aligned} H(z,\bar{z},0) &= -2\,\mathrm{Re}[\bar{q}^{*}(0)\,f_0(z,\bar{z})q(0)] + f_0(z,\bar{z}) \\ &= -(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots)q(0) \\ &\quad -(\bar{g}_{20}(\theta)\frac{z^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(0) + \begin{pmatrix} 0 \\ K_1z^2 + K_2z\bar{z} + K_3\bar{z}^2 + K_4z^2\bar{z} \end{pmatrix}. \end{aligned}$$

Comparing the coefficients with (41), we have

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\begin{pmatrix} 0 \\ K_1 \end{pmatrix}, \\ H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \begin{pmatrix} 0 \\ K_2 \end{pmatrix}. \end{aligned} \tag{43}$$

Using (39), (42), we obtain

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_1e^{2i\omega_0\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{11}}{\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_2. \end{aligned}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$, $E_2 = (E_2^{(1)}, E_2^{(2)})^T$.

From the definition of $A(0)$ and (39), we have

$$\int_{-\tau_0}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0W_{20}(0) - H_{20}(0),$$

$$\int_{-\tau_0}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0).$$

Notice that

$$(i\omega_0 I - \int_{-\tau_0}^{0} e^{i\omega_0\theta}d\eta(\theta))q(0) = 0$$

$$(-i\omega_0 I - \int_{-\tau_0}^{0} e^{-i\omega_0\theta}d\eta(\theta))\bar{q}(0) = 0.$$

Hence, we can get

$$(2i\omega_0 I - \int_{-\tau_0}^{0} e^{2i\omega_0} d\eta(\theta))E_1 = 2\begin{pmatrix} 0 \\ K_1 \end{pmatrix}$$

$$(\int_{-\tau_0}^{0} d\eta(\theta))E_2 = -\begin{pmatrix} 0 \\ K_2 \end{pmatrix}$$

Therefore, we have

$$\begin{cases} \begin{pmatrix} i2\omega_0 & -1 \\ ke^{-2i\omega_0\tau_0} - k + 1 & i2\omega_0 - lc \end{pmatrix}\begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \end{pmatrix} = 2\begin{pmatrix} 0 \\ K_1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ -1 & lc \end{pmatrix}\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \end{pmatrix} = -\begin{pmatrix} 0 \\ K_2 \end{pmatrix} \end{cases} \tag{44}$$

Then we can get

$$\begin{aligned} E_1^{(1)} &= \frac{2K_1}{ke^{-i2\omega_0\tau_0} - k + 1 - i2lc\omega_0 - 4\omega_0^2} \\ E_1^{(2)} &= i2\omega_0 E_1^{(1)}. \end{aligned} \tag{45}$$

Similarly, we have

$$\begin{aligned} E_2^{(1)} &= K_2 \\ E_2^{(2)} &= 0. \end{aligned} \tag{46}$$

Based on the above analysis, we have the following parameters [14-16]:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(0)\}}, \\ \beta_2 &= 2\,\mathrm{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2(\mathrm{Im}\{\lambda'(0)\})}{\omega_0}. \end{aligned} \tag{47}$$

which determine the quantities of bifurcating periodic solution in the manifold at the critical value $\tau = \tau_0$, now we have the following theorem for the system (3) [17-18].

Theorem 2.

a) The direction of the Hopf bifurcation is determined by the parameter μ_2. If $\mu_2 > 0$, the Hopf bifurcation is supercritical . If $\mu_2 < 0$, the Hopf bifurcation is subcritical .

b) β_2 determines the stability of the bifurcating periodic solution. If $\beta_2 < 0$, the bifurcating periodic solutions is stable; if $\beta_2 > 0$, the bifurcating periodic solutions is unstable.

c) The period of the bifurcating periodic solution is decided by the parameter T_2. If $T_2 > 0\,(< 0)$, the period increases(decreases).

IV. NUMERICAL SIMULATION

In this section, we present numerical results to confirm the analytical predictions obtained in the previous section. For system (3) , We take the parameters $a = -1, b = -1, c = -2, l = -2, k = 2$. By directly computing, we get that

$$\omega_0 = 0.406388, \tau_0 = 2.33496$$

From the above analysis in section 2, If we choose $\tau = 2 < \tau_0$, the equilibrium point $(0,0)$ of the system (3) is asymptotically stable which proved by numerical simulations (see Figs. 1-4.).

For convenient comparison, When $\tau = 2.33496 = \tau_0$, a Hopf bifurcation occurs, namely, there are periodic solutions bifurcating out from the equilibrium point $(0,0)$ (see Figs. 5-8.). When $\tau = 2.5 > \tau_0$, the equilibrium point $(0,0)$ of the system (3) loses its stability and the system is unstable(see Figs. 9-12.).

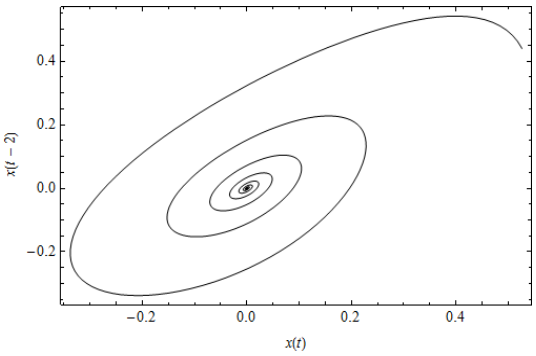

Figure 1. Phase plot of $x(t)$ with $\tau = 2$.

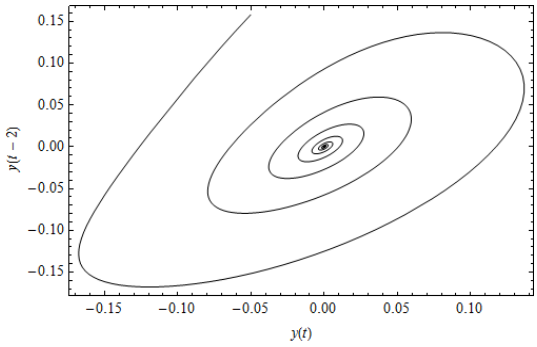

Figure 2. Phase plot of $y(t)$ with $\tau = 2$.

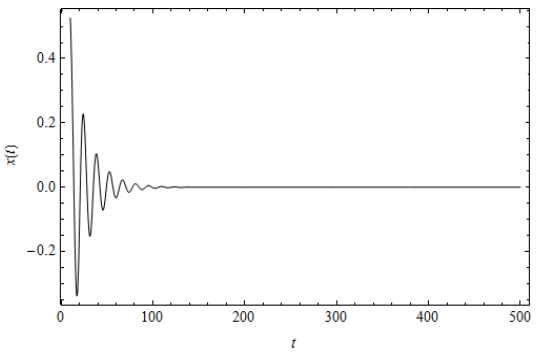

Figure 3. State plot of $x(t)$ with $\tau = 2$.

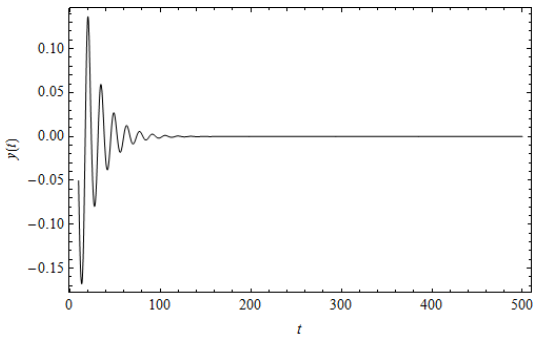

Figure 4. State plot of $y(t)$ with $\tau = 2$.

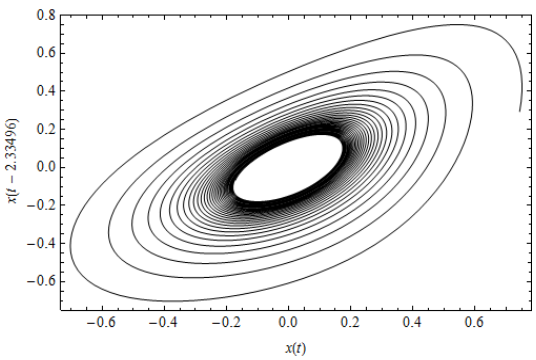

Figure 5. Phase plot of $x(t)$ with $\tau = 2.33496$

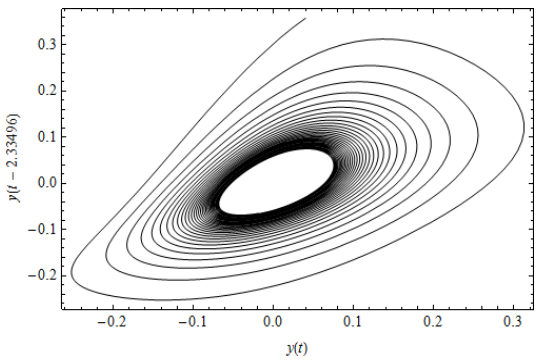

Figure 6. Phase plot of $y(t)$ with $\tau = 2.33496$.

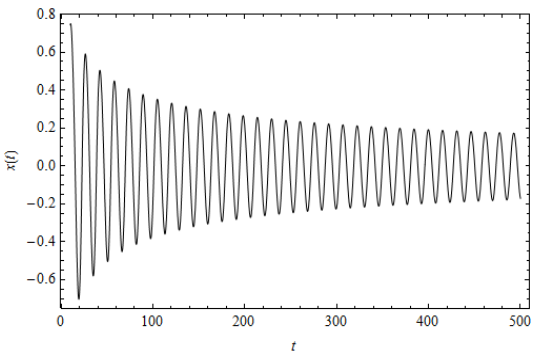

Figure 7. State plot of $x(t)$ with $\tau = 2.33496$.

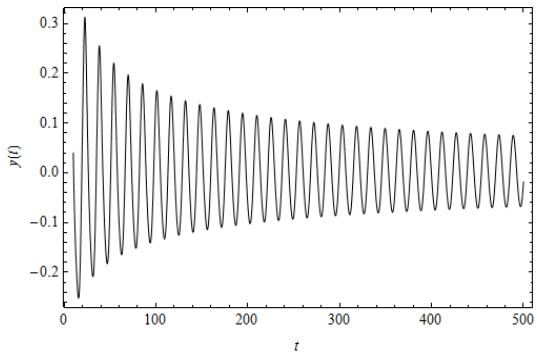

Figure 8. State plot of $y(t)$ with $\tau = 2.33496$.

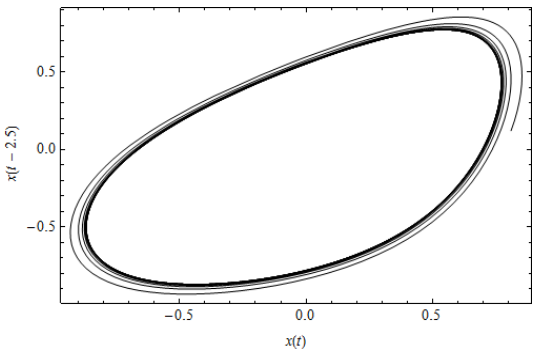

Figure 9. Phase plot of $x(t)$ with $\tau = 2.5$.

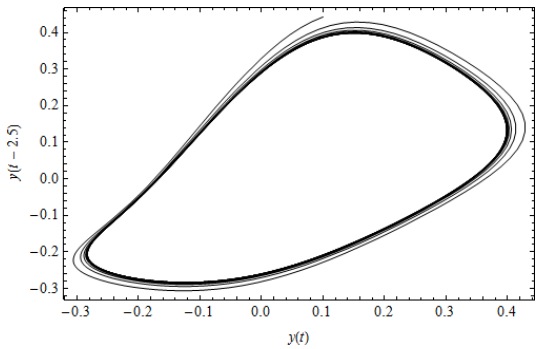

Figure 10. Phase plot of $y(t)$ with $\tau = 2.5$.

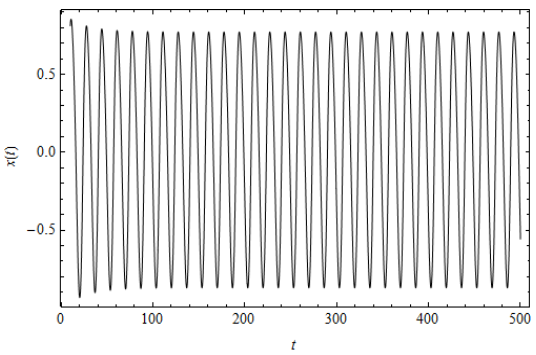

Figure 11. State plot of $x(t)$ with $\tau = 2.5$.

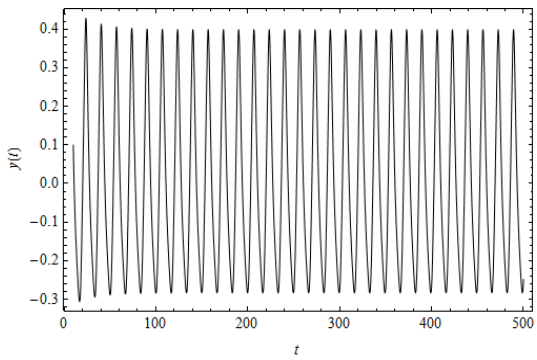

Figure 12. State plot of $y(t)$ with $\tau = 2.5$.

V. CONCLUSIONS

Based on the original Rayleigh price model , a time-delayed feedback price model was studied by this paper. Until now, there are few studies on price models with feedback delay and we provide an insight to unexplored aspects of them. By applying the control and bifurcation theory, We discussed the effect of the feedback delay on the system. Moreover, we derived the conditions for the stability and the existence of Hopf bifurcation at the equilibrium of the system. By employing the theory of functional differential equations and the theory and applications of Hopf bifurcation, we obtained the the direction of Hopf bifurcation and the stability of bifurcation periodic solutions. Some computer simulation results have been presented to illustate the validity of the theoretical analysis. The research of this paper enriches and develops the study on Rayleigh price model.

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