On Unification of Generalized Functions Representable by Mellin-Barnes Contour Integrals

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ABSTRACT
Th e present paper gives in brief the unification and extented picture about various generalized Beth-function defined by using Mellin-Barnes contour integral representaation. Thus it appears that the opportunity of any further generalization using Mellin-Barnes contour integrals closes for the moment.

KEYWORDS : Multivariable Beth-function, multiple integral contours, Jacobi polynomials, series representation, expansion serie.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables noted $\beth$. This function is a modification of the multivariable Aleph-function recently defined by Ayant [1].

$$\beth(z_1,\cdots,z_r) = \beth^{m_2,0;m_3,0;\cdots;m_r,0;m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right.$$

$$[(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,m_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{m_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,m_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{m_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,m_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{m_r+1,p_r}:[(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,p_i^{(1)}}$$
$$[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_r}:[(d_j^{(1)}),\delta_j^{(1)};D_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,q_i^{(1)}}$$

$$\left.\begin{array}{l};\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}} \\ ;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,n^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)}]_{n^{(r)}+1,q_i^{(r)}}\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{A_{2j}}(a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=m_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(1-a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{A_{3j}}(a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=m_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(1-a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{A_{rj}}(a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=m_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(1-a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \quad (1.2)$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{n^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{m^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=n^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=m^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.3)$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,\cdots,m_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant n_2,0\leqslant m_2\leqslant p_{i_2},\cdots,0\leqslant n_r,0\leqslant m_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant p_{i^{(r)}}$ and

$0\leqslant n^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant q_{i^{(r)}}$

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{R}^{+}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

5) $c_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, m^{(k)}); (k = 1, \cdots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, n^{(k)}); (k = 1, \cdots, r).$

$a_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, m_2), \Gamma^{A_{3j}}\left(a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, m_3)$ $, \cdots, \Gamma^{A_{rj}}\left(a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, m_r)$, $\Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$to the left of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$ lie to the right of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{n^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{m^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=n^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=m^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) +$$

$$\sum_{j=1}^{m_2} A_{2j} \beta_{2j}^{(k)} - \tau_{i_2} \left(\sum_{j=m_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) + \cdots +$$

$$\sum_{j=1}^{m_r} A_{rj} \beta_{rj}^{(k)} - \tau_{i_r} \left(\sum_{j=m_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant n^{(i)}} Re\left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]$ and $\beta_i = \max_{1 \leqslant j \leqslant m^{(i)}} Re\left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}}\right)\right]$

In your investigation, we shall use the following notations.

$\mathbb{A} = [(\mathrm{a}_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,m_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{m_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,m_3},$

$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{m_3+1,p_{i_3}}; \cdots; [(\mathrm{a}_{(r-1)j}; \alpha_{(r-1)j}^{(1)}, \cdots, \alpha_{(r-1)j}^{(r-1)}; A_{(r-1)j})]_{1,m_{r-1}},$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha^{(1)}_{(r-1)ji_{r-1}},\cdots,\alpha^{(r-1)}_{(r-1)ji_{r-1}};A_{(r-1)ji_{r-1}})]_{m_{r-1}+1,p_{i_{r-1}}} \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})]_{1,m_r},[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})]_{m+1,p_{i_r}} \tag{1.6}$$

$$A = [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)]_{1,m^{(1)}},[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})]_{m^{(1)}+1,p^{(1)}_i};\cdots;$$

$$[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)]_{1,m^{(r)}},[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})]_{m^{(r)}+1,p^{(r)}_i} \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta^{(1)}_{(r-1)ji_{r-1}},\cdots,\beta^{(r-1)}_{(r-1)ji_{r-1}};B_{(r-1)ji_{r-1}}]_{1,q_{i_{r-1}}} \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})]_{1,q_{i_r}} \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}^{(1)}_j,\delta^{(1)}_j;D^{(1)}_j)]_{1,n^{(1)}},[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})]_{n^{(1)}+1,q^{(1)}_i};\cdots;$$

$$[(\mathrm{d}^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)]_{1,m^{(r)}},[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})]_{m^{(r)}+1,q^{(r)}_i} \tag{1.10}$$

$$U = m_2,0;m_3,0;\cdots;m_{r-1},0;V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

we will take similar parameters, we add the notation a^* to differentiate them from the other parameters.

The serie representation of the multivariable Beth-function is given

$$\beth^*(Z_1,\cdots,Z_v) = \sum_{G_i=1}^{N^{*(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^v\phi_i Z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^v g_i}}{\prod_{i=1}^v\epsilon^{(i)}_{G_i}g_i!} \tag{1.13}$$

where

$$\phi_1 = \frac{\prod_{j=1}^{m_2^*}\Gamma^{A^*_{2j}}(a^*_{2j}+\sum_{k=1}^2\alpha^{*}_{2j}{}^{(k)}\eta_{G_i,g_i})}{\sum_{i_2=1}^{R_2^*}[\tau^*_{i_2}\prod_{j=m_2^*+1}^{p^*_{i_2}}\Gamma^{A^*_{2ji_2}}(1-a^*_{2ji_2}-\sum_{k=1}^2\alpha^{*}_{2ji_2}{}^{(k)}\eta_{G_i,g_i})\prod_{j=1}^{q^*_{i_2}}\Gamma^{B^*_{2ji_2}}(b^*_{2ji2}+\sum_{k=1}^2\beta^{*}_{2ji^2}{}^{(k)}\eta_{G_i,g_i})]}$$

$$\frac{\prod_{j=1}^{m_3^*}\Gamma^{A^*_{3j}}(a^*_{3j}+\sum_{k=1}^3\alpha^{*}_{3j}{}^{(k)}s\eta_{G_i,g_i})}{\sum_{i_3=1}^{R_3^*}[\tau_{i_3}\prod_{j=m_3^*+1}^{p^*_{i_3}}\Gamma^{A_{3ji_3}}(1-a^*_{3ji_3}-\sum_{k=1}^3\alpha^{*}_{3ji_3}{}^{(k)}\eta_{G_i,g_i})\prod_{j=1}^{q^*_{i_3}}\Gamma^{B^*_{3ji_3}}(b^*_{3ji3}+\sum_{k=1}^3\beta^{*}_{3ji^3}{}^{(k)}\eta_{G_i,g_i})]}$$

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$$\frac{\prod_{j=1}^{m_v^*}\Gamma^{A_{vj}^*}(a_{vj}^*+\sum_{k=1}^{r}\alpha_{vj}^{*}{}^{(k)}\eta_{Gi},g_i))}{\sum_{i_v=1}^{R_v^*}[\tau_{i_v}\prod_{j=m_v^*+1}^{p_{i_v}^*}\Gamma^{A_{vji_v}^*}(1-a_{vji_v}^*-\sum_{k=1}^{v}\alpha_{vji_v}^{*}{}^{(k)}\eta_{G_i,g_i})\prod_{j=1}^{q_{i_v}^*}\Gamma^{B_{vji_v}}(1b_{vjiv}+\sum_{k=1}^{v}\beta_{vjiv}^{*}{}^{(k)}\eta_{G_i,g_i})]} \tag{1.14}$$

and

$$\phi_i=\frac{\prod_{j=1}^{n^{*(k)}}\Gamma^{D_j^{*(k)}}(d_j^{*(k)}-\delta_j^{*(k)}\eta_{G_i,g_i})\prod_{j=1}^{m^{*(k)}}\Gamma^{C_j^{*(k)}}(1-c_j^{*(k)}+\gamma_j^{*(k)}\eta_{G_i,g_i})}{\sum_{i^{(k)}=1}^{R^{*(k)}}[\tau_{i^{(k)}}\prod_{j=n^{*(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{*}{}^{(k)}}(1-d_{ji^{(k)}}^{*}{}^{(k)}+\delta_{ji^{(k)}}^{*}{}^{(k)}\eta_{G_i,g_i})\prod_{j=m^{*(k)}+1}^{p_{i^{(k)}}^*}\Gamma^{C_{ji^{(k)}}^{*}{}^{(k)}}(c_{ji^{(k)}}^{*}{}^{(k)}-\gamma_{ji^{(k)}}^{*}{}^{(k)}\eta_{G_i,g_i})]} \tag{1.15}$$

where

$$\eta_{G_i,g_i}=\frac{1-d_{G_i}^{(i)}+g_i}{\epsilon_{G_i}^{(i)}},i=1,\cdots,v \tag{1.16}$$

which is valid under the following conditions : $\frac{1-d_{G_{j_l}}^{(i)}+g_l}{\epsilon_{G_{j_l}^{(i)}}}\neq\frac{1-d_{G_{h_l}}^{(i)}+g_l}{\epsilon_{G_{h_l}^{(i)}}}j_l\neq h_l$

2. Required results.

In this section, we cite two integrals due to (Erdelyi, [3], 1954 p. 284-285).

Lemma 1.

$$\int_0^a\frac{t^\rho(a-t)^\sigma}{(a+bt)^{\sigma+\rho+2}}P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right)\mathrm{d}t=\frac{\Gamma(\alpha+u+1)}{a(b+1)^{1+\rho}u!\Gamma(\alpha+\beta+u+1)}$$

$$\sum_{s=0}^{u}\frac{(-u)_s\Gamma(\alpha+\beta+u+s+1)}{u!\Gamma(a+s-1)}\frac{\Gamma(1+s+\rho)\Gamma(1+\sigma)}{\Gamma(2+s+\rho+\sigma)} \tag{2.1}$$

provided that $a>0,b\neq-1,Re(\alpha)>-1,Re(\beta)>1.$

Lemma 2 ;

$$\int_0^a\frac{t^\rho(a-t)^\sigma}{(a+bt)^{\sigma+\rho+2}}P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right)P_s^{(\gamma,\delta)}\left(\frac{a-bt-2t}{a+bt}\right)\mathrm{d}t$$

$$=\left[\begin{array}{c}0\text{ if s}\neq u\\ .\\ \frac{1}{a(b+1)^{1+a}u!}\frac{\Gamma(\alpha+u+1)\Gamma(\beta+u+1)}{(\alpha+\beta+2u+1)\Gamma(\alpha+\beta+u+1)}\text{ if s = u}\end{array}\right. \tag{2.2}$$

3. Main integral.

In this section, we evaluate one general integral involving the product of two multivariable Beth -functions and Jacobi polynomials.

Theorem 1.

$$\int_0^a\frac{t^\rho(a-t)^\sigma}{(a+bt)^{\sigma+\rho+2}}P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right)\beth^*\left(\frac{y_1(b+1)^{c_1}t^{c_1}(a-t)^{d_1}}{(a+bt)^{c_1+d_1}},\cdots,\frac{y_v(b+1)^{c_v}t^{c_v}(a-t)^{d_v}}{(a+bt)^{c_v+d_v}}\right)$$

$$\beth\left(\frac{z_1(b+1)^{e_1}t^{e_1}(a-t)^{f_1}}{(a+bt)^{e_1+f_1}},\cdots,\frac{y_r(b+1)^{e_r}t^{e_r}(a-t)^{f_r}}{(a+bt)^{e_r+f_r}}\right)\mathrm{d}t=\frac{\Gamma(\alpha+u+1)}{a(b+1)^{1+\rho}u!\Gamma(\alpha+\beta+u+1)}$$

$$\sum_{G_i=1}^{N^*(i)}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i y_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{G_i}^{(i)}g_i!}\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;m_r+2,0:V}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};(1+s+\rho+\sum_{j=1}^{v}c_j\eta_{G_j,g_j};e_1,\cdots,e_r;1),(1+\sigma+\sum_{j=1}^{v}d_j\eta_{G_j,g_j};f_1,\cdots,f_r;1)\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};(2+s+\rho+\sigma+\sum_{j=1}^{v}(c_j+d_j)\eta_{G_j,g_j};e_1+f_1,\cdots,e_r+f_r;1)\mathbf{B}:B \end{array}\right) \quad (3.1)$$

provided

$e_i,f_i>0(i)1,\cdots,r),c_j,d_j>0(j=1,\cdots,v),a>0,b\neq-1.$

$$Re\left(\rho+\sum_{j=1}^{v}c_j\eta_{G_j,g_j}\right)+\sum_{i=1}^{r}e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>-1$$

and

$\left|arg\left(z_i\frac{b+1)^{e_i}t^{e_i}(a-t)^{f_i}}{(a+bt)^{e_i+f_i}}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

In the left of (3.1) expressing the beth-function $\gimel^*$ of v variables in series form with rhe help of (1.13) and expressing the Beth-function of r variables in multiple integrals contour with the help of (1.1), and interchanging the order of integrations and summations which is justified under the conditions stated, evaluating the inner t-integral with the help of the lemma 1 and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

4. expansion formula.

By using of the previous section here we have obtained an expansion formula.

Theorem 2.

$$\frac{t^\rho(a-t)^\sigma}{(a+bt)^{\sigma+\rho}}\gimel^*\left(\frac{y_1(b+1)^{c_1}t^{c_1}(a-t)^{d_1}}{(a+bt)^{c_1+d_1}},\cdots,\frac{y_v(b+1)^{c_v}t^{c_v}(a-t)^{d_v}}{(a+bt)^{c_v+d_v}}\right)$$

$$\gimel\left(\frac{z_1(b+1)^{e_1}t^{e_1}(a-t)^{f_1}}{(a+bt)^{e_1+f_1}},\cdots,\frac{y_r(b+1)^{e_r}t^{e_r}(a-t)^{f_r}}{(a+bt)^{e_r+f_r}}\right)=$$

$$\sum_{s=0}^{u}\frac{(-u)_s\Gamma(\alpha+\beta+u+s+1)}{u!\Gamma(a+s-1)}P_u^{(\alpha,\beta)}\left(\frac{a-bt-2t}{a+bt}\right)$$

$$\sum_{G_i=1}^{N^*(i)}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i y_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{G_i}^{(i)}g_i!}\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;m_r+2,0:V}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};(1+s+\alpha+\rho+\sum_{j=1}^{v}c_j\eta_{G_j,g_j};e_1,\cdots,e_r;1),(1+\sigma+\beta+\sum_{j=1}^{v}d_j\eta_{G_j,g_j};f_1,\cdots,f_r;1)\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};(2+s+\rho+\sigma+\sum_{j=1}^{v}(c_j+d_j)\eta_{G_j,g_j};e_1+f_1,\cdots,e_r+f_r;1)\mathbf{B}:B \end{array}\right) \quad (4.1)$$

provided

$e_i, f_i > 0(i)1, \cdots, r), c_j, d_j > 0(j = 1, \cdots, v), a > 0, b \neq -1.Re'\alpha), Re(\beta) > -1.$

$$Re\left(\rho + \sum_{j=1}^{v} c_j \eta_{G_j,g_j}\right) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1$$

and

$$\left|arg\left(z_i \frac{b+1)^{e_i} t^{e_i}(a-t)^{f_i}}{(a+bt)^{e_i+f_i}}\right)\right| < \frac{1}{2} A_i^{(k)} \pi \text{ where } A_i^{(k)} \text{ is defined by (1.4).}$$

Proof

suppose when $0 < t < a$, left hand side of (4.1) is

$$\sum_{s=0}^{u} A_s P_s^{(\alpha,\beta)}\left(\frac{a - bt - 2t}{a + bt}\right) \tag{4.2}$$

Muliply (4.2) by $\dfrac{t^\alpha (a-t)^\beta}{(a+bt)^{\alpha+\beta+2}} P_u^{(\alpha,\beta)}\left(\dfrac{a-bt-2t}{a+bt}\right)$, integrate with respect t form 0 to a, change the order of integrations and summations in the right (which is permissible under the above conditions), use the theorem 1 in the left and the lemma 2 in the right to obtain A_u. Now substitute the value of A_u in (4.2), we obtain the the theorem 2.

Remark 1.
If $m_2 = \cdots = m_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable beth-function reduces in the modified multivariable Aleph- function. This function is a modification of the multivariable Aleph-function defined by Ayant [1]. We obtain the same relations obtained above.

Remark 2.
If $m_2 = \cdots = m_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Beth-function reduces in a modified multivariable I-function. This function is a modification of the multivariable I-function defined by Prathima et al. [5]. We obtain the same relations obtained above.

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)}$ $= \cdots = R^{(r)} = 1$, then the multivariable Beth-function reduces in modified multivariable I-function. This function is a modification of the multivariable I-function defined by Prasad [4]. We obtain the same relations obtained above.

Remark 4.
If the three above conditions are satisfied at the same time, then the multivariable Beth-function reduces in the modified multivariable H-function. This function is a modification of the multivariable H-function defined by Srivastava and Panda [6,7]. We obtain the same relations obtained above.

4. Conclusion.

The importance of our all the results lies in their manifold generality. By specialising the various parameters as well as variables in the multivariable Beth-functions, we get an integral formulae and an expansion serie in Jacobi polynomial involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

REFERENCES.

[1] F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

[2] B.L.J. Braaksma, Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[3] A. Erdelyi, W. Magnus, F. Oberhettinger transforms and F.G. Tricomi, Integral transforms, Vol I, McGraw-Hill, New York (1953).

[4] Y.N. Prasad, Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , 231-237.

[5] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 1-12.

[6] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment certain of. Math. Univ. St. Paul. 24 (1975),119-137.

[7] H.M. Srivastava and R.Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.