On Integrals of Multivariable Beth -Functions

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ABSTRACT
In the present paper we have evaluated four integrals given in the form of theorems. These integrals by virtue og the multivariable Beth-functions in the integrand, form the basic of obtaining most general integrals. Interesting special cases are also recorded.

KEYWORDS : Multivariable Beth-function, multiple integral contours, serie representation, integrals.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Introduction and preliminaries.

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Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

We define a generalized transcendental function of several complex variables noted $\beth$. This function is a modification of the multivariable Aleph-function recently defined by Ayant [1].

$$\beth(z_1,\cdots,z_r)=\beth^{m_2,0;m_3,0;\cdots;m_r,0;m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right|$$

$$\begin{array}{c}[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,m_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{m_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,m_3},\\ [\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};\end{array}$$

$$\begin{array}{l}[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{m_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})]_{1,m_r},\\ [\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\qquad\cdots\end{array}$$

$$\begin{array}{l}[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})]_{m_r+1,p_r}:[(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)]_{1,m^{(1)}},[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})]_{m^{(1)}+1,p^{(1)}_i}\\ [\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})]_{1,q_r}:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)]_{1,n^{(1)}},[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})]_{n^{(1)}+1,q^{(1)}_i}\end{array}$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)]_{1,m^{(r)}},[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})]_{m^{(r)}+1,p^{(r)}_i}\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)]_{1,n^{(r)}},[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}}]_{n^{(r)}+1,q^{(r)}_i}\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{A_{2j}}(a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=m_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(1-a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{A_{3j}}(a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=m_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(1-a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{A_{rj}}(a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=m_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(1-a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{n^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{m^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=n^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=m^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,\cdots,m_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant n_2,0\leqslant m_2\leqslant p_{i_2},\cdots,0\leqslant n_r,0\leqslant m_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant p_{i^{(r)}}$ and

$0\leqslant n^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant q_{i^{(r)}}$

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r).$

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k).$

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k).$

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k).$

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r).$

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r).$

5) $c_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, m^{(k)}); (k = 1, \cdots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, n^{(k)}); (k = 1, \cdots, r).$

$a_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right)(j = 1, \cdots, m_2), \Gamma^{A_{3j}}\left(a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, m_3)$ $, \cdots, \Gamma^{A_{rj}}\left(a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right)(j = 1, \cdots, m_r)$, $\Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right)(j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$to the left of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right)(j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$ lie to the right of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{n^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{m^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=n^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=m^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) +$$

$$\sum_{j=1}^{m_2} A_{2j} \beta_{2j}^{(k)} - \tau_{i_2} \left(\sum_{j=m_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) + \cdots +$$

$$\sum_{j=1}^{m_r} A_{rj} \beta_{rj}^{(k)} - \tau_{i_r} \left(\sum_{j=m_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(\,|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}\,), max(\,|z_1|, \cdots, |z_r|\,) \to 0$

$\aleph(z_1, \cdots, z_r) = 0(\,|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}\,), min(\,|z_1|, \cdots, |z_r|\,) \to \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant n^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]$ and $\beta_i = \max_{1 \leqslant j \leqslant m^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}}\right)\right]$

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(a_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,m_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{m_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{m_3+1,p_{i_3}}; \cdots; [(a_{(r-1)j}; \alpha_{(r-1)j}^{(1)}, \cdots, \alpha_{(r-1)j}^{(r-1)}; A_{(r-1)j})]_{1,m_{r-1}},$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}}; \alpha_{(r-1)ji_{r-1}}^{(1)}, \cdots, \alpha_{(r-1)ji_{r-1}}^{(r-1)}; A_{(r-1)ji_{r-1}})]_{m_{r-1}+1,p_{i_{r-1}}} \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})]_{1,m_r},[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{m+1,p_{i_r}} \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,p_i^{(1)}};\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}} \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}}]_{1,q_{i_{r-1}}} \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_{i_r}} \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,q_i^{(1)}};\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}} \tag{1.10}$$

$$U = m_2,0;m_3,0;\cdots;m_{r-1},0;V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

we will take similar parameters, we add the notation a^* to differentiate them from the other parameters.

The serie representation of the multivariable Beth-function is given

$$\beth^*(Z_1,\cdots,Z_v) = \sum_{G_i=1}^{N^{*(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^v\phi_i Z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^v g_i}}{\prod_{i=1}^v\epsilon_{G_i}^{(i)}g_i!} \tag{1.13}$$

where

$$\phi_1 = \frac{\prod_{j=1}^{m_2^*}\Gamma^{A_{2j}^*}(a_{2j}^*+\sum_{k=1}^2\alpha_{2j}^{*k}\eta_{G_i,g_i})}{\sum_{i_2=1}^{R_2^*}[\tau_{i_2}^*\prod_{j=m_2^*+1}^{p_{i_2}^*}\Gamma^{A_{2ji_2}^*}(1-a_{2ji_2}^*-\sum_{k=1}^2\alpha_{2ji_2}^{*k}\eta_{G_i,g_i})\prod_{j=1}^{q_{i_2}^*}\Gamma^{B_{2ji_2}^*}(b_{2ji2}^*+\sum_{k=1}^2\beta_{2ji^2}^{*k}\eta_{G_i,g_i})]}$$

$$\frac{\prod_{j=1}^{m_3^*}\Gamma^{A_{3j}^*}(a_{3j}^*+\sum_{k=1}^3\alpha_{3j}^{*k}s\eta_{G_i,g_i})}{\sum_{i_3=1}^{R_3^*}[\tau_{i_3}^*\prod_{j=m_3^*+1}^{p_{i_3}^*}\Gamma^{A_{3ji_3}^*}(1-a_{3ji_3}^*-\sum_{k=1}^3\alpha_{3ji_3}^{*k}\eta_{G_i,g_i})\prod_{j=1}^{q_{i_3}^*}\Gamma^{B_{3ji_3}^*}(b_{3ji3}^*+\sum_{k=1}^3\beta_{3ji^3}^{*k}\eta_{G_i,g_i})]}$$

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$$\frac{\prod_{j=1}^{m_v^*}\Gamma^{A_{vj}^*}(a_{vj}^*+\sum_{k=1}^{r}\alpha_{vj}^{k*}\eta_{G_i,g_i})}{\sum_{i_v=1}^{R_v^*}[\tau_{i_v}^*\prod_{j=m_v^*+1}^{p_{i_v}^*}\Gamma^{A_{vji_v}^*}(1-a_{vji_v}^*-\sum_{k=1}^{v}\alpha_{vji_v}^{*k}\eta_{G_i,g_i})\prod_{j=1}^{q_{i_v}^*}\Gamma^{B_{vji_v}^*}(b_{vjiv}^*+\sum_{k=1}^{v}\beta_{vjiv}^{*k}\eta_{G_i,g_i})]} \tag{1.14}$$

and

$$\phi_i=\frac{\prod_{j=1}^{n^{*(k)}}\Gamma^{D_j^{*(k)}}(d_j^{*(k)}-\delta_j^{*(k)}\eta_{G_i,g_i})\prod_{j=1}^{m^{*(k)}}\Gamma^{C_j^{*(k)}}(1-c_j^{*(k)}+\gamma_j^{*(k)}\eta_{G_i,g_i})}{\sum_{i^{*(k)}=1}^{R^{*k}}[\tau_{i^{(k)}}^*\prod_{j=n^{*k}+1}^{q_{i^{(k)}}^*}\Gamma^{D_{ji^{(k)}}^{*k}}(1-d_{ji^{(k)}}^{*k}+\delta_{ji^{(k)}}^{*k}\eta_{G_i,g_i})\prod_{j=m^{*k}+1}^{p_{i^{(k)}}^*}\Gamma^{C_{ji^{(k)}}^{*k}}(c_{ji^{(k)}}^{*k}-\gamma_{ji^{(k)}}^{*k}\eta_{G_i,g_i})]} \tag{1.15}$$

where

$$\eta_{G_i,g_i}=\frac{1-d_{G_i}^{(i*)}+g_i}{\epsilon_{G_i}^{(i)}}, i=1,\cdots,v \tag{1.16}$$

which is valid under the following conditions : $\frac{1-d_{G_{j_l}}^{(i*)}+g_l}{\epsilon_{G_{j_l}^{(i)}}}\neq\frac{1-d_{G_{h_l}}^{(i*)}+g_l}{\epsilon_{G_{h_l}^{(i)}}} j_l\neq h_l$

2. Required results.

In this section, we cite four integrals due to (Erdelyi, [3], 1954). These integrals will utilized in your investigation.

Lemma 1.

$$\int_0^1 x^{c-1}(1-x)^{d-1}[ax+b(1-x)]^{-c-d}e^{-\frac{zax}{ax+b(1-x)}}{}_2F_1\left[\alpha,\beta;c;\frac{ax}{ax+b(1-x)}\right]\mathrm{d}x=$$

$$e^{-z}\frac{\Gamma(c)\Gamma(d)\Gamma(c+d-\alpha-\beta)}{a^cb^d\Gamma(c+d-\alpha)\Gamma(c+d-\beta)}{}_2F_2\left[d,c+d-\alpha-\beta;c+d-\alpha,c+d-\beta;z\right] \tag{2.1}$$

provided $Re(c),Re(d),Re(c+d-\alpha-\beta)>0$, a,b are different at zero, $ax+b(1-x)\neq 0$. $0\leqslant x\leqslant 1$

Lemma 2.

$$\int_0^1 x^{c-1}(1-x)^{d-1}[ax+b(1-x)]^{-2c}{}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{ax}{ax+b(1-x)}\right]\mathrm{d}x=$$

$$\frac{\pi\Gamma(c)\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\frac{1-\alpha-\beta}{2}+c\right)}{2^{2c-1}(ab)^c\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(\frac{1-\alpha}{2}+c\right)\Gamma\left(\frac{1-\beta}{2}+c\right)} \tag{2.2}$$

provided $Re(c),Re\left(\frac{\alpha+\beta+1}{2}\right),Re\left(\frac{1-\alpha-\beta}{2}+c\right)>0$, a,b are different at zero, $ax+b(1-x)\neq 0$. $0\leqslant x\leqslant 1$

Lemma 3.

$$\int_0^1 x^{c-1}(1-x)^{d-1}[ax+b(1-x)]^{-2c+d-1}{}_2F_1\left[a,1-a;d;\frac{ax}{ax+b(1-x)}\right]\mathrm{d}x=$$

$$\frac{\pi 2^{1-2c}\Gamma(c)\Gamma(d)\Gamma(c-d+1)}{b^{c-d+1}a^c\Gamma\left(\frac{d+\alpha}{2}\right)\Gamma\left(c+\frac{a-d+1}{2}\right)\Gamma\left(\frac{1-a+d}{2}\right)\Gamma\left(1+c-\frac{a+d}{2}\right)} \tag{2.3}$$

provided $Re(c),Re(c-d+1),Re(d)>0$,a,b are different at zero, $ax+b(1-x)\neq 0, 0\leqslant x\leqslant 1$.

Lemma 4.

$$\int_0^\infty x^{s-1}e^{\frac{x}{2}}W_{k,u}(x)\mathrm{d}x=\frac{\Gamma\left(u+s+\frac{1}{2}\right)\Gamma\left(s-u+\frac{1}{2}\right)\Gamma(-k-s)}{\Gamma\left(u-k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-u-k\right)} \tag{2.4}$$

provided $|Re(u)|-\frac{1}{2}<Re(s)<-Re(k)$

3. Main integrals.

In this section, we evaluate four general integrals involving the product of two multivariable Beth -functions.

Theorem 1.

$$\int_0^1 x^{c-1}(1-x)^{d-1}[ax+b(1-x)]^{-c-d}e^{-\frac{zax}{ax+b(1-x)}}\,{}_2F_1\left[\alpha,\beta;c;\frac{ax}{ax+b(1-x)}\right]$$

$$\beth^*\left(Z_1\left(\frac{b(1-x)}{ax+b(1-x)}\right)^{a_1},\cdots,Z_v\left(\frac{b(1-x)}{ax+b(1-x)}\right)^{a_v}\right)\beth\left(z_1\left(\frac{b(1-x)}{ax+b(1-x)}\right)^{b_1},\cdots,z_r\left(\frac{b(1-x)}{ax+b(1-x)}\right)^{b_r}\right)\mathrm{d}x=$$

$$\frac{e^{-z}\Gamma(c)}{a^cb^d}\sum_{u=0}^{\infty}\sum_{G_i=1}^{N^{*(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^v\phi_iZ_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^v g_i}}{\prod_{i=1}^v\epsilon_{G_i}^{(i)}g_i!}\frac{z^u}{u!}\beth^{U;m_r+2,0:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};(d+\sum_{j=1}^v a_j\eta_{G_j,g_j};b_1,\cdots,b_r;1),(c+d+u-\alpha-\beta+\sum_{j=1}^v a_j\eta_{G_j,g_j};b_1,\cdots,b_r;1)\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\mathbf{B},(c+d+u-\alpha+\sum_{j=1}^v d_j\eta_{G_j,g_j};b_1,\cdots,b_r;1),(c+d+u-\beta+\sum_{j=1}^v d_j\eta_{G_j,g_j};b_1,\cdots,b_r;1):B\end{array}\right) \tag{3.1}$$

Provided

$a_j>0(j=1,\cdots,v);b_i>0(i=1,\cdots,r),Re(c+d-\alpha-\beta)>0,Re(c)>0,Re(d)>0,$ a,b are different at zero, $ax+b(1-x)\neq 0, 0\leqslant x\leqslant 1.$

$$Re\left(c+(c+d-\alpha-\beta)u+\sum_{j=1}^v a_j\eta_{G_j,g_j}\right)+\sum_{i=1}^r e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$Re\left(d+(c+d-\alpha-\beta)u+\sum_{j=1}^v a_j\eta_{G_j,g_j}\right)+\sum_{i=1}^r e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$\left|arg\left(z_i\frac{b(1-x)}{(x+b(1-x))}\right)^{b_i}\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

In the left of (3.1) expressing the beth-function $\beth^*$ of v variables in series form with rhe help of (1.13) and expressing the Beth-function of r variables in multiple integrals contour with the help of (1.1), and interchanging the order of integrations and summations which is justified under the conditions stated, we obtain (say I)

$$I=\sum_{G_i=1}^{N^{*(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^v\phi_iZ_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^v g_i}}{\prod_{i=1}^v\epsilon_{G_i}^{(i)}g_i!}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^r\theta_k(s_k)z_k^{s_k}$$

$$\int_0^1 x^{c-1}(1-x)^{d+\sum_{j=1}^v\eta_{G_j,g_j}a_j+\sum_{i=1}^r b_is_i}[ax+b(1-x)]^{-c-d-\sum_{j=1}^v\eta_{G_j,g_j}a_j-\sum_{i=1}^r b_is_i}$$

$$e^{-\frac{zax}{ax+b(1-x)}}\,{}_2F_1\left[\alpha,\beta;c;\frac{ax}{ax+b(1-x)}\right]\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.2}$$

evaluating the inner x-integral with the help of the lemma 1 and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

Theorem 2.

$$\int_0^1 x^{c-1}(1-x)^{d-1}[ax+b(1-x)]^{-2c}{}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{ax}{ax+b(1-x)}\right]$$

$$\gimel^*\left(Z_1\left(\frac{b(1-x)}{[ax+b(1-x)]^2}\right)^{a_1},\cdots,Z_v\left(\frac{b(1-x)}{(ax+b(1-x)]^2}\right)^{a_v}\right)\gimel\left(z_1\left(\frac{b(1-x)}{[ax+b(1-x)]^2}\right)^{b_1},\cdots,z_r\left(\frac{b(1-x)}{[ax+b(1-x)]^2}\right)^{b_r}\right)\mathrm{d}x=$$

$$\frac{2\pi\Gamma\left(\frac{\alpha+\beta+1}{2}\right)}{(4ab)^c\Gamma\left(\frac{1}{2}+\alpha\right)\Gamma\left(\frac{1+\beta}{2}\right)}\sum_{G_i=1}^{N^{*(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^v\phi_i(4^{w_j}Z_i)^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^v g_i}}{\prod_{i=1}^v\epsilon_{G_i}^{(i)}g_i!}\gimel^{U;m_r+2,0:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c}4^{b_1}z_1 & \mathbb{A};(c+4\sum_{j=1}^v\eta_{G_j,g_j};b_1,\cdots,b_r;1),\left(\frac{1-\alpha-\beta}{2}+4\sum_{j=1}^v\eta_{G_j,g_j};b_1,\cdots,b_r;1\right)\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ 4^{b_r}z_r & \mathbb{B};\mathrm{B},\left(\frac{1-\alpha}{2}+c+\sum_{j=1}^v a_j\eta_{G_j,g_j};b_1,\cdots,b_r;1\right),(c+\frac{1-\beta}{2}+\sum_{j=1}^v a_j\eta_{G_j,g_j};b_1,\cdots,b_r;1):B\end{array}\right) \tag{3.3}$$

Provided

$a_j>0(j=1,\cdots,v);b_i>0(i=1,\cdots,r),Re\left(\frac{1+\alpha+\beta}{2}\right)>0,$, a,b are different at zero, $ax+b(1-x)\neq 0$, $0\leqslant x\leqslant 1$.

$$Re\left(c+\sum_{j=1}^v a_j\eta_{G_j,g_j}\right)+\sum_{i=1}^r e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$Re\left(d+\sum_{j=1}^v a_j\eta_{G_j,g_j}\right)+\sum_{i=1}^r e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$\left|arg\left(z_i\frac{b(1-x)}{[x+b(1-x)]^2}\right)^{b_i}\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 3.

$$\int_0^1 x^{c-1}(1-x)^{d-1}[ax+b(1-x)]^{-2c+d-1}{}_2F_1\left[a,1-a;d;\frac{ax}{ax+b(1-x)}\right]$$

$$\gimel^*\left(Z_1\left(\frac{b(1-x)}{[ax+b(1-x)]^2}\right)^{a_1},\cdots,Z_v\left(\frac{b(1-x)}{(ax+b(1-x)]^2}\right)^{a_v}\right)\gimel\left(z_1\left(\frac{b(1-x)}{[ax+b(1-x)]^2}\right)^{b_1},\cdots,z_r\left(\frac{b(1-x)}{[ax+b(1-x)]^2}\right)^{b_r}\right)\mathrm{d}x=$$

$$\frac{2\pi\Gamma(d)}{(4ab)^c b^{1-d}\Gamma\left(\frac{a+d}{2}\right)\Gamma\left(\frac{1-a+d}{2}\right)}\sum_{G_i=1}^{N^{*(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^v\phi_i\left(\frac{Z_i}{4w_j}\right)^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^v g_i}}{\prod_{i=1}^v\epsilon_{G_i}^{(i)}g_i!}\gimel^{U;m_r+2,0:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} 4^{-b_1}z_1 & \mathbb{A};(c+\sum_{j=1}^{v}a_j\eta_{G_j,g_j};b_1,\cdots,b_r;1),\left(c+\sum_{j=1}^{v}a_j\eta_{G_j,g_j}-d+1;b_1,\cdots,b_r;1\right)\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{-b_r}z_r & \mathbb{B};\mathrm{B},\left(\frac{1+a-d}{2}+c+\sum_{j=1}^{v}\eta_{G_j,g_j}a_j;b_1,\cdots,b_r;1\right),(1+c-\frac{a+d}{2}+\sum_{j=1}^{v}a_j\eta_{G_j,g_j};b_1,\cdots,b_r;1):B \end{array}\right) \quad (3.4)$$

Provided

$a_j>0(j=1,\cdots,v);b_i>0(i=1,\cdots,r),Re(c)>0,Re(c-d+1)>0,Re(d)>0$
, a,b are different at zero, $ax+b(1-x)\neq 0$, $0\leqslant x\leqslant 1$.

$$Re\left(c+\sum_{j=1}^{v}a_j\eta_{G_j,g_j}\right)+\sum_{i=1}^{r}e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$Re\left(d+\sum_{j=1}^{v}a_j\eta_{G_j,g_j}\right)+\sum_{i=1}^{r}e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$\left|arg\left(z_i\frac{b(1-x)}{[x+b(1-x)]^2}\right)^{b_i}\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 4.

$$\int_0^{\infty}x^{s-1}e^{\frac{x}{2}}W_{k,u}(x)\daleth^*\left(Z_1x^{a_1},\cdots,Z_vx^{a_v}\right)\daleth\left(z_1x^{b_1},\cdots,z_rx^{b_r}\right)\mathrm{d}x=\frac{1}{\Gamma\left(u-s+\frac{1}{2}\right)\Gamma\left(-u-s+\frac{1}{2}\right)}$$

$$\sum_{G_i=1}^{N^{*(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{v}\phi_i\left(-Z_i\right)^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{v}g_i}}{\prod_{i=1}^{v}\epsilon_{G_i}^{(i)}g_i!}\quad\daleth_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;m_r+2,1:V}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\left(u+s+\sum_{j=1}^{v}a_j\eta_{G_j,g_j}+\frac{1}{2};b_1,\cdots,b_r;1\right),\left(\frac{1}{2}-u-s+\sum_{j=1}^{v}a_j\eta_{G_j,g_j};b_1,\cdots,b_r;1\right)\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathrm{B},\left(1+k+s+\sum_{j=1}^{v}a_j\eta_{G_j,g_j};b_1,\cdots,b_r;1\right):B \end{array}\right) \quad (3.5)$$

Provided

$a_j>0(j=1,\cdots,v);b_i>0(i=1,\cdots,r),|Re(u)|-\frac{1}{2}<Re(s)<-Re(k)$

$$Re\left(s+\frac{1}{2}+\sum_{j=1}^{v}c_j\eta_{G_j,g_j}\right)+\sum_{i=1}^{r}e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>[Re(u)|$$

$$Re\left(s-k+\sum_{j=1}^{v}c_j\eta_{G_j,g_j}\right)+\sum_{i=1}^{r}e_i\min_{1\leqslant j\leqslant n^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]<0$$

$|arg\left(z_ix^{b_i}\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

To prove the theorems 2 to 4, we use the similar lines to the theorem 1 by using the lemmae 2 to 4 respectively, instead of lemma 1.

Remark 1.
If $m_2 = \cdots = m_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable beth-function reduces in the modified multivariable Aleph- function. This function is a modification of the multivariable Aleph-function defined by Ayant [1].

Remark 2.
If $m_2 = \cdots = m_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Beth-function reduces in a modified multivariable I-function. This function is a modification of the multivariable I-function defined by Prathima et al. [5].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Beth-function reduces in modified multivariable I-function. This function is a modification of the multivariable I-function defined by Prasad [4].

Remark 4.
If the three above conditions are satisfied at the same time, then the multivariable Beth-function reduces in the modified multivariable H-function. This function is a modification of the multivariable H-function defined by Srivastava and Panda [6,7].

Remark 5.
We obtain easily the same integrals about the above functions.

4. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these single integrals, we can obtain a large simpler single finite integrals, Secondly by specialising the various parameters as well as variables in the multivariable Beth-functions, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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