

# A Study of Unified Finite Integrals Involving Generalized Modified Bessel Function of Third Kind, $(\lambda_{\mu, \nu}^{\eta})$ General Class of Polynomials and the Multivariable Gimel-Function

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**ABSTRACT**

In this paper, we first evaluate a unified and general finite integral whose integrand involves the product of the function  $\lambda_{\mu, \nu}^{\eta}$ , the class of polynomials  $S_N^M$  and the multivariable Gimel function. The arguments of the function occurring in the integrand involve the product of factors of the form  $x^{\rho-1}(a - x^{\sigma}(1 + (bx)^{\lambda})^{-\lambda})$ . On account of the most general nature of our main findings, a large number of new and known integrals can easily be obtained from it merely by specializing the functions and parameters involved therein. At the end of this paper, we shall give two particular cases.

**Keywords:** Generalized Bessel function of third kind, multivariable Gimel-function, class of polynomials, H-function.

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## 1. Introduction

We throughout this paper, let  $\mathbb{C}$ ,  $\mathbb{R}$  and  $\mathbb{N}$  be set of complex numbers, real numbers and positive integers respectively. Also  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . We define a generalized transcendental function of several complex variables.

$$\mathfrak{J}(z_1, \dots, z_r) = \mathfrak{J}_{p_2, q_2, \tau_2; R_2; p_3, q_3, \tau_3; R_3; \dots; p_r, q_r, \tau_r; R_r; p_i(1), q_i(1), \tau_i(1); R_i(1); \dots; p_i(r), q_i(r), \tau_i(r); R_i(r)}^{0, n_2; 0, n_3; \dots; 0, n_r; m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \dots; m^{(r)}, n^{(r)}} \left( \begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \right)$$

$$[(a_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1, n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1, p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1, n_3},$$

$$[\tau_{i_2}(b_{2ji_2}; \beta_{2ji_2}^{(1)}, \beta_{2ji_2}^{(2)}; B_{2ji_2})]_{1, q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1, p_{i_3}}; \dots; [(a_{rj}; \alpha_{rj}^{(1)}, \dots, \alpha_{rj}^{(r)}; A_{rj})]_{1, n_r},$$

$$[\tau_{i_3}(b_{3ji_3}; \beta_{3ji_3}^{(1)}, \beta_{3ji_3}^{(2)}, \beta_{3ji_3}^{(3)}; B_{3ji_3})]_{1, q_{i_3}}; \dots;$$

$$[\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \dots, \alpha_{rji_r}^{(r)}; A_{rji_r})]_{n_r+1, p_{i_r}} : [(c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})]_{1, n^{(1)}}, [\tau_{i(1)}(c_{ji(1)}^{(1)}, \gamma_{ji(1)}^{(1)}; C_{ji(1)}^{(1)})]_{n^{(1)}+1, p_i^{(1)}}]$$

$$[\tau_{i_r}(b_{rji_r}; \beta_{rji_r}^{(1)}, \dots, \beta_{rji_r}^{(r)}; B_{rji_r})]_{1, q_r} : [(d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})]_{1, m^{(1)}}, [\tau_{i(1)}(d_{ji(1)}^{(1)}, \delta_{ji(1)}^{(1)}; D_{ji(1)}^{(1)})]_{m^{(1)}+1, q_i^{(1)}}]$$

$$\left. \begin{matrix} \dots; [(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})]_{1, m^{(r)}}, [\tau_{i(r)}(c_{ji(r)}^{(r)}, \gamma_{ji(r)}^{(r)}; C_{ji(r)}^{(r)})]_{m^{(r)}+1, p_i^{(r)}}] \\ \dots; [(d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})]_{1, n^{(r)}}, [\tau_{i(r)}(d_{ji(r)}^{(r)}, \delta_{ji(r)}^{(r)}; D_{ji(r)}^{(r)})]_{n^{(r)}+1, q_i^{(r)}}] \end{matrix} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \theta_k(s_k) z_k^{s_k} ds_1 \dots ds_r \tag{1.1}$$

with  $\omega = \sqrt{-1}$

$$\psi(s_1, \dots, s_r) = \frac{\prod_{j=1}^{n_2} \Gamma^{A_{2j}} (1 - a_{2j} + \sum_{k=1}^2 \alpha_{2j}^{(k)} s_k)}{\sum_{i_2=1}^{R_2} [\tau_{i_2} \prod_{j=n_2+1}^{p_{i_2}} \Gamma^{A_{2ji_2}} (a_{2ji_2} - \sum_{k=1}^2 \alpha_{2ji_2}^{(k)} s_k) \prod_{j=1}^{q_{i_2}} \Gamma^{B_{2ji_2}} (1 - b_{2ji_2} + \sum_{k=1}^2 \beta_{2ji_2}^{(k)} s_k)]}$$

$$\frac{\prod_{j=1}^{n_3} \Gamma^{A_{3j}} (1 - a_{3j} + \sum_{k=1}^3 \alpha_{3j}^{(k)} s_k)}{\sum_{i_3=1}^{R_3} [\tau_{i_3} \prod_{j=n_3+1}^{p_{i_3}} \Gamma^{A_{3ji_3}} (a_{3ji_3} - \sum_{k=1}^3 \alpha_{3ji_3}^{(k)} s_k) \prod_{j=1}^{q_{i_3}} \Gamma^{B_{3ji_3}} (1 - b_{3ji_3} + \sum_{k=1}^3 \beta_{3ji_3}^{(k)} s_k)]}$$

$$\dots$$

$$\frac{\prod_{j=1}^{n_r} \Gamma^{A_{rj}} (1 - a_{rj} + \sum_{k=1}^r \alpha_{rj}^{(k)} s_k)}{\sum_{i_r=1}^{R_r} [\tau_{i_r} \prod_{j=n_r+1}^{p_{i_r}} \Gamma^{A_{rji_r}} (a_{rji_r} - \sum_{k=1}^r \alpha_{rji_r}^{(k)} s_k) \prod_{j=1}^{q_{i_r}} \Gamma^{B_{rji_r}} (1 - b_{rji_r} + \sum_{k=1}^r \beta_{rji_r}^{(k)} s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k) = \frac{\prod_{j=1}^{m^{(k)}} \Gamma^{D_j^{(k)}} (d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{n^{(k)}} \Gamma^{C_j^{(k)}} (1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=m^{(k)}+1}^{q_{i^{(k)}}} \Gamma^{D_{ji^{(k)}}^{(k)}} (1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=n^{(k)}+1}^{p_{i^{(k)}}} \Gamma^{C_{ji^{(k)}}^{(k)}} (c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]} \tag{1.3}$$

The contour  $L_k$  is in the  $s_k (k = 1, \dots, r)$ - plane and run from  $\sigma - i\infty$  to  $\sigma + i\infty$  where  $\sigma$  is a real number with loop, if necessary to ensure that the poles of  $\Gamma^{A_{2j}} \left( 1 - a_{2j} + \sum_{k=1}^2 \alpha_{2j}^{(k)} s_k \right) (j = 1, \dots, n_2), \Gamma^{A_{3j}} \left( 1 - a_{3j} + \sum_{k=1}^3 \alpha_{3j}^{(k)} s_k \right) (j = 1, \dots, n_3), \dots, \Gamma^{A_{rj}} \left( 1 - a_{rj} + \sum_{i=1}^r \alpha_{rj}^{(i)} s_k \right) (j = 1, \dots, n_r), \Gamma^{C_j^{(k)}} (1 - c_j^{(k)} + \gamma_j^{(k)} s_k) (j = 1, \dots, n^{(k)}) (k = 1, \dots, r)$  to the right of the contour  $L_k$  and the poles of  $\Gamma^{D_j^{(k)}} (d_j^{(k)} - \delta_j^{(k)} s_k) (j = 1, \dots, m^{(k)}) (k = 1, \dots, r)$  lie to the left of the contour  $L_k$ . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left( \sum_{j=m^{(k)}+1}^{q_{i^{(k)}}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_{i^{(k)}}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right)$$

$$- \tau_{i_2} \left( \sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \dots - \tau_{i_r} \left( \sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\alpha_1}, \dots, |z_r|^{\alpha_r}), \max(|z_1|, \dots, |z_r|) \rightarrow 0$$

$$\aleph(z_1, \dots, z_r) = O(|z_1|^{\beta_1}, \dots, |z_r|^{\beta_r}), \min(|z_1|, \dots, |z_r|) \rightarrow \infty \text{ where } i = 1, \dots, r :$$

$$\alpha_i = \min_{1 \leq j \leq m^{(i)}} \operatorname{Re} \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] \text{ and } \beta_i = \max_{1 \leq j \leq n^{(i)}} \operatorname{Re} \left[ C_j^{(i)} \left( \frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$$

**Remark 1.**

If  $n_2 = \dots = n_{r-1} = p_{i_2} = q_{i_2} = \dots = p_{i_{r-1}} = q_{i_{r-1}} = 0$  and  $A_{2j} = A_{2ji_2} = B_{2ji_2} = \dots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ , then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

**Remark 2.**

If  $n_2 = \dots = n_r = p_{i_2} = q_{i_2} = \dots = p_{i_r} = q_{i_r} = 0$  and  $\tau_{i_2} = \dots = \tau_{i_r} = \tau_{i(1)} = \dots = \tau_{i(r)} = R_2 = \dots = R_r = R^{(1)} = \dots = R^{(r)} = 1$ , then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

**Remark 3.**

If  $A_{2j} = A_{2ji_2} = B_{2ji_2} = \dots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$  and  $\tau_{i_2} = \dots = \tau_{i_r} = \tau_{i(1)} = \dots = \tau_{i(r)} = R_2 = \dots = R_r = R^{(1)} = \dots = R^{(r)} = 1$ , then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

**Remark 4.**

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [10,11]. For more details about The Gimel function, see Ayant [2].

The generalization of the modified integral Bessel function of the third kind or Macdonald function will be represented by the following form [4,p.152, eq.(1.2)]

$$\lambda_{\mu,v}^{(\eta)}(z) = \frac{\eta}{\Gamma\left(\mu + 1 - \frac{1}{\eta}\right)} \int_1^\infty (t^\eta - 1)^{\mu - \frac{1}{\eta}} t^v e^{-zt} dt \tag{1.5}$$

Srivastava [8] introduced the general class of polynomials :

$$S_N^M(x) = \sum_{k=0}^{[N/M]} \frac{(-N)_{Mk}}{k!} A_{N,k} x^k, N = 0, 1, 2, \dots \tag{1.6}$$

Where  $M$  is an arbitrary positive integer and the coefficient  $A_{N,k}$  are arbitrary constants, real or complex.

By suitably specialized the coefficient  $A_{N,k}$  the polynomials  $S_N^M(x)$  can be reduced to the classical orthogonal polynomials such as Jacobi, Hermite, Legendre and Laguerre polynomials etc, see Srivastava and Singh [12].

**2. Main integral**

In your investigation, we shall use the following notations.

$$\begin{aligned} \mathbf{A} = & [(a_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3}, \\ & [\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1,p_{i_3}}; \dots; [(a_{(r-1)j}; \alpha_{(r-1)j}^{(1)}, \dots, \alpha_{(r-1)j}^{(r-1)}; A_{(r-1)j})]_{1,n_{r-1}}, \\ & [\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}}; \alpha_{(r-1)ji_{r-1}}^{(1)}, \dots, \alpha_{(r-1)ji_{r-1}}^{(r-1)}; A_{(r-1)ji_{r-1}})]_{n_{r-1}+1,p_{i_{r-1}}} \end{aligned} \tag{2.1}$$

$$\begin{aligned} \mathbf{A} = & [(a_{rj}; \alpha_{rj}^{(1)}, \dots, \alpha_{rj}^{(r)}, 0, 0; A_{rj})]_{1,n_r}, [\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \dots, \alpha_{rji_r}^{(r)}, 0, 0; A_{rji_r})]_{n+1,p_{i_r}} \\ A_1 = & (1 - \lambda - R\lambda'; \lambda_1, \dots, \lambda_{r+1}, 1; 1), (1 - \rho - R\rho'; \rho_1, \dots, \rho_{r+1}, l; 1), (-\sigma - R\sigma'; \sigma_1, \dots, \sigma_{r+1}, 0; 1) \end{aligned} \tag{2.2}$$

$$\begin{aligned} A = & [(c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})]_{1,n^{(1)}}, [\tau_{i(1)}(c_{ji(1)}^{(1)}, \gamma_{ji(1)}^{(1)}; C_{ji(1)}^{(1)})]_{n^{(1)}+1,p_i^{(1)}}; \dots; \\ & [(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})]_{1,m^{(r)}}, [\tau_{i(r)}(c_{ji(r)}^{(r)}, \gamma_{ji(r)}^{(r)}; C_{ji(r)}^{(r)})]_{m^{(r)}+1,p_i^{(r)}}, \left(1 - \frac{v+1}{\eta}; \frac{1}{\eta}; 1\right) \end{aligned} \tag{2.3}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2j_{i_2}}; \beta_{2j_{i_2}}^{(1)}, \beta_{2j_{i_2}}^{(2)}; B_{2j_{i_2}})]_{1, q_{i_2}}, [\tau_{i_3}(b_{3j_{i_3}}; \beta_{3j_{i_3}}^{(1)}, \beta_{3j_{i_3}}^{(2)}, \beta_{3j_{i_3}}^{(3)}; B_{3j_{i_3}})]_{1, q_{i_3}}; \dots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)j_{i_{r-1}}}; \beta_{(r-1)j_{i_{r-1}}}^{(1)}, \dots, \beta_{(r-1)j_{i_{r-1}}}^{(r-1)}; B_{(r-1)j_{i_{r-1}}})]_{1, q_{i_{r-1}}} \quad (2.4)$$

$$\mathbf{B} = [\tau_{i_r}(b_{rj_{i_r}}; \beta_{rj_{i_r}}^{(1)}, \dots, \beta_{rj_{i_r}}^{(r)}, 0; 0; B_{rj_{i_r}})]_{1, q_{i_r}} \quad (2.5)$$

$$B_1 = (-\rho - \sigma - R(\rho' + \sigma'); \rho_1 + \sigma_1, \dots, \rho_{r+1} + \sigma_{r+1}, l; 1), (1 - \lambda - R\lambda'; \lambda_1, \dots, \lambda_{r+1}, 0; 1) \quad (2.6)$$

$$B = [(d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})]_{1, m^{(1)}}, [\tau_{i^{(1)}}(d_{j_{i^{(1)}}}^{(1)}, \delta_{j_{i^{(1)}}}^{(1)}; D_{j_{i^{(1)}}}^{(1)})]_{m^{(1)}+1, q_i^{(1)}; \dots;$$

$$[(d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})]_{1, m^{(r)}}, [\tau_{i^{(r)}}(d_{j_{i^{(r)}}}^{(r)}, \delta_{j_{i^{(r)}}}^{(r)}; D_{j_{i^{(r)}}}^{(r)})]_{m^{(r)}+1, q_i^{(r)}, \left(-\mu - \frac{v}{\eta}; \frac{1}{\eta}; 1\right); (0, 1; 1) \quad (2.7)$$

$$U = 0, n_2; 0, n_3; \dots; 0, n_{r-1}; V = m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \dots; m^{(r)}, n^{(r)} \quad (2.8)$$

$$X = p_{i_2}, q_{i_2}, \tau_{i_2}; R_2; \dots; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}}; R_{r-1}; Y = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \dots; p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \quad (2.9)$$

We have the following resulting

**Theorem**

$$\int_0^a x^\rho (a-x)^\sigma \lambda_{\mu, v}^{(\eta)}(z_{r+1}) x^{\rho_{r+1}} (a-x)^{\sigma_{r+1}} [1 + (bx)^l]^{-\lambda_{r+1}} S_N^M [Y x^{\rho'} (a-x)^{\sigma'} [1 + (bx)^l]^{-\lambda'}]$$

$$\int \begin{pmatrix} z_1 x^{\rho_1} (a-x)^{\sigma_1} [1 + (bx)^l]^{-\lambda_1} \\ \vdots \\ z_r x^{\rho_r} (a-x)^{\sigma_r} [1 + (bx)^l]^{-\lambda_r} \end{pmatrix} dx = a^{\rho+\sigma} \sum_{R=0}^{[N/M]} \frac{-N_{MR} A_{N,R} (Y a^{\rho+\sigma})^R}{R!}$$

$$\int_{X: p_{i_r}+3, q_{i_r}+2, \tau_{i_r}; R_r; Y: 1, 2, 0, 1}^{U: 0, n_r+3; V: 2, 0, 1, 0} \left( \begin{array}{c|c} z_1 a^{\rho_1+\sigma_1} & \mathbb{A}, \mathbf{A}_1, \mathbf{A} : A \\ \vdots & \vdots \\ z_r a^{\rho_r+\sigma_r} & \vdots \\ z_{r+1} a^{\rho_{r+1}+\sigma_{r+1}} & \vdots \\ (ab)^l & \mathbb{B}, \mathbf{B}, B_1 : B \end{array} \right) \quad (2.10)$$

Provided that

$$Re() \geq 0, \min\{\rho', \sigma', \lambda', \rho_i, \sigma_i, \lambda_i\} \geq 0, i = 1, \dots, r+1$$

$$Re(\rho - \rho_{r+1}(\mu\eta + v)) + \sum_{i=1}^r \rho_i \min_{1 \leq j \leq m^{(i)}} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0$$

$$Re(\sigma - \sigma_{r+1}(\mu\eta + v)) + \sum_{i=1}^r \sigma_i \min_{1 \leq j \leq m^{(i)}} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0$$

$\arg(z_i x^{\rho_i} (a-x)^{\sigma_i} [1+(bx)^l]^{-\lambda_i}) < \frac{1}{2} A_i^{(k)} \pi, A_i^{(k)}$  is given in (1.13).

**Proof**

First we express the modified integral Bessel function of the third kind  $\lambda_{\mu, \nu}^{(\eta)}(z)$  in terms of the H-function of one variable [4, p. 155, eq. (2.6)] and the generalized polynomial with the help of (1.5). Now, we express the multivariable Gimel-function and H-function of one variable in term of their respective Mellin-Barnes type integrals contour. Then we change the order of the series and integrals contour with the x-integral (which is permissible under the conditions stated above). The left-hand side of (2.10) writes (say I)

$$\begin{aligned}
 I &= \sum_{R=0}^{[N/M]} \frac{-N_{MR} A_{N,R} Y^R}{R!} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \dots, s_r) \prod_{k=1}^r \theta_k(s_k) z_k^{s_k} \\
 &\frac{1}{2\pi\omega} \int_{L_{r+1}} \frac{\Gamma(-s_{r+1}) \Gamma\left(-\mu - \frac{\nu+s_{r+1}}{\eta}\right)}{\Gamma\left(1 - \frac{\nu+1+s_{r+1}}{\eta}\right)} z_{r+1}^{s_{r+1}} \\
 &\int_0^a x^{\sum_{i=1}^{r+1} \rho_i s_i + \rho + r \rho' - 1} (a-x)^{\sigma + R\sigma' + \sum_{i=1}^{r+1} \sigma_i s_i} [1+(bx)^l]^{-\lambda - R\lambda' - \sum_{i=1}^{r+1} \sigma_i s_i} dx ds_1 \cdots ds_{r+1}
 \end{aligned} \tag{2.11}$$

Finally, on evaluating the x-integral obtained above with the help of a special case of [9, p. 61, eq. (5.2.1)] and expressing the H-function thus obtained in terms of its Mellin-Barnes integral contour. Now interpreting the result in Mellin-Barnes integrals contour in multivariable Gimel-function of (r+2)-variables, we obtain the equation (2.10)

### 3. Particular cases.

In this section, the multivariable Gimel-function reduce in multivariable H-function.

If we reduce  $\lambda_{\mu, \nu}^{(\eta)}(z)$  into  $K_\mu$  [3, p.152, eq(1.3)],  $S_N^M$  into Laguerre polynomials and the multivariable H-function into generalized hypergeometric function by taking  $r = 1$  [9, p.18, eq.(2.6.3)] in the integral (2.10), we arrive at the following relation, see Harjule and Jain [5].

**Corollary 1.**

$$\begin{aligned}
 &\int_0^a x^{\rho - \rho_2 \mu - 1} (a-x)^{\sigma - \sigma_2 \mu} [1+(bx)^l]^{-\lambda + \lambda_2 \mu} K_{-\mu}(z_2 x^{\rho_2} (a-x)^{\sigma_2} [1+(bx)^l]^{-\lambda_2}) L_N(Y x^{\rho'} (a-x)^{\sigma'} [1+(bx)^l]^{-\lambda'}) \\
 &{}_p F_q[(c_j)_p; (d_j)_q; -z_1 x^{\rho_1 - 1} (a-x)^{\sigma_1} [1+(bx)^l]^{-\lambda_1}] dx = 2^{-\mu - 1} \sqrt{\pi} z_2^{\mu} \frac{\prod_{j=1}^q \Gamma(d_j)}{\prod_{j=1}^p \Gamma(c_j)} a^{\rho + \sigma} \sum_{R=0}^N \frac{(-N)_R (Y a^{\rho' + \sigma'})^R}{(\alpha + 1)_R R!}
 \end{aligned}$$

$$\left( \begin{matrix} N + \alpha \\ N \end{matrix} \right) H_{3,2;p,q+1;1,2;01}^{0,3;1,p;2,0;1;0} \left( \begin{matrix} z_1 a^{\rho_1 + \sigma_1} \\ z_2 a^{\rho_2 + \sigma_2} \\ (ab)^l \end{matrix} \middle| \begin{matrix} \mathbb{A}: (1-c_j; 1)_{1,p}; \left(\frac{1}{2}; \frac{1}{2}\right); - \\ \mathbb{B}: (0; 1), (1-d_j; 1)_{1,q}; (0; 1), \left(-\mu; \frac{1}{2}\right); (0, 1) \end{matrix} \right) \tag{3.1}$$

where

$$\mathbb{A} = (1 - \lambda - R\lambda'; \lambda_1, \lambda_2, 1), (1 - \rho - R\rho'; \rho_1, \rho_2, l), (-\sigma - R\sigma'; \sigma_1, \sigma_2, 0) \tag{3.2}$$

$$\mathbb{B} = (-\rho - R(\rho' + \sigma') - \sigma; \rho_1 + \sigma_1, \rho_2 + \sigma_2, l), (1 - \lambda - R\lambda'; \lambda_1, \lambda_2, 0) \tag{3.3}$$

the existence conditions of (2.10) are satisfied.

Consider a second particular case, see Harjule and Jain [5].

If we take  $r = 2, l = 1, \lambda' = \sigma' = \sigma_i = \lambda_i = (i = 1, \dots, r + 1)$  in the main integral and further reduce  $\lambda_{\mu, v}^{(\eta)}(z)$  to Meijer's function  $G_{1,0}^{2,0}$  by taking  $\eta = 1, S_N^M$  into Jacobi polynomial  $P_N^{(\alpha, \beta)}$  see Srivastava and Singh [12] and the multivariable Gimel -function into Appell's function  $F_3$ , we obtain the following integral after algebraic manipulations :

**Corollary 2.**

$$\int_0^a x^{\rho-1} (a-x)^\sigma G_{1,0}^{2,0} \left( z_3 x^{\rho_3} \left| \begin{matrix} v \\ 0, -\mu - v \end{matrix} \right. \right) [1 + (bx)]^{-\lambda} P_N^{(\alpha, \beta)}(1 - 2Yx^{\rho'}) F_3[k_1, h_1, k_2, h_2; s; z_1 x_1^{\rho_1}, z_2 x_2^{\rho_2}] dx$$

$$= \frac{\Gamma(s)\Gamma(1+\sigma)}{\Gamma(k_1)\Gamma(k_2)\Gamma(h_1)\Gamma(h_2)\Gamma(\lambda)} a^{\rho+\sigma} \sum_{R=0}^N \frac{(-N)_R (\alpha + \beta + N + 1)_R (Y a^{\rho'})^R}{(\alpha + 1)_R R!} \binom{\alpha + N}{N}$$

$$H_{1,2;2,1;2,1;1,2;1,1}^{0,1;1,2;1,2;2,0;1,1} \left( \begin{matrix} -z_1 a^{\rho_1} \\ -z_2 a^{\rho_2} \\ z_3 a^{\rho_3} \\ ab \end{matrix} \left| \begin{matrix} \mathbb{A}' : (1-k_1; 1), (1-k_2; 1); (1-h_1; 1), (1-h_2; 1); (-v; 1), (1-\lambda; 1) \\ \cdot \\ \cdot \\ \mathbb{B}' : (0;1), (0;1); (0;1), (-\mu - v; 1); (0, 1) \end{matrix} \right. \right) \tag{3.4}$$

where

$$\mathbb{A}' = (1 - \rho - \rho' R; \rho_1, \rho_2, \rho_3; 1), \mathbb{B}' = (-\rho - \sigma - \rho' R; \rho_1, \rho_2, \rho_3; 1), (1 - s; 1, 1, 0, 0) \tag{3.5}$$

the existence conditions of (2.10) are satisfied.

**4. Conclusion**

The integral formulae involving in this paper are double fold generality in term of variables and parameters. By specializing the various parameters and variables involved, these formulae can suitably be applied to derive the corresponding results involving wide variety of useful functions (or product of several such functions) which can be expressed in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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