Finite Double Integrals Involving Multivariable Gimel-Function

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ABSTRACT

In this paper, we establish three finite double integrals involving the multivariable Gimel function with general arguments, general class of polynomials, special functions and Aleph-function. Importance of our findings lies in the fact that they involve the multivariable Gimel function, which are the sufficiently general in nature and are capable of yielding a large number of simpler and useful results merely by specializing the parameters in them. For the sake of illustration, only one particular case of this integral obtained has been given which is also new and is of interest.

Keywords:Multivariable Gimel-function, Mellin-Barnes integrals contour , finite double integrals, class of multivariable polynomials, Aleph-function.

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I. Introduction.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a generalized transcendental function of several complex variables.

We consider a generalized transcendental function called Gimel function of several complex variables.

$$\beth(z_1, \cdots, z_r) = \beth_{X;p_{i_r},q_{i_r},\tau_{i_r}:R_r:Y}^{U;0,n_r:V} \left(\begin{array}{c|cc} z_1 & \mathbb{A};\mathbf{A}: A \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ z_r & \mathbb{B};\mathbf{B}: B \end{array} \right) = \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \quad (1.1)$$

with $\omega = \sqrt{-1}$

The following quantities $\mathbb{A}, \mathbf{A}, A, \mathbb{B}, \mathbf{B}, B, X, Y, U, V, \psi(s_1, \cdots, s_r)$ and $\theta_k(s_k)(k = 1, \cdots, r)$ are defined by Ayant [2].

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}) , max(|z_1|, \cdots, |z_r|) \to 0$$

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}) , min(|z_1|, \cdots, |z_r|) \to \infty \text{ where } i = 1, \cdots, r :$$

$$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] \text{ and } \beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$$

The generalized polynomials defined by Srivastava ([12],p. 251, Eq. (C.1)), is given in the following manner :

$$S_{N_1, \cdots, N_s}^{M_1, \cdots, M_s}[y_1, \cdots, y_s] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_s)_{M_s K_s}}{K_s!} A[N_1, K_1; \cdots ; N_s, K_s] y_1^{K_1} \cdots y_s^{K_s} \quad (1.2)$$

we shall note $a_1 = \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_s)_{M_s K_s}}{K_s!} A[N_1, K_1; \cdots ; N_s, K_s]$

Where $M_1, \cdots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_s, K_s]$ are arbitrary constants, real or complex. I we take $s = 1$ in the (1.13) and denote $A[N, K]$ thus obtained by $A_{N,K}$, we arrive at general class of polynomials $S_N^M(x)$ study by Srivastava ([11],p. 1, Eq. 1).

The Aleph- function , introduced by Südland et al. [16,17], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z \left| \begin{array}{c} (\ a_j, A_j)_{1,N}, [c_i(a_{ji}, A_{ji})]_{N+1,P_i;r'} \\ (\ b_j, B_j)_{1,M}, [c_i(b_{ji}, B_{ji})]_{M+1,Q_i;r'} \end{array} \right. \right) = \frac{1}{2\pi\omega} \int_L \ \Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)z^s \mathrm{d}s \quad (1.3)$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s) = \frac{\prod_{j=1}^M \Gamma(b_j - B_j s)\prod_{j=1}^N \Gamma(1 - a_j + A_j s)}{\sum_{i=1}^{r'} c_i \prod_{j=N+1}^{P_i} \Gamma(a_{ji} - A_{ji}s) \prod_{j=M+1}^{Q_i} \Gamma(1 - b_{ji} + B_{ji}s)} \qquad (1.4)$$

with $\left|argz\right| < \dfrac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^M B_j + \sum_{j=1}^N A_j - c_i\left(\sum_{j=N+1}^{P_i} A_{ji} + \sum_{j=M+1}^{Q_i} B_{ji} \right) > 0, i = 1, \cdots, r'$

For convergence conditions and other details of Aleph-function , see Südland et al [16,17]. The serie representation of Aleph-function is given by Chaurasia and Singh [4].

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z) = \sum_{G=1}^M \sum_{g=0}^\infty \frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!} z^s \qquad (1.5)$$

With $s = \eta_{G,g} = \dfrac{b_G + g}{B_G}, P_i < Q_i, |z| < 1$ and $\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)$ is given in (1.15) $\qquad (1.6)$

2. Required results.

The following integrals ([18], p.33), ([5],p.172, eq.27), ([6], p. 46, eq. (5) and ([8] p.71) will be required to establish our main results.

Lemma 1.

a) $\displaystyle\int_0^1 x^{c-1}(1-x)^{-\frac{1}{2}}{}_2F_1\left(a, b, a + b + \frac{1}{2}; x \right)\mathrm{d}x = \frac{\Gamma(c)\Gamma\left(a + b + \frac{1}{2} \right)\Gamma(c - a - b + \frac{1}{2})}{\Gamma\left(a + \frac{1}{2} \right) \Gamma\left(b + \frac{1}{2} \right) \Gamma\left(c - a + \frac{1}{2} \right) \Gamma\left(c - b + \frac{1}{2} \right)}$ $\qquad (2.1)$

where $Re(c) > 0, Re(2c - a - b) > -1$

Lemma 2.

b)$\displaystyle\int_0^\pi \sin^{\alpha-1}\theta P_v^{-\mu}(\cos\theta)\mathrm{d}\theta = \frac{2^{-\mu}\pi\Gamma\left(\frac{\alpha\pm\mu}{2} \right)}{\Gamma\left(\frac{\alpha+v+1}{2} \right)\Gamma\left(\frac{\alpha-v}{2} \right)\Gamma\left(\frac{\mu+v+2}{2} \right)\Gamma\left(\frac{\mu-v+1}{2} \right)}$ $\qquad (2.2)$

where $Re(\alpha \pm \mu) > 0$

Lemma 3.

c)$\displaystyle\int_0^\pi J_\mu(\theta)(\cos\theta)^{2\rho+1}\mathrm{d}\theta = 2^\rho\Gamma(\rho + 1)\alpha^{-\rho+1}J_{\rho+\mu+1}(\alpha)$ $\qquad (2.3)$

where $Re(\rho), Re(\mu) > -1$

Lemma 4.

d) $\int_0^\pi e^{i(\alpha+\beta)\theta} \sin^{\alpha-1}\theta \cos^{\beta-1}\theta\, {}_2F_1\left(\gamma, \delta; \beta; e\cos\theta\right)\mathrm{d}\theta = \dfrac{e^{\frac{i\pi\alpha}{2}}\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-\delta-\gamma)}{\Gamma(\alpha+\beta-\gamma)\Gamma(\alpha+\beta-\delta)}$ (2.4)

where $min\{Re(\alpha), Re(\beta), Re(\alpha+\beta-\gamma-\delta)\} > 0$

3. Main integrals.

Theorem 1.

$$\int_0^1 \int_0^\pi x^{c-1}(1-x)^{-\frac{1}{2}} {}_2F_1\left(a, b, a+b+\frac{1}{2}; x\right)\sin^{\alpha-1}\theta P_v^{-\mu}(\cos\theta) N_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}\sin^d\theta)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1 x^{e_1}\sin^{f_1}\theta. \\ . \\ \mathrm{y}_s x^{e_s}\sin^{f_s}\theta \end{array}\right) \beth \left(\begin{array}{c} \mathrm{z}_1 x^{c_1}\sin^{d_1}\theta. \\ . \\ \mathrm{z}_r x^{c_s}\sin^{d_s}\theta \end{array}\right)\mathrm{d}x\mathrm{d}\theta = \dfrac{\pi^2 2^{-\mu}\Gamma(c)\Gamma\left(a+b+\frac{1}{2}\right)\Gamma(c-a-b+\frac{1}{2})}{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma\left(\frac{\mu+v+2}{2}\right)\Gamma\left(\frac{\mu-v+1}{2}\right)}$$

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1 \dfrac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!} z^s\, y_1^{K_1}\cdots y_s^{K_s}\, \beth_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}$$

$$\left(\begin{array}{c} \mathrm{z}_1 \\ \cdots \\ \cdots \\ \mathrm{z}_r \end{array}\right.\left|\begin{array}{c} \mathbb{A}; (1\text{-}c\text{-}c'\eta_{G,g} - \sum_{i=1}^s K_i e_i; c_1,\cdots,c_r;1),(\frac{1}{2}-c+a+b-c'\eta_{G,g}-\sum_{i=1}^s K_i e_i; c_1,\cdots,c_r;1) \\ . \\ . \\ \mathbb{B}; \mathbf{B}, \left(\frac{1}{2}-c+a-c'\eta_{G,g}-\sum_{i=1}^s K_i e_i; c_1,\cdots,c_r;1\right), (\frac{1}{2}-c+b-c'\eta_{G,g}-\sum_{i=1}^s K_i e_i; c_1,\cdots,c_r;1) \end{array}\right.$$

$$\left.\begin{array}{c} \left(1-\frac{\alpha+\mu+d\eta_{G,g}+\sum_{i=1}^s K_i f_i}{2}; \frac{d_1}{2},\cdots,\frac{d_r}{2};1\right),\left(1-\frac{\alpha-\mu+d\eta_{G,g}+\sum_{i=1}^s K_i f_i}{2}; \frac{d_1}{2},\cdots,\frac{d_r}{2};1\right), \mathbf{A}:A \\ . \\ . \\ \left(\frac{1}{2}-\frac{\alpha+v+d\eta_{G,g}+\sum_{i=1}^s K_i f_i}{2}; \frac{d_1}{2},\cdots,\frac{d_r}{2};1\right)\quad, \left(1-\frac{\alpha-v+d\eta_{G,g}+\sum_{i=1}^s K_i f_i}{2}; \frac{d_1}{2},\cdots,\frac{d_r}{2};1\right):B \end{array}\right)$$ (3.1)

Provided that ; $e_j, f_j, c_i, d_i > 0, i = 1, \cdots, r; j = 1, \cdots, s$

$$Re\left(c + c'\eta_{G,g}\right) + \sum_{i=1}^r e_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1.$$

$$Re\left(c - a - b + c'\eta_{G,g}\right) + \sum_{i=1}^r e_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > \frac{1}{2}.$$

$$Re\left(c - a + c'\eta_{G,g}\right) + \sum_{i=1}^r e_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\frac{1}{2}.$$

$$Re\left(c - b + c'\eta_{G,g}\right) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{e_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\frac{1}{2}.$$

$$Re\left(\alpha \pm \mu + d\eta_{G,g}\right) + \sum_{i=1}^{r} f_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0.$$

$$Re\left(\alpha - v + d\eta_{G,g}\right) + \sum_{i=1}^{r} f_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0.$$

$$Re\left(\alpha \pm v + d\eta_{G,g}\right) + \sum_{i=1}^{r} f_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1.$$

$|arg(z_i x^{c_j} \sin^{d_i} \theta)| < \frac{1}{2} A_i^{(k)} \pi$ where $A_i^{(k)}$ is defined by (1.4)

and $|arg(zx^{c'} \sin^d \theta)| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} A_j - c_i \left(\sum_{j=N+1}^{P_i} A_{ji} + \sum_{j=M+1}^{Q_i} B_{ji}\right) > 0, i = 1, \cdots, r'.$

Proof

We first express the general class of multivariable polynomial occurring on the L.H.S of (3.1) in series form with the help of (1.13), the Aleph-function in serie form with the help of (1.16) and replace the multivariable Gimel-function by its Mellin-Barnes integrals contour with the help of (1.1). Now we interchange the order of summations and integrations , we obtain (say I)

$$I = \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!} z^s \, y_1^{K_1} \cdots y_s^{K_s} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k}$$

$$\left(\int_0^1 x^{c+c\eta_{G,g}+\sum_{j=1}^{s} K_j e_j + \sum_{i=1}^{r} c_i s_i - 1}(1-x)^{-\frac{1}{2}} {}_2F_1\left(a, b, a+b+\frac{1}{2}; x\right)dx\right)$$

$$\left(\int_0^{\pi} \sin^{\alpha+d\eta_{G,g}+\sum_{j=1}^{s} K_j e_j + \sum_{i=1}^{r} c_i s_i - 1} \theta P_v^{-\mu}(\cos\theta)d\theta\right)ds_1 \cdots ds_r \qquad (3.2)$$

Now using the lemmae 1 and 2 to evaluate the x-integral and θ- integral and reinterpreting the Mellin-Barnes multiple integrals contour so obtained in the form of multivariable Gimel-function, we obtain the desired result.

Theorem 2

$$\int_0^1 \int_0^{\pi} x^{c-1}(1-x)^{-\frac{1}{2}} {}_2F_1\left(a, b, a+b+\frac{1}{2}; x\right)\sin^{\mu+1}\theta(\cos\theta)^{2\rho+1} J_\mu(\alpha\sin\theta) N_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'} \sin^d \theta)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} y_1 x^{e_1} \sin^{2f_1}\theta. \\ . \\ y_s x^{e_s} \sin^{2f_s}\theta \end{array}\right) \beth \left(\begin{array}{c} z_1 x^{c_1} \sin^{2d_1}\theta. \\ . \\ z_r x^{c_s} \sin^{2d_s}\theta \end{array}\right)dxd\theta = \frac{\pi 2^\rho \alpha^{-\rho}\Gamma\left(a+b+\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)}$$

$$\sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!} z^s \, y_1^{K_1} \cdots y_s^{K_s} \quad \beth_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}$$

$$\left(\begin{array}{c|c} z_1 \\ \cdots \\ \cdots \\ z_r \end{array} \quad \begin{array}{c} \mathbb{A}; \ (1\text{-c-c'}\eta_{G,g} - \sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r; 1), \\ . \\ . \\ \mathbb{B}; \ \mathbf{B}, \ \left(\frac{1}{2} - c + a - c'\eta_{G,g} - \sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r; 1\right), \end{array}\right.$$

$$\left.\begin{array}{c} \left(-c + a + \frac{1}{2} + b - c'\eta_{G,g} - \sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r; 1\right), \left(-\rho - d\eta_{G,g} - \sum_{j=1}^{s} K_j f_j; d_1, \cdots, d_i; 1\right), \mathbf{A} : A \\ . \\ . \\ \left(\frac{1}{2} - c + b - c'\eta_{G,g} - \sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r; 1\right), \left(g - \rho - \mu - 1 - d\eta_{G,g} - \sum_{j=1}^{s} K_j f_j; d_1, \cdots, d_i; 1\right) : B \end{array}\right) \tag{3.3}$$

Provided that ; $e_j, f_j, c_i, d_i > 0, i = 1, \cdots, r; j = 1, \cdots, s$

$$Re\left(c + c'\eta_{G,g}\right) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1.$$

$$Re\left(c - a - b + c'\eta_{G,g}\right) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\frac{1}{2}.$$

$$Re\left(c - a + c'\eta_{G,g}\right) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\frac{1}{2}.$$

$$Re\left(c - b + c'\eta_{G,g}\right) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\frac{1}{2}.$$

$$Re\left(\rho + d\eta_{G,g}\right) + \sum_{i=1}^{r} d_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0.$$

$$Re\left(g - \mu + \rho + d\eta_{G,g}\right) + \sum_{i=1}^{r} d_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -2.$$

$|arg(z_i x^{c_i} \sin^{2d_i} \theta)| < \frac{1}{2} A_i^{(k)} \pi$ where $A_i^{(k)}$ is defined by (1.4) where $i = 1, \cdots, r$.

and $|arg(zx^{c'} \sin^d \theta)| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} A_j - c_i \left(\sum_{j=N+1}^{P_i} A_{ji} + \sum_{j=M+1}^{Q_i} B_{ji}\right) > 0, i = 1, \cdots, r'.$

To prove the theorem 2, we use the similar method above but we use the lemmae 1 and 3.

Theorem 3.

$$\int_0^1 \int_0^\pi x^{c-1}(1-x)^{-\frac{1}{2}} {}_2F_1\left(a, b, a + b + \frac{1}{2}; x\right) {}_2F_1\left(\gamma, \delta; \beta, e^{\omega\theta} \cos\theta\right) \sin^{\alpha-1}\theta(\cos\theta)^{\beta-1} N_{P_i, Q_i, c_i; r'}^{M,N}(zx^{c'} \sin^d \theta)$$

$$S_{N_1, \cdots, N_s}^{M_1, \cdots, M_s}\left(\begin{array}{c} y_1 x^{e_1} e^{\omega f_1 \theta} \sin^{2f_1}\theta. \\ . \\ y_s x^{e_s} e^{\omega f_s \theta} \sin^{2f_s}\theta \end{array}\right)\beth\left(\begin{array}{c} z_1 x^{c_1} e^{\omega\theta d_1} \sin^{2d_1}\theta. \\ . \\ z_r x^{c_r} e^{\omega\theta d_1} \sin^{2d_r}\theta \end{array}\right) dx d\theta = \frac{\pi e^{\frac{i\pi\alpha}{2}} \Gamma(\beta)\left(a + b + \frac{1}{2}\right)}{\Gamma\left(a + \frac{1}{2}\right)\Gamma\left(b + \frac{1}{2}\right)}$$

$$\sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-)^g \Omega_{P_i, Q_i, c_i, r'}^{M,N}(s)}{B_G g!} z^s y_1^{K_1} \cdots y_s^{K_s} \quad \beth_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}$$

$$\left(\begin{array}{c|c} z_1 \\ \cdots \\ \cdots \\ z_r \end{array}\right. \begin{array}{l} \mathbb{A};\ (1\text{-}c\text{-}c'\eta_{G,g} - \sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r; 1), \qquad (1\text{-}\alpha - d\eta_{G,g} - \sum_{j=1}^{s} K_j f_j; d_1, \cdots, d_r; 1), \\ \qquad\qquad\qquad . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \mathbb{B};\ \mathbf{B},\ \left(\frac{1}{2} - c + a - c'\eta_{G,g} - \sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r; 1\right), (1\text{-}\alpha - \beta + \gamma - d\eta_{G,g} - \sum_{j=1}^{s} K_j f_j; d_1, \cdots, d_r; 1), \end{array}$$

$$\left.\begin{array}{l} \left(-c + a + b + \frac{1}{2} - c'\eta_{G,g} - \sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r; 1\right), (1\text{-}\alpha - \beta + \gamma\delta - d\eta_{G,g} - \sum_{j=1}^{s} K_j f_j; d_1, \cdots, d_r; 1), \mathbf{A}:A \\ \qquad\qquad\qquad . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \left(c + b + \frac{1}{2} - c'\eta_{G,g} - \sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r; 1\right) \qquad , (1\text{-}\alpha - \beta + \delta - d\eta_{G,g} - \sum_{j=1}^{s} K_j f_j; d_1, \cdots, d_r; 1):B \end{array}\right) \quad (3.4)$$

Provided that ; $e_j, f_j, c_i, d_i > 0, i = 1, \cdots, r; j = 1, \cdots, s$

$$Re\,(c + c'\eta_{G,g}) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -1.$$

$$Re\,(c - a - b + c'\eta_{G,g}) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\frac{1}{2}.$$

$$Re\,(c - a + c'\eta_{G,g}) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\frac{1}{2}.$$

$$Re\,(c - b + c'\eta_{G,g}) + \sum_{i=1}^{r} e_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > -\frac{1}{2}.$$

$$Re\,(\alpha + \beta - \delta - \gamma + d\eta_{G,g}) + \sum_{i=1}^{r} d_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0.$$

$$Re\,(\alpha + \beta - \delta + d\eta_{G,g}) + \sum_{i=1}^{r} d_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0.$$

$$Re\,(\alpha + \beta - \gamma + d\eta_{G,g}) + \sum_{i=1}^{r} d_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0.$$

$\left|arg(z_i x^{c_i} e^{\omega\theta d_i} \sin^{2d_i}\theta)\right| < \frac{1}{2} A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) where $i = 1, \cdots, r$

and $\left|arg(zx^{c'}\sin^d\theta)\right| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} A_j - c_i\left(\sum_{j=N+1}^{P_i} A_{ji} + \sum_{j=M+1}^{Q_i} B_{ji}\right) > 0, i = 1, \cdots, r'.$

To prove the theorem 2, we use the similar method above but we use the lemmae 1 and 4.

4. Special case.

Let $S_{N_1, \cdots, N_s}^{M_1, \cdots, M_s}[y_1, \cdots, y_s] \to S_N^M(x)$, see [10] ,we obtain the following formula.

Corollary1.

$$\int_0^1 \int_0^\pi x^{c-1}(1-x)^{-\frac{1}{2}}{}_2F_1\left(a, b, a + b + \frac{1}{2}; x\right)\sin^{\alpha-1}\theta P_v^{-\mu}(\cos\theta)N_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}\sin^d\theta)$$

$$S_N^M(yx^e \sin^f \theta) \lrcorner \begin{pmatrix} z_1 x^{c_1} \sin^{d_1} \theta. \\ . \\ z_r x^{c_s} \sin^{d_s} \theta \end{pmatrix} dx d\theta = \frac{\pi^2 2^{-\mu} \Gamma(c) \Gamma\left(a+b+\frac{1}{2}\right) \Gamma(c-a-b+\frac{1}{2})}{\Gamma\left(a+\frac{1}{2}\right) \Gamma\left(b+\frac{1}{2}\right) \Gamma\left(\frac{\mu+v+2}{2}\right) \Gamma\left(\frac{\mu-v+1}{2}\right)}$$

$$\sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K=0}^{[N/M]} A_{N,K} \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!} z^s \, y^K \quad \lrcorner_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}$$

$$\begin{pmatrix} z_1 \\ \cdots \\ \cdots \\ z_r \end{pmatrix} \begin{array}{|l} \mathbb{A}; (1\text{-c-c'}\eta_{G,g} - Ke; c_1,\cdots,c_r;1), (\frac{1}{2} - c + a + b - c'\eta_{G,g} - Ke; c_1,\cdots,c_r;1) \\ \hspace{1cm} . \\ \hspace{1cm} . \\ \mathbb{B}; \mathbf{B}, \left(\frac{1}{2} - c + a - c'\eta_{G,g} - Ke; c_1,\cdots,c_r;1\right), \left(\frac{1}{2} - c + b - c'\eta_{G,g} Ke; c_1,\cdots,c_r;1\right) \end{array}$$

$$\left.\begin{array}{l} \left(1 - \frac{\alpha+\mu+d\eta_{G,g}+Kf}{2}; \frac{d_1}{2},\cdots,\frac{d_r}{2};1\right), \left(1 - \frac{\alpha-\mu+d\eta_{G,g}+Kf}{2}; \frac{d_1}{2},\cdots,\frac{d_r}{2};1\right), \mathbf{A}:A \\ \hspace{3cm} . \\ \hspace{3cm} . \\ \left(\frac{1}{2} - \frac{\alpha+v+d\eta_{G,g}+Kf}{2}; \frac{d_1}{2},\cdots,\frac{d_r}{2};1\right) \quad, \left(1 - \frac{\alpha-v+d\eta_{G,g}+Kf}{2}; \frac{d_1}{2},\cdots,\frac{d_r}{2};1\right):B \end{array}\right) \tag{4.1}$$

under the same existence conditions that (3.1)

Corollary 2.

$$\int_0^1 \int_0^\pi x^{c-1}(1-x)^{-\frac{1}{2}} {}_2F_1\left(a,b,a+b+\frac{1}{2};x\right) \sin^{\mu+1}\theta(\cos\theta)^{2\rho+1} J_\mu(\alpha\sin\theta) N_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}\sin^d\theta)$$

$$S_N^M(yx^e \sin^f \theta) \lrcorner \begin{pmatrix} z_1 x^{c_1} \sin^{2d_1} \theta. \\ . \\ z_r x^{c_s} \sin^{2d_s} \theta \end{pmatrix} dx d\theta = \frac{\pi 2^\rho \alpha^{-\rho} \Gamma\left(a+b+\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right) \Gamma\left(b+\frac{1}{2}\right)}$$

$$\sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K=0}^{[N/M]} A_{N,K} \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!} z^s \, Y^K \quad \lrcorner_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}$$

$$\begin{pmatrix} z_1 \\ \cdots \\ \cdots \\ z_r \end{pmatrix} \begin{array}{|l} \mathbb{A}; (1\text{-c-c'}\eta_{G,g} - Ke; c_1,\cdots,c_r;1), \hspace{1cm} \left(-c+a+\frac{1}{2}+b-c'\eta_{G,g}-Ke; c_1,\cdots,c_r;1\right), \\ \hspace{2cm} . \hspace{4cm} . \\ \hspace{2cm} . \hspace{4cm} . \\ \mathbb{B}; \mathbf{B}, \left(\frac{1}{2}-c+a-c'\eta_{G,g}-Ke; c_1,\cdots,c_r;1\right), \quad \left(\frac{1}{2}-c+b-c'\eta_{G,g}-Ke; c_1,\cdots,c_r;1\right), \end{array}$$

$$\left.\begin{array}{l} (\text{-}\rho - d_{\eta G,g} - Kf; d_1,\cdots,d_i;1), \mathbf{A}:A \\ \hspace{2cm} . \\ \hspace{2cm} . \\ (\text{g-}\rho - \mu - 1 - d_{\eta G,g} - Kf; d_1,\cdots,d_i;1):B \end{array}\right) \tag{4.2}$$

under the same existence conditions that (3.1).

Corollary 3.

$$\int_0^1 \int_0^\pi x^{c-1}(1-x)^{-\frac{1}{2}}{}_2F_1\left(a,b,a+b+\frac{1}{2};x\right){}_2F_1(\gamma,\delta;\beta,e^{\omega\theta}\cos\theta)\ \sin^{\alpha-1}\theta(\cos\theta)^{\beta-1}\ N_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}\sin^d\theta)$$

$$S_N^M(yx^e e^{\omega f\theta}\sin^{2f}\theta)\big]\begin{pmatrix} z_1 x^{c_1}e^{\omega\theta d_1}\sin^{2d_1}\theta. \\ . \\ z_r x^{c_r}e^{\omega\theta d_1}\sin^{2d_r}\theta \end{pmatrix}\mathrm{d}x\mathrm{d}\theta = \frac{\pi e^{\frac{i\pi\alpha}{2}}\Gamma(\beta)\left(a+b+\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)}$$

$$\sum_{G=1}^M \sum_{g=0}^\infty \sum_{K=0}^{[N/M]} A_{N,K}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!}z^s y^K \quad \big]_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}$$

$$\begin{pmatrix} z_1 \\ \cdots \\ \cdots \\ z_r \end{pmatrix}\begin{array}{l} \mathbb{A};\ (1\text{-}c\text{-}c'\eta_{G,g}-Ke;c_1,\cdots,c_r;1), \qquad (1\text{-}\alpha-d\eta_{G,g}-Kf;d_1,\cdots,d_r;1), \\ \qquad\qquad\qquad\qquad . \qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \qquad\qquad\qquad\qquad . \qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \mathbb{B};\ \mathbf{B},\ \left(\frac{1}{2}-c+a-c'\eta_{G,g}-\sum_{i=1}^s Ke;c_1,\cdots,c_r;1\right),(1\text{-}\alpha-\beta+\gamma-d\eta_{G,g}-Kf;d_1,\cdots,d_r;1), \end{array}$$

$$\begin{pmatrix} \left(-c+a+b+\frac{1}{2}-c'\eta_{G,g}-Ke;c_1,\cdots,c_r;1\right),(1\text{-}\alpha-\beta+\gamma\delta-d\eta_{G,g}-Kg;d_1,\cdots,d_r;1),\mathbf{A}:A \\ \qquad\qquad . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \qquad\qquad . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \left(c+b+\frac{1}{2}-c'\eta_{G,g}-Ke;c_1,\cdots,c_r;1\right)\qquad,(1\text{-}\alpha-\beta+\delta-d\eta_{G,g}-Kg;d_1,\cdots,d_r;1):B \end{pmatrix} \qquad (3.4)$$

under the same existence conditions that (3.1).

On suitably specializing the coefficients $A_{N,K}$, $S_N^M(x)$ yields a number of known polynomials, these include the Jacobi polynomials, Laguerre polynomials and others polynomials ([15],p. 158-161).

Remark :
We obtain the same double finites integrals concerning the multivariable Aleph- function defined by Ayant [1], the multivariable I-function defined by Prathima et al. [11], the multivariable I-function defined by Prasad [10] and the multivariable H-function defined by Srivastava and Panda [14,15], see Garg et al. [7] for more details concerning the multivariable H-function.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of the double finite integrals with general class of polynomials and general arguments utilized in this study, we can obtain a large variety of single, double finites integrals. Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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