Representation of the Translational Hull of a Proper Type A Semigroup

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Abstract
The structure of proper type A semigroup constructed by Lawson(1986) is visited and some results obtained. The structure is married up with the structure maps of type A semigroup and an alternative structure of proper type A semigroup is obtained in line with Armstrong(1988) analysis of concordant semigroups. By using Munn semigroup, the representation of the left translational hull of the alternative proper type A semigroup is obtained.

Keywords: **-Prehomomorphism, *-Prehomomorphism triple, Translational Hull, Representation.*

I. INTRODUCTION

*-Prehomomorphism triple is an alternative name we have, in this paper, chosen to call the structure of proper type A semigroup due to the pivotal position of *-prehomomorphism in the structure as constructed by Lawson(1986). The structure, whose operation is the direct product, takes the form $T = T(S, C, \psi)$, where C is a cancellative monoid, ψ is a *-prehomomorphism from C to a type A semigroup W , and S is a type A *-subsemigroup of of W.

Let W be a type A semigroup and C a cancellative monoid. A mapping $\psi: C \to W$ is called a $*$-$Prehomomorphism$ if the following conditions are satisfied:

i. $\forall g, h \in C,\ \psi(g)\psi(h) \leq \psi(gh)$
ii. $\psi(1)$ is an idempotent
iii. If $s \in W$ and $s \leq \psi(g)$ $[g \in C]$, then $s^*, s^\dagger \leq \psi(1)$

where $\leq$ is the natural partial order on W.

Now, let $\psi: C \to W$ be a $*$-prehomomorphism and let S be a type A $*$-subsemigroup of W. We define $*$-$prehomomorphism\ triple$ by $T = T(S, C, \psi) = \{(s, g) \in S \times C \mid s \leq \psi(g)\}$ endowed with $direct\ product$.

If $(s, g), (t, h) \in T$, then $s \leq \psi(g)$ and $t \leq \psi(h)$ and therefore $st \leq \psi(g)\psi(h) \leq \psi(gh)$. Hence, $(st, gh) \in T$. Thus, if T is non-empty, T is a subsemigroup of $S \times C$.

If S is type A with semilattice E of idempotents and assuming for $e \in E,\ a \in S,\ a \leq e$. Then $\exists f \in E$ such that $a = fe = ef \in E$. That is $a \leq e \ \Rightarrow\ a \in E$. For references, we can formally state this fact.

Lemma 1.1: If S is type A with semilattice E of idempotents and assuming for $\in E$, $a \in S$, then $a \leq e \Rightarrow a \in E$.

Still on S a type A semigroup, if $a, b \in S$ such that $a \leq b$. Then $\exists e \in E$ such that $a = be$. This implies that $ae = be = a$. Therefore, $ae = a$.
If $a\mathcal{L}^*b$, then $a^* = b^*$. Now, assuming $a \leq b$ and $a\mathcal{L}^*b$, then $ae = a$ and $\forall x, y \in S^1$, $ax = ay \ \Leftrightarrow\ bx = by$. If we take $x = e$, $y = b^* = a^*$, then we have that $[ae = aa^* = a] \ \Leftrightarrow\ [be = bb^* = b]$ and since $be = a,\ ae = a \ \Leftrightarrow\ a = b$.
So that if $a \leq b$ and $a\mathcal{L}^*b$, then $a = b$. Similarly or dually, if $a \leq b$ and $a\mathcal{R}^*b$, then $a = b$.

If C is a cancellative monoid with set of idempotents $E(C)$ and suppose $e \in E(C)$. Then $e.e = e = 1.e$ and by cancellation, $e = 1$. That is, the only idempotent in a cancellative monoid is 1.
If $(s, 1) \in T = T(S, C, \psi)$, then $s \leq \psi(1)$ and since by definition $\psi(1)$ is an idempotent,$s \in E(S)$.

Assuming $(s,g) \in T$ is an idempotent, then $(s,g)^2 = (s^2,g^2) = (s,g)$. That is $s^2 = s$ and $g^2 = g$. C being a cancellative monoid implies that $g = 1$.
Thus, the set of idempotents of T is given by $E(T) = \{(e,1) |\ e \leq \psi(1), e \in E(S)\}$
That is $E(T) = \{\omega[\psi(1)] \cap E(S)\} \times \{1\}$.
We know that direct product of two semigroups is again a semigroup. Now, what about when the semigroups are type A?

Proposition 1.2: The direct product of two type A semigroups is again type A.
Proof: Let S and T be type A semigroups and assuming $(s,t) \in S \times T,\ s \in S, t \in T$.
For all $s_1, s_2 \in S, ss_1 = ss_2 \Leftrightarrow s^*s_1 = s^*s_2$
Similarly, $\forall t_1, t_2 \in T, tt_1 = tt_2 \Leftrightarrow t^*t_1 = t^*t_2$
So that, $(ss_1, tt_1) = (ss_2, tt_2) \Leftrightarrow (s^*s_1, t^*t_1) = (s^*s_2, t^*t_2)$
That is, $(s,t)(s_1,t_1) = (s,t)(s_2,t_2) \Leftrightarrow (s^*,t^*)(s_1,t_1) = (s^*,t^*)(s_2,t_2)$
Therefore, $\forall (s,t) \in S \times T,\ (s,t)\mathcal{L}^*(S \times T)(s^*,t^*)$
Similarly, $\forall (s,t) \in S \times T,\ (s,t)\mathcal{R}^*(S \times T)(s^\dagger,t^\dagger)$.
Thus, every $\mathcal{L}^*$ and $\mathcal{R}^*$ contains an idempotent.
If $e_s \in E(S)$ and $e_t \in E(T)$, then $(e_s,e_t) \in E(S \times T)$. Let also $(f_s,f_t) \in E(S \times T)$, then $(e_s,e_t)(f_s,f_t) = (e_sf_s, e_tf_t) = (f_se_s, f_te_t) = (f_s,f_t)(e_s,e_t) \in E(S \times T)$. Therefore, $E(S \times T)$ is a semilattice.
Now, $(e_s,e_t)(s,t) = (e_ss, e_tt) = [s(e_ss)^*, t(e_tt)^*] = (s,t)[(e_ss)^*, (e_tt)^*]$
$= (s,t)(e_ss, e_tt)^* = (s,t)[(e_s,e_t)(s,t)]^*$.
And $(s,t)(e_s,e_t) = (se_s, te_t) = [(se_s)^\dagger s, (te_t)^\dagger t] = [(se_s)^\dagger, (te_t)^\dagger](s,t)$
$= (se_s, te_t)^\dagger(s,t) = [(s,t)(e_s,e_t)]^\dagger(s,t)$
Hence, $S \times T$ is type A.

Lemma 1.3(Lawson 1986): Let S be a type A semigroup and let T be an abundant subsemigroup of S with $E(T)$ an order ideal of $E(S)$. Then, T is type A.

Proposition 1.4(Lawson 1986): Let $T = T(S,C,\psi)$. Then T is a type A semigroup.

Proposition 1.5(Lawson 1986): $T = T(S,C,\psi)$ is a proper type A semigroup.

Now, the projection $\alpha: T = T(S,C,\psi) \to S$, defined by $\alpha: (s,g) \mapsto s$ is clearly a homomorphism as we can see that for $(s,g)\alpha(t,h)\alpha = st = (st,gh)\alpha = [(s,g)(t,h)]\alpha$.
α is one-to-one on the idempotents of T since if $(e,1),(f,1) \in E(T)$, $(e,1)\alpha = (f,1)\alpha \Rightarrow e = f \Rightarrow (e,1) = (f,1)$. Thus, α is idempotent – separating.

We can easily show that α is good. For $(s,g),(t,h) \in T$, $(s,g)\mathcal{L}^*(T)(t,h) \Rightarrow \exists (s_1,g_1),(s_2,g_2) \in T$ such that $(s,g)(s_1,g_1) = (s,g)(s_2,g_2) \Leftrightarrow (t,h)(s_1,g_1) = (t,h)(s_2,g_2)$
$(ss_1, gg_1) = (ss_2, gg_2) \Leftrightarrow (ts_1, hg_1) = (ts_2, hg_2)$
$ss_1 = ss_2 \Leftrightarrow ts_1 = ts_2$. This implies that $s\mathcal{L}^*(S)t = (s,g)\alpha\mathcal{L}^*(S)(t,h)\alpha$
Similarly, $(s,g)\mathcal{R}^*(T)(t,h) \Rightarrow (s,g)\alpha\mathcal{R}^*(S)(t,h)\alpha$
$ker\ \alpha = \{(s,g),(t,h) \in T \times T |\ s = t\}$

The projection $\beta: T = T(S,C,\psi) \to C$, defined by $\beta: (s,g) \mapsto g$ is equally a homomorphism.
$ker\ \beta = \{(s,g),(t,h) \in T \times T |\ g = h\}$.

Proposition 1.6: The kernel of the projection $\beta: T \to C$ coincides with the minimum cancellative monoid congruence on T.
Proof: Let $(s,g),(t,h) \in ker\ \beta$. This implies that $g = h$ and $s \leq \psi(g)$ and $t \leq \psi(h) = \psi(g)$.
This implies that $\exists u,v \in E(W)$ such that $s = \psi(g)u$ and $t = \psi(g)v$.
Therefore, $svu = \psi(g)vu$ and $tvu = \psi(g)vu$
Hence, $svu = tvu$ and since $E(W)$ is a semilattice, $\exists e \in E(W)$ such that $vu = e$. So that $se = te$. Since S is a *-subsemigroup of W, $e \in S$.
Now, we have: $se = te$ and $g = h$. That is $(s,g)(e,1) = (t,h)(e,1)$; $(e,1) \in E(T)$
This implies that $(s,g)\sigma_T(t,h)$. Reversing the argument gives the converse.

Theorem 1.7: The composition of the kernels, $\beta\ o\ ker\ \alpha\ o\ ker\ \beta$, is transitive.
Proof: Let $[(s,g),(t,h)],\ [(t,h),(x,k)] \in\ ker\ \beta\ \ o\ \ ker\ \alpha\ \ o\ \ ker\ \beta$. Then, we wish to show that $[(s,g),(x,k)] \in\ ker\ \beta\ \ o\ \ ker\ \alpha\ \ o\ \ ker\ \beta$.
To be more explicit, we need to show that if there exist $(s_1,g_1),(t_1,h_1),(t_2,h_2),(x_1,k_1)\ \in T$ such that

$[(s,g),(s_1,g_1)] \in\ ker\ \beta,\ \ [(s_1,g_1),(t_1,h_1)]\ \in\ ker\ \alpha,\ \ [(t_1,h_1),(t,h)]\ \in\ ker\ \beta$

and

$[(t,h),(t_2,h_2)] \in ker\ \beta,\ \ [(t_2,h_2),(x_1,k_1)]\ \in\ ker\ \alpha,\ \ \ [(x_1,k_1),(x,k)]\ \in\ ker\ \beta$

Then, we can find $u,v \in T$ such that

$[(s,g),u] \in ker\ \beta,\ \ (u,v) \in ker\ \alpha,\ \ [v,(x,k)] \in ker\ \beta.$

If we accept the premise, then we know from the definitions of $ker\ \beta$ and $ker\ \alpha$ that

$g = g_1,\ \ s_1 = t_1,\ \ h = h_1 = h_2,\ t_2 = x_1$ and $k = k_1$.

And since $ker\ \beta = \sigma_T$, there exist $(e,1)$ and $(n,1) \in E(T)$ such that

$(e,1)(s,g)\ = (e,1)(s_1,g)$ ……………….(1)

$(n,1)(x,k)\ = (n,1)(x,k)$ ……………….(2)

For $[(s,g),u] \in\ ker\ \beta$, we need to find $e_1 \in E(T)$ such that

$e_1(s,g) =\ \ e_1 u$ ………………………….(3)

Similarly, for $[v,(x,k)] \in ker\ \beta$, we need some $f_1 \in E(T)$ such that

$f_1 v\ =\ f_1(x,k)$ ………………………….(4)

We know that our $e,\ f,\ m,$ and n lie in $E(S)$ and since $E(S)$ is a semilattice, $efm,\ nfm\ \in E(S)$.
If we take $e_1 = (efm,1)\ \in E(T)$ and $f_1 = (nfm,1) \in E(T)$, then equations (3) and (4) will respectively give

$(efm,1)(s,g) =\ \ (efm,1)u$ ……………..(5)

and $(nfm,1)v\ =\ (nfm,1)(x,k)$ ……………..(6)

Now, if we choose $u = (fms_1,g)$ where $s_1 = fs$ and $e \le f$, and if we choose $v = (fmx_1,k)$ where $x_1 = mx$ and $n \le m$, then our equations are satisfied and therefore our goal achieved.

II. THE STRUCTURE MAPS OF TYPE *A* SEMIGROUP

The structure maps of a semigroup S are maps between $\mathcal{L}$-classes and $\mathcal{R}$-classes of the idempotents $E(S)$ of the semigroup. Interestingly, Armstrong(1988) broadened the definition to maps between $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes of the idempotents $E(S)$ for abundant semigroups. Lawson(1986) characterised type A semigroup as proper if and only if these structure maps are injective. Our goal this section is to characterize type A semigroup in structure maps framework in line with Armstrong's (1988) analysis of *concordant* semigroups in terms of their traces and structure maps. A *concordant* semigroup is an idempotent connected abundant semigroup in which the idempotents generate a regular semigroup.

Let S be a type A semigroup and $e,f \in E(S)$ with $f\omega e$. Suppose $a \in \mathcal{R}_e^*$, then $(fa)^\dagger = (fa^\dagger)^\dagger = (fe)^\dagger = f^\dagger = f$

Thus, $fa \in \mathcal{R}_f^*$. So that corresponding to each pair of $\mathcal{R}^*$-classes $\mathcal{R}_e^*$ and $\mathcal{R}_f^*$ with $f\omega e$, we can associate a map:

$$\phi_{e,f}\colon\ \mathcal{R}_e^* \to \mathcal{R}_f^*\ \text{ defined by } a \mapsto fa$$

Similarly, for $\mathcal{L}^*$-classes $\mathcal{L}_e^*$ and $\mathcal{L}_f^*$ with $f\omega e$ we have $\psi_{e,f}\colon\ \mathcal{L}_e^* \to \mathcal{L}_f^*$ defined by $a \mapsto af$

The maps $\phi_{e,f}$ and $\psi_{e,f}$, given that $f\omega e$, are called the $structure\ maps\ of\ S$.

Suppose S is a type A semigroup endowed with one-one structure maps and assuming $(a,b) \in \sigma \cap \mathcal{R}^*$, so that for some $f \in E(S)$, $fa = fb$ and $a^\dagger = b^\dagger$. We therefore have $fa = fb = fb^\dagger b = fa^\dagger b$.
We know that $fa^\dagger \omega a^\dagger$ and $(fa)^\dagger = (fa)^\dagger fa^\dagger$, $(fa)^\dagger =\ (fa^\dagger)^\dagger$ and therefore, $fa\mathcal{R}^* fa^\dagger$. Let us put $e = a^\dagger$ and $h = fa^\dagger$. Then we have a structure map $\phi_{e,h}\colon\ \mathcal{R}_e^* \to \mathcal{R}_h^*$ with $a\phi_{e,h} = ha,\ a \in \mathcal{R}_e^*$.
$a \in \mathcal{R}_e^*$ implies that $e = a^\dagger = b^\dagger$. So that $b \in \mathcal{R}_e^*$ as well.
$a\phi_{e,h} = ha = fa^\dagger a = fa = fa^\dagger b = hb = b\phi_{e,h}$ and since by assumption, $\phi_{e,h}$ is one-one, $a = b$. Thus, $\sigma \cap \mathcal{R}^* = \iota$. Similar argument produces $\sigma \cap \mathcal{L}^* = \iota$.

On the other hand, assuming the type A semigroup S is proper with $\phi_{e,f}\colon \mathcal{R}_e^* \to \mathcal{R}_f^*$, $f\omega e$, $e, f \in E(S)$. Suppose that for $a, b \in \mathcal{R}_e^*$, $a\phi_{e,f} = b\phi_{e,f}$. This implies that $a\mathcal{R}^*b$ and $fa = fb$. That is, $(a, b) \in \sigma \cap \mathcal{R}^*$ and since S is proper, $a = b$.
Thus, a type A semigroup is proper if and only if its structure maps are one-one.

Let us recall that $\omega = \omega^r \cap \omega^l$. Now, suppose $f\omega^r \cup \omega^l e$. Then we have $f\omega^r e$ or $f\omega^l e$. $f\omega^r e$ implies that $ef = f$. So that $fe.f = f$ and $f.e = fe$ and therefore, $fe\mathcal{R}f$. In the same vein, $f\omega^l e$ produces $fe = f$ and $ef\mathcal{L}f$.
Now, let Φ and Ψ be collections of structure maps on S, given by
$\Phi = \{\phi_{e,f}\colon \mathcal{R}_e^* \to \mathcal{R}_f^* \mid f\omega e\}$ and $\Psi = \{\psi_{e,f}\colon \mathcal{L}_e^* \to \mathcal{L}_f^* \mid f\omega e\}$

With $g \in E(S)$, suppose $g\omega f\omega e$, then for $a \in \mathcal{R}_e^*$, $a\phi_{e,f}\phi_{f,g} = fa\phi_{f,g} = gfa = gfea$ [since $f\omega e$]
$= a\phi_{e,gfe}$
and with $\text{dom}[\phi_{e,f}\phi_{f,g}] = \text{dom}[\phi_{e,gfe}]$, we have $\phi_{e,f}\phi_{f,g} = \phi_{e,gfe}$.
Thus, if $g\omega f\omega e$, then $\phi_{e,f}\phi_{f,g} = \phi_{e,gfe}$.

Lemma 2.1: $\phi_{e,f} = 1_{\mathcal{R}_e^*}$ if $f \in \mathcal{R}_e^* \cap E(S)$.
Proof: Suppose $f \in \mathcal{R}_e^* \cap E(S)$, $e \in E(S)$. Since S is adequate, $f = e$. With $a \in \mathcal{R}_e^*$, $a\phi_{e,f} = fa = ea = a$.

The following theorem was adapted from lemma 8.1 of Armstrong(1988) where it basically had to do with concordant semigroup. We recast the proof to particularly suit type A semigroup.

Theorem 2.2: Given that $e\mathcal{L}^*a\mathcal{R}^*f$, $(a \in S)$ $[e, f \in E(S)]$. Then there exist isomorphisms
$\theta_{f,a,e}\colon \omega(f) \to \omega(e)$ satisfying $a\phi_{f,h} = a\psi_{e,h\theta_{f,a,e}}$ for $h \in \omega(f)$
$\theta_{e,a,f}\colon \omega(e) \to \omega(f)$ satisfying $a\psi_{e,g} = a\phi_{f,g\theta_{e,a,f}}$ for $g \in \omega(e)$
such that $\theta_{e,a,f}\theta_{f,a,e} = 1_{dom\,\theta_{e,a,f}}$ and $\theta_{f,a,e}\theta_{e,a,f} = 1_{dom\,\theta_{f,a,e}}$

Proof: Type A semigroup is idempotent – connected. Hence, for all $a \in S$, and for all $a^\dagger, a^* \in S$, there exists isomorphisms $\alpha\colon \langle a^\dagger\rangle \to \langle a^*\rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^\dagger\rangle$ and $\alpha^{-1}\colon \langle a^*\rangle \to \langle a^\dagger\rangle$ satisfying $ay = (y\alpha^{-1})a$, for all $y \in \langle a^*\rangle$.
We recall that $\langle a^\dagger\rangle = \{x \mid x\omega a^\dagger\} = \omega(a^\dagger)$.
$e\mathcal{L}^*a\mathcal{R}^*f$ implies that $a^\dagger = f$ and $a^* = e$ since S is adequate. We can therefore give our connecting isomorphisms as
$\alpha\colon \omega(f) \to \omega(e)$ such that $ha = a(h\alpha)$ with $h \in \omega(f)$
and $\beta\colon \omega(e) \to \omega(f)$ such that $ag = (g\beta)a$ with $g \in \omega(e)$.
Since we are talking about type A semigroup here, $h\alpha = (ha)^*$ and $g\beta = (ag)^\dagger$.
We can see that $ha = a(h\alpha)$ implies that $a\phi_{f,h} = a\psi_{e,h\alpha}$
and $ag = (g\beta)a$ implies that $a\psi_{e,g} = a\phi_{f,g\beta}$.
To satisfy the demands, we can now take $\alpha = \theta_{f,a,e}$ and $\beta = \theta_{e,a,f}$.
$\beta\alpha\colon \omega(e) \to \omega(e)$ and $a(g\beta\alpha) = a[(ag)^\dagger\alpha] = a[(ag)^\dagger a]^* = (ag)^\dagger a = ag$
So that $a(g\beta\alpha) = ag$. The fact that $a\psi_{e,g} = a\phi_{f,g\beta}$ tells us that $a \in \mathcal{L}_e^*$
Therefore, $a(g\beta\alpha) = ag \Leftrightarrow e(g\beta\alpha) = eg$.
Now, $g\beta\alpha \in \omega(e)$ so that $e(g\beta\alpha) = g\beta\alpha$. $g \in \omega(e)$, therefore $eg = g$
Hence, with $e(g\beta\alpha) = eg$, we have $g\beta\alpha = g$. Thus, $\theta_{e,a,f}\theta_{f,a,e} = 1_{dom\,\theta_{e,a,f}}$.
Furthermore, $\alpha\beta\colon \omega(f) \to \omega(f)$, $(h\alpha\beta)a = [(h\alpha)^*\beta]a = [a(h\alpha)^*]^\dagger a = (ha)^\dagger a = ha^\dagger a = ha$.
From $a\phi_{f,h} = a\psi_{e,h\alpha}$, we notice that $a \in \mathcal{R}_f^*$. Therefore, $(h\alpha\beta)a = ha \Leftrightarrow (h\alpha\beta)f = hf$.
$h\alpha\beta \in \omega(f)$ so that $(h\alpha\beta)f = h\alpha\beta$, $h \in \omega(f)$ so that $hf = h$. Hence, $h\alpha\beta = h$.
Thus, $\theta_{f,a,e}\theta_{e,a,f} = 1_{dom\,\theta_{f,a,e}}$. All as required.

Let $a, b \in S$, and suppose $\mathcal{L}_a^* \cap \mathcal{R}_b^* \cap E(S) \neq \emptyset$. Then $\exists h \in \mathcal{L}_a^* \cap \mathcal{R}_b^* \cap E(S)$, and since $\mathcal{L}^*$ is a right congruence and h is left identity in $\mathcal{R}_b^*$, $ab\mathcal{L}^*hb = b$. Similarly, $a = ah\mathcal{R}^*ab$.
Thus, $\mathcal{L}_a^* \cap \mathcal{R}_b^* \cap E(S) \neq \emptyset$ implies that $a\mathcal{R}^*ab\mathcal{L}^*b$. In fact, this is obviously true of all semigroups.

Suppose, $h\omega^r f$. This implies that $fh = h$ and therefore $(h, hf) \in \mathcal{R} \subseteq \mathcal{R}^*$. Our S here is adequate and $h, hf \in E(S)$, so that $h = hf$. Thus, if $h\omega^r f$, then $\phi_{f,hf} = \phi_{f,h}$. Dually, if $k\omega^l f$, then $\psi_{f,fk} = \psi_{f,k}$.
Lawson(1985) described the structure of proper type A semigroup in terms of *-prehomomorphism. Now, we wish to give an alternative description of this structure in terms of the structure maps of type A semigroup. We make use of the product given by Armstrong(1988) on the structure maps of concordant semigroup.

To avoid ambiguity, we hereby replace our *-prehomomorphism ψ by π and therefore the structure $T(S, C, \psi)$ becomes $T(S, C, \pi)$. We need to do this because we have also used ψ as one of the structure maps of type A semigroup and right here, the *-prehomomorphism and the structure maps are coming together.

We have given Φ and Ψ as collections of structure maps on S. That is

$\Phi = \{\phi_{e,f} : \mathcal{R}_e^* \to \mathcal{R}_f^* \mid f\omega e\}$ and $\Psi = \{\psi_{e,f} : \mathcal{L}_e^* \to \mathcal{L}_f^* \mid f\omega e\}$

Now, define a product ‘’$_\circ$’’ on T by:
For all $(s, g), (t, h)$ in T, $(s, g)_\circ(t, h) = [(s, g)\psi_{(e,1),(k,1)}][(t, h)\phi_{(f,1),(k,1)}]$
Where $(e, 1) \in \mathcal{L}_{(s,g)}^*(T) \cap E(T)$, $(f, 1) \in \mathcal{R}_{(t,h)}^*(T) \cap E(T)$, $(k, 1) \in S[(e, 1), (f, 1)]$
$S[(e, 1), (f, 1)]$ is the sandwich set of $(e, 1)$ and $(f, 1)$ defined by

$$S[(e, 1), (f, 1)] = \{(k, 1) \in E(T) : (ke, 1) = (fk, 1) = (k, 1);\ (ekf, 1) = (ef, 1)\}$$

We denote the structure by $\hat{T} = \hat{T}(T, \Phi, \Psi)$ and show that the binary operation $''_\circ''$ in $\hat{T}$ coincides with the direct product in $T = T(S, C, \pi)$.

Theorem 2.3: $\hat{T} = \hat{T}(T, \Phi, \Psi)$ is a proper type A semigroup with only one $\mathcal{D}^*$-class.
We first show that the operation in $\hat{T} = \hat{T}(T, \Phi, \Psi)$ coincides with the direct product in $T = T(S, C, \pi)$.
Proof: $(s, g), (t, h) \in \hat{T}$. This implies that $(s, g), (t, h) \in T$.

$$(s, g)_\circ(t, h) = [(s, g)\psi_{(e,1),(k,1)}][(t, h)\phi_{(f,1),(k,1)}] = (s, g)(k, 1)(k, 1)(t, h) = (skt, gh)$$

From the product, we notice that $(s, g) \in \mathcal{L}_{(e,1)}^*[dom\ \psi_{(e,1),(k,1)}]$ and therefore $s\mathcal{L}^*e$.
So that $se = s$ ……………. (i)
We also have that $(k, 1) \in S[(e, 1), (f, 1)]$ which implies that $ke = k$ ………… (ii)
Equations (i) and (ii) give that $s\mathcal{L}^*k$ ……….(iii)
Hence, $sk = s$.
In the same vein, $(t, h) \in \mathcal{R}_{(f,1)}^*$ implying that $t\mathcal{R}^*f$. So that $ft = t$ and $(k, 1) \in S[(e, 1), (f, 1)]$ implies that $fk = k$ and therefore $t\mathcal{R}^*k$ ....………..…(iv)
That is $kt = t$.
Now, $(skt, gh) = (st, gh)$ which is the direct product of (s, g) and (t, h) in $T = T(S, C, \pi)$.
Evidently, $\hat{T}$ is a subsemigroup of T and therefore a proper type A semigroup.
(iii) and (iv) give $s\mathcal{D}^*t$ ………..(v)
With $(s, g) \in \mathcal{L}_{(e,1)}^*$, $g \in \mathcal{L}_1^*$ and with $(t, h) \in \mathcal{R}_{(f,1)}^*$, we have $h \in \mathcal{R}_1^*$. So that $g\mathcal{D}^*h$ …………(vi)
Thus, (v) and (vi) give $(s, g)\mathcal{D}^*(\hat{T})(t, h)$.

III. THE TRANSLATIONAL HULL

3.1 The Translational Hull of a Semigroup

A map λ from a semigroup S to itself is a *left translation* of S if for all elements $a, b \in S$, $\lambda(ab) = (\lambda a)b$. A map ρ from a semigroup S to itself is a *right translation* of S if $(ab)\rho = a(b\rho)$ for all elements $a, b \in S$. A left translation λ and a right translation ρ are linked if $a(\lambda b) = (a\rho)b$ for all $a, b \in S$. The set of all linked pairs (λ, ρ) of left and right translations is called the *translational hull* of S and it is denoted by $\Omega(S)$. We denote the set of all the idempotents of $\Omega(S)$ by $E_{\Omega(S)}$. The set of the left translations of S is denoted by $\Lambda(S)$ and the set of the right translations of S is denoted by $P(S)$. $\Omega(S)$ is a subsemigroup of the direct product $\Lambda(S) \times P(S)$. For $(\lambda, \rho)(\lambda', \rho') \in \Omega(S)$, the multiplication is given by $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$
where $\lambda\lambda'$ denotes the composition of the left maps λ and λ' (that is, first λ' and then λ) and $\rho\rho'$ denotes the composition of the right maps ρ and ρ' (that is, first ρ and then ρ'). For each a in S, there is a linked pair (λ_a, ρ_a)

within $\Omega(S)$ defined by $\lambda_a x = ax$ and $x\rho_a = xa$, and called the $inner\, part$ of $\Omega(S)$ and for all $a, b \in S$, the following is obvious $(\lambda_a, \rho_a)(\lambda_b, \rho_b) = (\lambda_{ab}, \rho_{ab})$.

Theorem 3.2: The translational hull of proper type$A$ semigroup is a proper type$A$ semigroup
Proof:
Suppose S is a proper type$A$ semigroup let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ such that
$(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \mathcal{L}^* \cap \sigma$……………….(i)
Then, $(\lambda_1, \rho_1)\sigma(\lambda_2, \rho_2) \Rightarrow \exists(\lambda, \rho) \in E_{\Omega(S)}$ such that $(\lambda, \rho)(\lambda_1, \rho_1) = (\lambda, \rho)(\lambda_2, \rho_2)$
$\Rightarrow \lambda\lambda_1 = \lambda\lambda_2$ and therefore, $\forall e \in E(s)$, $\lambda\lambda_1(e) = \lambda\lambda_2(e)$
$\lambda[\lambda_1(e)]^\dagger\lambda_1(e) = \lambda[\lambda_2(e)]^\dagger\lambda_2(e)$
Since S is type$A$, we have $[\lambda\lambda_1(e)]^\dagger\lambda\lambda_1(e) = [\lambda\lambda_2(e)]^\dagger\lambda\lambda_2(e) = [\lambda\lambda_1(e)]^\dagger\lambda\lambda_2(e)$
and since $[\lambda\lambda_2(e)]^\dagger\lambda \in E(S)$, $\lambda_1(e)\sigma\lambda_2(e)$ ……………………..(ii)
From (i), $(\lambda_1, \rho_1)\mathcal{L}^*(\lambda_2, \rho_2)$. Now, assuming $(\forall a, b \in S^1)[\forall e \in E(S)]$
$\lambda_1(e)a = \lambda_2(e)b$ ………………….(iii)
Then, $[\forall f \in E(S)]$, $\lambda_1(e)af = \lambda_2(e)bf$ and we have $\lambda_1\lambda_{ea}f = \lambda_1\lambda_{eb}f$
We note that if $\lambda|_{E(S)} = \lambda'|_{E(S)}$, then $\lambda = \lambda'$
Thus, $\lambda_1\lambda_{ea} = \lambda_1\lambda_{eb}$ and $(\lambda_1\lambda_{ea}, \rho_1\rho_{ea}) = (\lambda_1\lambda_{eb}, \rho_1\rho_{eb})$
This implies that $(\lambda_1, \rho_1)(\lambda_{ea}, \rho_{ea}) = (\lambda_1, \rho_1)(\lambda_{eb}, \rho_{eb})$ and with $(\lambda_1, \rho_1)\mathcal{L}^*(\lambda_2, \rho_2)$ we have
$(\lambda_2, \rho_2)(\lambda_{ea}, \rho_{ea}) = (\lambda_2, \rho_2)(\lambda_{eb}, \rho_{eb}) \Rightarrow \lambda_2\lambda_{ea} = \lambda_2\lambda_{eb}$
$(\lambda_2 e)a = \lambda_2 eaa^* = \lambda_2\lambda_{ea}a^* = \lambda_2\lambda_{eb}a^* = (\lambda_2 e)ba^* \Rightarrow (\lambda_2 e)a \le (\lambda_2 e)b$
In the same vein, $(\lambda_2 e)b \le (\lambda_2 e)a$

Thus, $(\lambda_2 e)a = (\lambda_2 e)b$ …………………. (iv)

(iii) and (iv) give $\lambda_1(e)\mathcal{L}^*\lambda_2(e)$ ……………..(v)

(ii) and (v) give $[\lambda_1(e), \lambda_2(e)] \in \mathcal{L}^* \cap \sigma$

and since S is a proper type$A$ semigroup, $\lambda_1(e) = \lambda_2(e)\ \forall e \in E(S)$

Hence, $\lambda_1 = \lambda_2$. In the same vein, $\rho_1 = \rho_2$. So that $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$

Thus, $\mathcal{L}^*[\Omega(S)] \cap \sigma_{\Omega(S)} = \iota_{\Omega(S)}$ and dually, $\mathcal{R}^*[\Omega(S)] \cap \sigma_{\Omega(S)} = \iota_{\Omega(S)}$.

Let S be a monoid with a set of idempotents E. S is said to be $left\, E\text{-}cancellable$ if $(\forall a \in S)$ $(\forall e \in E)$, $ae = a$ and $(\forall s, t \in S)$ $as = at$ implies $es = et$.

Theorem 3.3: If $\Lambda(\hat{T})$ is left $E_{\Lambda(\hat{T})}$ −cancellable, then the map $\theta: \Lambda(\hat{T}) \to \Lambda(\hat{T})/\sigma_{\Lambda(\hat{T})} \times E_{\Lambda(\hat{T})}$ defined by $\theta(\lambda) = (\lambda\sigma_{\Lambda(\hat{T})}, \lambda')$, $\lambda' \in E_{\Lambda(\hat{T})}$ is one-one.
Proof:
Let $(\lambda\sigma_{\Lambda(\hat{T})}, \lambda') = (\bar{\lambda}\sigma_{\Lambda(\hat{T})}, \lambda')$, $\lambda, \bar{\lambda} \in \Lambda(\hat{T}), \lambda' \in E_{\Lambda(\hat{T})}$
$\Rightarrow \lambda\sigma_{\Lambda(\hat{T})} = \bar{\lambda}\sigma_{\Lambda(\hat{T})}$ and since $\Lambda(\hat{T})$ is left $E_{\Lambda(\hat{T})}$ −cancellable, $\lambda\mathcal{L}^*\lambda'\mathcal{L}^*\bar{\lambda}$
Thus, $(\lambda, \bar{\lambda}) \in \sigma_{\Lambda(\hat{T})} \cap \mathcal{L}^*[\Lambda(\hat{T})]$
$\Lambda(\hat{T})$ is proper and therefore $(\lambda, \bar{\lambda}) \in \sigma_{\Lambda(\hat{T})} \cap \mathcal{L}^*[\Lambda(\hat{T})] = \iota_{\Omega(\hat{T})}$
Hence, $\lambda = \bar{\lambda}$.

IV. THE REPRESENTATION

Let X be a set, and denote by T_X the set of all functions $\alpha: X \to X$. T_X is called the $full\ transformation\ semigroup$ on X with the operation of composition of functions. A homomorphism $\phi: S \to T_X$ is a $representation$ of the semigroup S.

In this section, we use the Munn semigroup to obtain the representation of $\Lambda(\hat{T})$

4.1 The Munn Semigroup

Given a semilattice E, the Munn semigroup T_E consists of all isomorphisms between principal ideals of E. It is an inverse subsemigroup of the *symmetric inverse semigroup* $\mathfrak{T}_E$ which is a subsemigroup of the full transformation semigroup. Munn showed that for any inverse semigroup S, there is a homomorphism from S into $T_{E(S)}$ which maps $E(S)$ isomorphically onto $E(T_{E(S)})$ and induces the maximum idempotent separating congruence on S. The maximum idempotent separating congruence on an inverse semigroup is the largest congruence contained in Green's relation $\mathcal{H}$(details in Howie 1995).
Fountain(1979) extended Munn's result to type A semigroup S to give a homomorphism from S into $T_{E(S)}$ mapping $E(S)$ isomorphically onto $E(T_{E(S)})$ and inducing the largest congruence contained in $\mathcal{H}^*$.

By analogy with this Fountain's result, we wish to obtain a representation of $\Lambda(\hat{T})$ on a full subsemigroup of a type A semigroup of mapping using the Munn semigroup.

So, for an arbitrary semilattice $\Lambda(E)$, $T_{\Lambda(E)}$ consists of all isomorphisms between principal ideals of $\Lambda(E)$, and $T_{\Lambda(E)}$ is an inverse semigroup.

For $\lambda \in \Lambda(\hat{T})$, define the maps ϑ_λ and Υ_λ as follows:

$\vartheta_\lambda\colon \langle\lambda^\dagger\rangle \to \langle\lambda^*\rangle$ and $\Upsilon_\lambda\colon \langle\lambda^*\rangle \to \langle\lambda^\dagger\rangle$ by $\lambda'\vartheta_\lambda = (\lambda'\lambda)^*$ and $\lambda'\Upsilon_\lambda = (\lambda\lambda')^\dagger$

For $\lambda' \in \langle\lambda^\dagger\rangle$, $\lambda'\vartheta_\lambda\Upsilon_\lambda = [\lambda(\lambda'\lambda)^*]^\dagger$

Now, $\lambda' \cdot [\lambda'\vartheta_\lambda\Upsilon_\lambda] = \lambda'[\lambda(\lambda'\lambda)^*]^\dagger = [\lambda'\lambda(\lambda'\lambda)^*]^\dagger = (\lambda'\lambda)^\dagger = \lambda'\lambda^\dagger = \lambda'$ since $\lambda' \in \langle\lambda^\dagger\rangle$.

Therefore,$\forall\lambda' \in \langle\lambda^\dagger\rangle$, $\lambda' \le \lambda'\vartheta_\lambda\Upsilon_\lambda$. In the same vein, for $\lambda' \in \langle\lambda^*\rangle$, $\lambda' \le \lambda'\Upsilon_\lambda\vartheta_\lambda$

For $\lambda' \in \langle\lambda^\dagger\rangle$, suppose $\lambda'\vartheta_\lambda = \lambda''\vartheta_\lambda$. Then we have $(\lambda'\lambda)^* = (\lambda''\lambda)^*$. Since $\Lambda(\hat{T})$ is type A, $\lambda'\lambda = \lambda(\lambda'\lambda)^* = \lambda(\lambda''\lambda)^* = \lambda''\lambda$. Therefore, $\lambda' = \lambda'\lambda^\dagger = (\lambda'\lambda)^\dagger = (\lambda''\lambda)^\dagger = \lambda''\lambda^\dagger = \lambda''$

Thence, ϑ_λ is one-one for each $\lambda \in \Lambda(\hat{T})$. Similarly, Υ_λ is one-one for each $\lambda \in \Lambda(\hat{T})$.

Thus, $\vartheta_\lambda, \Upsilon_\lambda \in T_{E_{\Lambda(\hat{T})}}$ $\forall\lambda \in \Lambda(\hat{T})$. For $\bar{\lambda} \in \langle\lambda^\dagger\rangle$, assume $\bar{\lambda}\vartheta_\lambda\Upsilon_\lambda = \lambda'$ ………………... (i)

where $\lambda' \in \langle\lambda^\dagger\rangle$. Then, $\bar{\lambda}\lambda' = \bar{\lambda}[\bar{\lambda}\vartheta_\lambda\Upsilon_\lambda] = \bar{\lambda}[\lambda(\bar{\lambda}\lambda)^*]^\dagger = [\bar{\lambda}\lambda(\bar{\lambda}\lambda)^*]^\dagger = (\bar{\lambda}\lambda)^\dagger = \bar{\lambda}\lambda^\dagger = \bar{\lambda}$

So that $\bar{\lambda} \le \lambda' = \bar{\lambda}\vartheta_\lambda\Upsilon_\lambda$, $\forall\bar{\lambda} \in \langle\lambda^\dagger\rangle$ ………………(ii)

The domain of $\vartheta_{\lambda'\lambda} = \langle(\lambda'\lambda)^\dagger\rangle = \langle\lambda'\lambda^\dagger\rangle = \langle\lambda'\rangle$. Now, $\lambda'' \in dom\vartheta_{\lambda'\lambda} = \langle\lambda'\rangle$,

$\lambda''\vartheta_{\lambda'\lambda} = (\lambda''\lambda'\lambda)^* = [(\lambda''\lambda')^*\lambda]^*$ and $\lambda''\vartheta_{\lambda'}\vartheta_\lambda = (\lambda''\lambda')^*\vartheta_\lambda = [(\lambda''\lambda')^*\lambda]^*$

So that $\vartheta_{\lambda'\lambda} = \vartheta_{\lambda'}\vartheta_\lambda|_{\langle\lambda'\rangle}$ ………………….. (iii)

Similarly, $\Upsilon_{\lambda'\lambda} = \Upsilon_\lambda\Upsilon_{\lambda'}|_{\langle(\lambda'\lambda)^*\rangle}$. Therefore,

$\lambda'\vartheta_{\lambda'\lambda}\Upsilon_{\lambda'\lambda} = \lambda'\vartheta_{\lambda'}\vartheta_\lambda\Upsilon_\lambda\Upsilon_{\lambda'} = (\lambda'\vartheta_\lambda\Upsilon_\lambda)\lambda' = \lambda'[\lambda(\lambda'\lambda)^*]^\dagger = [\lambda'\lambda(\lambda'\lambda)^*]^\dagger = (\lambda'\lambda)^\dagger = \lambda'\lambda^\dagger = \lambda'$…(iv)

$\bar{\lambda}\vartheta_{\lambda'\lambda}\Upsilon_{\lambda'\lambda} = \bar{\lambda}\vartheta_{\lambda'}\vartheta_\lambda\Upsilon_\lambda\Upsilon_{\lambda'} = \bar{\lambda}\vartheta_\lambda\Upsilon_\lambda\Upsilon_{\lambda'}$ since $\bar{\lambda} \le \lambda'$ from (ii)

$= \lambda'\Upsilon_{\lambda'}$ since $\lambda' = \bar{\lambda}\vartheta_\lambda\Upsilon_\lambda$ from (ii)

$= \lambda'$ ……………………….. (v)

So that (iii) = (iv). That is, $\lambda'\vartheta_{\lambda'\lambda}\Upsilon_{\lambda'\lambda} = \bar{\lambda}\vartheta_{\lambda'\lambda}\Upsilon_{\lambda'\lambda}$

$\Rightarrow$ $\lambda' = \bar{\lambda}$ since $\vartheta_{\lambda'\lambda}$ & $\Upsilon_{\lambda'\lambda}$ are one – one.

So from (i), we see that $\vartheta_\lambda\Upsilon_{\bar{\lambda}} = 1_{\langle\lambda^\dagger\rangle}$, the identity map on $\langle\lambda^\dagger\rangle$. In the same vein, $\Upsilon_\lambda\vartheta_\lambda = 1_{\langle\lambda^*\rangle}$.

Thus, ϑ_λ and Υ_λ are mutually inverse isomorphisms for each $\lambda \in \Lambda(\hat{T})$.

Theorem 4.2: $\mu = \{(\lambda,\bar{\lambda}) \in \Lambda(\hat{T}) \times \Lambda(\hat{T}):\ \vartheta_\lambda = \vartheta_{\bar{\lambda}},\ \Upsilon_\lambda = \Upsilon_{\bar{\lambda}}\}$ is the largest idempotent separating congruence contained in $\mathcal{H}^*$, and there is a homomorphism from $\Lambda(\hat{T})$ onto a full subsemigroup of $T_{E_{\Lambda(\hat{T})}}$ whose kernel equals μ, with semilattice of idempotents of $T_{E_{\Lambda(\hat{T})}}$ isomorphic to $E_{\Lambda(\hat{T})}$.

Proof: The straightforward definition clearly makes μ an equivalence. To show compatibility of μ with the operations on $\Lambda(\hat{T})$, let $(\lambda_1,\bar{\lambda}_1), (\lambda_2,\bar{\lambda}_2) \in \mu$

$\Rightarrow \vartheta_{\lambda_1} = \vartheta_{\bar{\lambda}_1}, \Upsilon_{\lambda_1} = \Upsilon_{\bar{\lambda}_1},\ \vartheta_{\lambda_2} = \vartheta_{\bar{\lambda}_2}, \Upsilon_{\lambda_2} = \Upsilon_{\bar{\lambda}_2}$

$\Rightarrow \vartheta_{\lambda_1}\vartheta_{\lambda_2} = \vartheta_{\bar{\lambda}_1}\vartheta_{\bar{\lambda}_2}$ and $\Upsilon_{\lambda_1}\Upsilon_{\lambda_2} = \Upsilon_{\bar{\lambda}_1}\Upsilon_{\bar{\lambda}_2}$

From (iii) above, we have $\vartheta_{\lambda_1\lambda_2} = \vartheta_{\bar{\lambda}_1\bar{\lambda}_2}$ and $\Upsilon_{\lambda_1\lambda_2} = \Upsilon_{\bar{\lambda}_1\bar{\lambda}_2}$

$\Rightarrow (\lambda_1\lambda_2,\ \bar{\lambda}_1\bar{\lambda}_2) \in \mu$. Thus, μ is a congruence.

Furthermore, let $\lambda', \bar{\lambda}' \in E_{\Lambda(\hat{T})}$ with $\lambda'\mu\bar{\lambda}'$

$\Rightarrow \vartheta_{\lambda'} = \vartheta_{\bar{\lambda}'}$ and $\Upsilon_{\lambda'} = \Upsilon_{\bar{\lambda}'}$

$\Rightarrow \langle\lambda'\rangle = \langle\bar{\lambda}'\rangle$ and evidently, $\lambda' = \bar{\lambda}'$. Therefore, μ is idempotent separating.

Now, let $(\lambda,\bar{\lambda}) \in \mu$. This implies that $\vartheta_\lambda = \vartheta_{\bar{\lambda}}$ and $\Upsilon_\lambda = \Upsilon_{\bar{\lambda}}$

$\Rightarrow \langle\lambda^\dagger\rangle = \langle\bar{\lambda}^\dagger\rangle$ and $\langle\lambda^*\rangle = \langle\bar{\lambda}^*\rangle$

$\Rightarrow \lambda^\dagger = \bar{\lambda}^\dagger$ and $\lambda^* = \bar{\lambda}^*$ so that $\lambda\mathcal{R}^*\bar{\lambda}$ and $\lambda\mathcal{L}^*\bar{\lambda}$

and therefore $\lambda\mathcal{H}^*\bar{\lambda}$. Hence $\mu \in \mathcal{H}^*$.

Suppose τ is a congruence on $\Lambda(\hat{T})$ with $\mu \in \mathcal{H}^*$ and suppose $(\lambda_1,\bar{\lambda}_1) \in \tau$.

Then $\forall\lambda' \in \langle\lambda^\dagger\rangle$ and $\forall\lambda'' \in \langle\lambda^*\rangle$, $(\lambda'\lambda_1, \lambda'\bar{\lambda}_1) \in \tau$ and $(\lambda_1\lambda'', \bar{\lambda}_1\lambda'') \in \tau$

$\Rightarrow \lambda'\lambda_1\mathcal{H}^*\lambda'\bar{\lambda}_1$ and $\lambda_1\lambda''\mathcal{H}^*\bar{\lambda}_1\lambda''$

That is $(\lambda'\lambda_1)^* = (\lambda'\bar{\lambda}_1)^*$ and $(\lambda_1\lambda'')^\dagger = (\bar{\lambda}_1\lambda'')^\dagger$

$\Rightarrow$ $\forall\lambda' \in \langle\lambda^\dagger\rangle$, $\lambda'\vartheta_{\lambda_1} = \lambda'\vartheta_{\bar{\lambda}_1}$ and $\forall\lambda'' \in \langle\lambda^*\rangle$, $\lambda''\Upsilon_{\lambda_1} = \lambda''\Upsilon_{\bar{\lambda}_1}$

$\Rightarrow (\lambda_1,\bar{\lambda}_1) \in \mu$. So that, $\tau \subseteq \mu$.

Define a map $\xi: \Lambda(\hat{T}) \to T_{E_{\Lambda(\hat{T})}}$ by $\lambda\xi = \vartheta_\lambda$ and suppose $\lambda, \bar{\lambda} \in \Lambda(\hat{T})$.

$dom\ \vartheta_\lambda\vartheta_{\bar{\lambda}} = (im\ \vartheta_\lambda \cap dom\ \vartheta_{\bar{\lambda}})\vartheta_\lambda^{-1}$ (Howie 1995, pg 148)

Where $\vartheta_\lambda: \langle\lambda^\dagger\rangle \to \langle\lambda^*\rangle$, $\vartheta_{\bar\lambda}: \langle\bar\lambda^\dagger\rangle \to \langle\bar\lambda^*\rangle$. So that $dom\ \vartheta_\lambda\vartheta_{\bar\lambda} = (\langle\lambda^*\rangle \cap \langle\bar\lambda^\dagger\rangle)\vartheta_\lambda^{-1}$

Every type A semigroup S with semilattice E is characterized by $eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae$ $\forall a \in S,\ \forall e \in E$. (Fountain 1979).

Therefore, $dom\ \vartheta_\lambda\vartheta_{\bar\lambda} = \langle\lambda^*\bar\lambda^\dagger\rangle\vartheta_\lambda^{-1}$ and since ϑ_λ and Υ_λ are mutually inverse isomorphisms

$dom\ \vartheta_\lambda\vartheta_{\bar\lambda} = \langle\lambda^*\bar\lambda^\dagger\rangle\Upsilon_\lambda = \langle(\lambda^*\bar\lambda^\dagger)\Upsilon_\lambda\rangle = \langle(\lambda\lambda^*\bar\lambda^\dagger)^\dagger\rangle = \langle(\lambda\bar\lambda^\dagger)^\dagger\rangle = \langle(\lambda\bar\lambda)^\dagger\rangle = dom\ \vartheta_{\lambda\bar\lambda}$

For $\lambda' \in \langle(\lambda\bar\lambda)^\dagger\rangle$, $\lambda'\vartheta_\lambda\vartheta_{\bar\lambda} = (\lambda'\lambda)^*\vartheta_{\bar\lambda} = [(\lambda'\lambda)^*\bar\lambda]^* = (\lambda'\lambda\bar\lambda)^* = \lambda'\vartheta_{\lambda\bar\lambda}$

Hence, $\vartheta_\lambda\vartheta_{\bar\lambda} = \vartheta_{\lambda\bar\lambda}$. So that, $\lambda\bar\lambda\xi = \lambda\xi\bar\lambda\xi$ and thus,ξ is a homomorphism.

For each $' \in E_{\Lambda(\hat T)}$, $\vartheta_{\lambda'}\cdot\vartheta_{\lambda'} = \vartheta_{\lambda'}$ and $(\lambda')^\dagger = (\lambda')^* = \lambda'$ so that $\vartheta_{\lambda'}: \langle\lambda'\rangle \to \langle\lambda'\rangle$. For $\lambda \in \langle\lambda'\rangle$, $\lambda\vartheta_{\lambda'} = \lambda\lambda' = \lambda$.

Thus, the idempotents of $T_{E_{\Lambda(\hat T)}}$ have the form $1_{\langle\lambda'\rangle}$ - the identical map of $\langle\lambda'\rangle$ onto itself, $\lambda' \in E_{\Lambda(\hat T)}$.

Thus, the one – one translation $\lambda' \mapsto 1_{\langle\lambda'\rangle}$ is an isomorphism since $1_{\langle\lambda'\lambda''\rangle} = \vartheta_{\lambda'\lambda''} = \vartheta_{\lambda'}\vartheta_{\lambda''} = 1_{\langle\lambda'\rangle}1_{\langle\lambda''\rangle}$. That is, $E_{\Lambda(\hat T)}$ is isomorphic to the semilattice of idempotents of $T_{E_{\Lambda(\hat T)}}$.

Moreover, $\lambda'\xi = \vartheta_{\lambda'} = 1_{\langle\lambda'\rangle}$ and therefore, the $im\ \xi$ is a full subsemigroup of $T_{E_{\Lambda(\hat T)}}$.

Kernel of ξ is given by $\xi\ o\ \xi^{-1}$. For $\lambda, \bar\lambda \in \Lambda(\hat T)$,

$$[(\lambda,\bar\lambda) \in \mu] \Leftrightarrow [\vartheta_\lambda = \vartheta_{\bar\lambda} \text{ and } \Upsilon_\lambda = \Upsilon_{\bar\lambda}] \Leftrightarrow [\vartheta_\lambda = \vartheta_{\bar\lambda} \text{ and } \vartheta_\lambda^{-1} = \vartheta_{\bar\lambda}^{-1}] \Leftrightarrow [\vartheta_\lambda = \vartheta_{\bar\lambda}]$$

$$\Leftrightarrow [\lambda\xi = \bar\lambda\xi] \Leftrightarrow [(\lambda,\bar\lambda) \in \xi\ o\ \xi^{-1}]. \text{ Thus, } ker\ \xi = \mu.$$

V. CONCLUSION

In this article, we married up the structure of proper type A semigroup constructed by Lawson(1986) with the structure maps of type A semigroup and we obtained an alternative structure of proper type A semigroup in line with Armstrong(1988) analysis of concordant semigroups. We obtained the representation of the left translational hull of the alternative proper type A semigroup using the Munn semigroup. Some other results associated proper type A semigroup were also obtained.

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