Faithful Representation of Translational Hull of Type *A* Semigroup

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Abstract

Fountain(1979) obtained monoid embedding of type A semigroup into an inverse semigroup. In this piece of work, we extend this result to translational hulls and, in effect, give a faithful representation of translational hull of type A semigroup with some accompanying results.

Keywords: *faithful representation, translational hull*

I. INTRODUCTION AND PRELIMINARIES

Let X be a set, and denote by T_X the set of all functions $\alpha: X \to X$. T_X is called the *full transformation semigroup* on X with the operation of composition of functions. A homomorphism $\phi: S \to T_X$ is a *representation* of the semigroup S. We say that ϕ is a *faithful* representation, if it is an embedding.

It is well known that the set of all partial one-one maps of any non-empty set X is an inverse semigroup and it is called *symmetric inverse semigroup* usually denoted by $\mathfrak{I}_X$.

*A. The *-Equivalences*

Let S be a semigroup and $a, b \in S$. $(a,b) \in \mathcal{L}^*$ if $\forall x, y \in S^1$, $ax = ay$ if and if $bx = by$. $\mathcal{R}^*$ is dual to $\mathcal{L}^*$ and this definition of $\mathcal{L}^*$ apply in dual manner to $\mathcal{R}^*$. The intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{H}^*$. The join of $\mathcal{L}^*$ and $\mathcal{R}^*$ on S is the equivalence $\mathcal{D}^*$. In general, $\mathcal{L}^* o\, \mathcal{R}^* \neq \mathcal{R}^* o\, \mathcal{L}^*$ and neither equals $\mathcal{D}^*$. Basically, $a\mathcal{D}^*b$ if and only if there exists elements $x_1, x_2, x_3, \dots, x_n$ in S such that $a\mathcal{L}^*x_1\mathcal{R}^*x_2\mathcal{L}^*x_3 \dots x_{n-1}\mathcal{L}^*x_n\mathcal{R}^*b$. $\mathcal{D} \subseteq \mathcal{D}^*$ and $\mathcal{H} \subseteq \mathcal{H}^*$. If S is regular, then $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$.

Let S be a semigroup and I an ideal of S. Then I is called *-ideal if $\mathcal{L}_a^* \subseteq I$ and $\mathcal{R}_a^* \subseteq I$ for all $a \in I$. The smallest *-ideal containing a is the principal *-ideal generated by a and is denoted by $\mathcal{J}^*(a)$. For $a, b \in S$, $a\mathcal{J}^*b$ if and only if $\mathcal{J}^*(a) = \mathcal{J}^*(b)$. The $\mathcal{J}^*$-class containing the element $a \in S$ is denoted by $\mathcal{J}_a^*$. $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence.

A semigroup is called *left(right) abundant* if each $\mathcal{R}^*$- ($\mathcal{L}^*$-) class contains an idempotent and *abundant* if it is both left and right abundant.

A semigroup S is said to be *superabundant* if each $\mathcal{H}^*$-class contains an idempotent. If the idempotents of a left (right) abundant semigroup form a semilattice, it is called *left(right) adequate*. It is called *adequate* if it is both left and right adequate.

In an adequate semigroup, the idempotents in each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class are unique. If S is adequate, and a is an element of S, then $a^*(a^\dagger)$ will denote the unique idempotent in the $\mathcal{L}^*$-($\mathcal{R}^*$-) class of a .

A left(right) adequate semigroup S is called *left(right) type A* if $ae = (ae)^\dagger a[ea = a(ea)^*]$ for all $a \in S$ and all idempotents $e \in S$. An adequate semigroup is called *type A* if it is both left and right type$A$.

Fountain (1979) characterised a type A semigroup in terms of certain embeddings into an inverse semigroup. In particular, we have the following:

Theorem 1.2 (Fountain1979): Let S be an adequate semigroup, then the following conditions are equivalent:

i. S is a type A semigroup
ii. $\forall a \in S$ and $\forall e \in E(S)$, $eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae$.
iii. there are inverse semigroups S_1, S_2, and embeddings $\phi_1: S \to S_1$, $\phi_2: S \to S_2$, such that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a)$, $\phi_2 a^\dagger = (\phi_2 a)^\dagger = (\phi_1 a)(\phi_1 a)^{-1}$.

The embedding in (iii) of the above theorem will be useful in our study here. Meanwhile, we outline more properties of type A semigroup below.

If S is an adequate semigroup with semilattice E of idempotents, then $\forall a, b \in S$, if $a\mathcal{L}^*b$ then $\mathcal{L}_a^* = \mathcal{L}_b^*$ and since a^* is the unique idempotent in $\mathcal{L}_a^*$, $a^* = b^*$.

Conversely, if $a^* = b^*$ then $a^*\mathcal{L}^*b^*$ and we have $a\mathcal{L}^*a^*\mathcal{L}^*b^*\mathcal{L}^*b$. That is, $a\mathcal{L}^*b$.
Hence, $a\mathcal{L}^*b$ if and only if $a^* = b^*$ $(\forall a, b \in S)$. Dually, $(\forall a, b \in S)$ $a\mathcal{R}^*b$ if and only if $a^\dagger = b^\dagger$.
$a\mathcal{L}^*a^* \Rightarrow ab\mathcal{L}^*a^*b$ and therefore $(ab)^* = (a^*b)^*$. $b\mathcal{R}^*b^\dagger \Rightarrow ab\mathcal{R}^*ab^\dagger$ and therefore $(ab)^\dagger = (ab^\dagger)^\dagger$.
It is therefore obvious that for $e \in E$, $(ae)^* = a^*e$ and $(ea)^\dagger = ea^\dagger$. $(ab)^*$ and b^* are idempotents. Therefore, $(ab)^*b^* = [(ab)^*b^*]^* = (abb^*)^* = (ab)^*$. Thus, $b^*(ab)^* = (ab)^*b^* = (ab)^*$
Therefore, $(ab)^* \leq b^*$, where $\leq$ is the usual ordering on E. Similarly, $(ab)^\dagger a^\dagger = a^\dagger(ab)^\dagger = [a^\dagger(ab)^\dagger]^\dagger = (a^\dagger ab)^\dagger = (ab)^\dagger$. Therefore, $(ab)^\dagger \leq a^\dagger$.
Furthermore

i. $(a^\dagger b)^\dagger = a^\dagger b^\dagger = b^\dagger a^\dagger$
ii. $(ab^*)^* = a^*b^* = b^*a^*$
iii. $(ab^\dagger)^* = a^*b^\dagger = b^\dagger a^*$ $\{a^*b^\dagger \in E$ since E is a semilattice, so that $(ab^\dagger)^* = (a^*b^\dagger)^* = a^*b^\dagger\}$
iv. $(a^*b)^\dagger = a^*b^\dagger = b^\dagger a^*$
v. $a(ab^\dagger)^* = (ab^\dagger)^\dagger a$ $\{a(ab^\dagger)^* = aa^*b^\dagger = ab^\dagger = (ab^\dagger)^\dagger a$ (left type$A$)$\}$
vi. $a^{\dagger\dagger} = a^\dagger$
vii. $a^*b = b(ab)^*$ $\{a^*b = b(a^*b)^*$ (right type$A$) $= b(ab)^*\}$
viii. $(a^*)^\dagger = a^*$
ix. $(a^\dagger)^* = a^\dagger$

B. The Translational Hull of a Semigroup

A map λ from a semigroup S to itself is a $left\ translation$ of S if for all elements $a, b \in S$, $\lambda(ab) = (\lambda a)b$. A map ρ from a semigroup S to itself is a $right\ translation$ of S if $(ab)\rho = a(b\rho)$ for all elements $a, b \in S$. A left translation λ and a right translation ρ are linked if $a(\lambda b) = (a\rho)b$ for all $a, b \in S$. The set of all linked pairs (λ, ρ) of left and right translations is called the $translational\ hull$ of S and it is denoted by $\Omega(S)$. We denote the set of all the idempotents of $\Omega(S)$ by $E_{\Omega(S)}$. The set of the left translations of S is denoted by $\Lambda(S)$ and the set of the right translations of S is denoted by $P(S)$. $\Omega(S)$ is a subsemigroup of the direct product $\Lambda(S) \times P(S)$. For $(\lambda, \rho)(\lambda', \rho') \in \Omega(S)$, the multiplication is given by $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$ where $\lambda\lambda'$ denotes the composition of the left maps λ and λ' (that is, first λ' and then λ) and $\rho\rho'$ denotes the composition of the right maps ρ and ρ' (that is, first ρ and then ρ'). For each a in S, there is a linked pair (λ_a, ρ_a) within $\Omega(S)$ defined by $\lambda_a x = ax$ and $x\rho_a = xa$, and called the $inner\ part$ of $\Omega(S)$ and for all $a, b \in S$, the following is obvious $(\lambda_a, \rho_a)(\lambda_b, \rho_b) = (\lambda_{ab}, \rho_{ab})$. $a \mapsto (\lambda_a, \rho_a)$ is a map of S into $\Omega(S)$ is denoted by Π_S. $\Pi_S(S) = \{(\lambda_a, \rho_a) \,|\, a \in S, \lambda_a x = ax, x\rho_a = xa, \forall x \in S\}$

Theorem 1.4 (Ault 1972): Translational hull of an inverse semigroup is an inverse semigroup.
For $(\lambda, \rho) \in \Omega(S)$, the inverse $(\lambda, \rho)^{-1}$ is denoted by $(\lambda^{-1}, \rho^{-1})$ and satisfies the property -

$$\lambda^{-1}x = (x^{-1}\rho)^{-1}, \text{ and } x\rho^{-1} = (\lambda x^{-1})^{-1} \quad \forall x \in S$$

Lemma 1.5: Let S be a type A semigroup. λ, λ' (ρ, ρ') are left (right) translations of S whose restrictions to the set of idempotents of S are equal, then $\lambda = \lambda'$ $(\rho = \rho')$.

If $\Omega(S)$ is adequate, and (λ, ρ) is an element of $\Omega(S)$, then (λ^*, ρ^*) denotes the unique idempotent in the $\mathcal{L}^*$- class of (λ, ρ), and $(\lambda^\dagger, \rho^\dagger)$ denotes the unique idempotent in the $\mathcal{R}^*$- class of (λ, ρ).
For $e \in E(S)$, $\lambda^\dagger e = (\lambda e)^\dagger$; $\lambda^* e = (\lambda e)^*$; $e\rho^\dagger = (e\rho)^\dagger$; $e\rho^* = (e\rho)^*$
$\lambda^\dagger, \lambda^*, \rho^\dagger, \rho^*$ satisfy the following properties –
For $a \in S$, $\lambda^\dagger a = (a^\dagger\rho)^\dagger a$; $\lambda^* a = (\lambda a^\dagger)^* a$; $a\rho^\dagger = a(a^*\rho)^\dagger$; $a\rho^* = a(\lambda a^*)^*$

We notice from the definition that $\lambda^\dagger e$, $\lambda^* e$, $e\rho^\dagger$ and $e\rho^*$ are idempotents. We also note the following:

i. $\lambda^\dagger e = (e\rho)^\dagger e$ from the definition
$= e(e\rho)^\dagger$ idempotents commute
$= e(e^*\rho)^\dagger$
$= e\rho^\dagger$ by definition.

ii. $\lambda^* e = (\lambda e)^* e$ by definition
$= e(\lambda e)^*$ commutativity of idempotents
$= e\rho^*$ by definition.

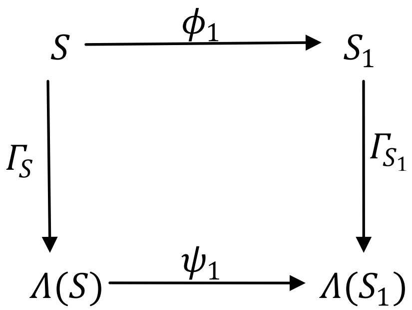
In particular therefore, $\lambda^\dagger b^\dagger$ and $a^*\rho^\dagger$ are idempotent of S.

Theorem 1.6: (Fountain & Lawson 1985): The translational hull of a type A semigroup is type$A$.

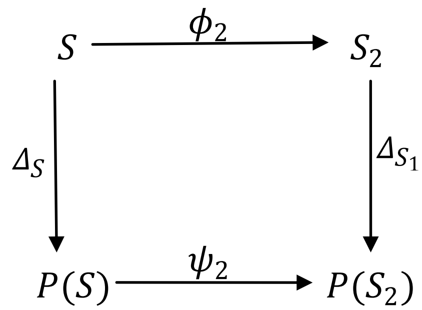
In theorem 1.2 above, Fountain(1979) obtained an embedding of type A semigroup into an inverse semigroup. In the next section, this embedding will be extended to that of the translational hulls.

II. REPRESENTATION OF TRANSLATIONAL HULL OF TYPE *A* SEMIGROUP

We will denote the set of left translations by $\Lambda(S)$ and right translations by $\Gamma(S)$. The left and the right translations will be assumed to be linked.
We let $\Gamma_S: a \mapsto \lambda_a$, and $\Gamma(S) = \{\lambda_a: a \in S\}$. We also let $\Delta_S: a \mapsto \rho_a$ and $\Delta(S) = \{\rho_a: a \in S\}$.
Let us consider the following theorem.
Theorem 2.1: Given a type$A$ monoid S, there are inverse semigroups S_1, S_2, and embeddings $\phi_1: S \to S_1$, $\phi_2: S \to S_2$, such that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a)$, $\phi_2 a^\dagger = (\phi_2 a)^\dagger = (\phi_1 a)(\phi_1 a)^{-1}$, and there are also embeddings $\psi_1: \Lambda(S) \to \Lambda(S_1)$, $\psi_2: P(S) \to P(S_2)$ such that each of the diagrams



(i)



(ii)

commutes and $\psi_1(\lambda^*) = [\psi_1(\lambda)]^* = [\psi_1(\lambda)]^{-1}\psi_1(\lambda)$, $\psi_2(\rho^\dagger) = [\psi_2(\rho)]^\dagger = \psi_2(\rho)[\psi_2(\rho)]^{-1}$.

The theorem is proved through the following propositions and corollaries. Diagram (i) is dual to diagram (ii) and therefore every fact established about diagram (i) applies in dual manner to diagram (ii).

Proposition 2.2: Given a type$A$ monoid S, there are inverse semigroups S_1, S_2, and embeddings $\phi_1: S \to S_1$, $\phi_2: S \to S_2$, such that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a)$, $\phi_2 a^\dagger = (\phi_2 a)^\dagger = (\phi_2 a)(\phi_2 a)^{-1}$.
Proof:
For each $a \in S$, $(a^*, a) \in \mathcal{L}^*$ since S is abundant.
Define a map $\eta_a: a^*S \to aS$ defined by $\eta_a(a^*s) = as$, $s \in S$.
Let $a^*s' = a^*s''$ $s', s'' \in S$.

$\eta_a(a^*s') = as' = aa^*s'$ $[a^*$ is a right identity to $a]$
$= aa^*s'' = as'' = \eta_a(a^*s'')$. Thus, η_a is well defined.

Let $\eta_a(a^*s_1) = \eta_a(a^*s_2)$, $s_1, s_2 \in S$. This implies that $as_1 = as_2$. Since $(a^*, a) \in \mathcal{L}^*$, $as_1 = as_2$ implies that $a^*s_1 = a^*s_2$. Thus, η_a is one – one and therefore, a member of the *symmetric inverse semigroup* $\mathfrak{T}_S$ on S.
So, $\mathfrak{T}_S$ becomes the S_1.
η_a is as well surjective since $\forall as \in aS$, $as = aa^*s = \eta_a(a^*s)$, which implies that every element $as \in aS$ has a pre – image $a^*s \in a^*S$.
Thus, η_a is a bijection.
Hence, $\forall a \in S$, there is a bijection $\eta_a: a^*S \to aS$ defined by $\eta_a(a^*s) = as$, $(s \in S)$ which maps a^* to a.

Now, define a map $\phi_1: S \to S_1$ by $\phi_1(a) = \eta_a$, $(a \in S)$
For $a, b \in S$, the domain of $\eta_b\eta_a$ is $\eta_a^{-1}(b^*S \cap aS)$

$\eta_a^{-1}(b^*S \cap aS) = \eta_a^{-1}(b^*aS)$ [since for a type A semigroup S, $eS \cap aS = eaS$, $(\forall a \in S)(\forall e \in E)$].
$= \eta_a^{-1}[a(b^*a)^*S] = \eta_a^{-1}\eta_a[(b^*a)^*S] = (b^*a)^*S$.

Now, since $ba\mathcal{L}^*(ba)^* = (b^*a)^*$, $dom(\eta_{ba}) = (b^*a)^*S = dom(\eta_b\eta_a)$.
For $(b^*a)^*s \in (b^*a)^*S = dom(\eta_{ba})$

$\eta_{ba}[(b^*a)^*s] = bas = bb^*as = \eta_b b^*as$
$= \eta_b a(b^*a)^*s$ since S is a type A semigroup
$= \eta_b\eta_a[(b^*a)^*s]$

Thus, $\eta_{ba} = \eta_b\eta_a$ $\forall a, b \in S$.
Hence, $\forall a, b \in S$, $\phi_1 ba = \eta_{ba} = \eta_b\eta_a = \phi_1 b\phi_1 a$.
Therefore, ϕ_1 is a homomorphism.

Let $\phi_1 a = \phi_1 b$. This implies that $\eta_a = \eta_b$. That is, $dom\,\eta_a = dom\,\eta_b = eS$ (say).
Then, $a = \eta_a(e) = \eta_b(e) = b$.
Thus, ϕ_1 is one – one and hence an embedding.
We show that $\phi_1 a^* = (\phi_1 a)^*$.
Let $a, x, y \in S$. $[\phi_1 a \phi_1 x = \phi_1 a \phi_1 y] \Leftrightarrow [\phi_1(ax) = \phi_1(ay)]$ since ϕ_1 is a homomorphism
$\Leftrightarrow [ax = ay]$ since ϕ_1 is one – one
$\Leftrightarrow [a^* x = a^* y]$ since $(a^*, a) \in \mathcal{L}^*$
$\Leftrightarrow [\phi_1(a^* x) = \phi_1(a^* y)] \Leftrightarrow [\phi_1(a^*)\phi_1(x) = \phi_1(a^*)\phi_1(y)]$
$\Leftrightarrow \phi_1(a)\mathcal{L}^*\phi_1(a^*)$.
Since ϕ_1 is a homomorphism, $\phi_1(a^*)$ is an idempotent in $\mathfrak{T}_S$ and since idempotent in $\mathcal{L}_{\phi_1 a}$ must be unique, $\phi_1(a^*) = (\phi_1 a)^*$.

We show next that if $(a, b) \in \mathcal{L}^*(S)$, then $[\phi_1(a), \phi_1(b)] \in \mathcal{L}^*[\phi_1(S)]$.
Let $(a, b) \in \mathcal{L}^*(S)$. This implies that $a^* = b^*$, and therefore $\phi_1 a^* = \phi_1 b^*$.
$\phi_1(a)\mathcal{L}^*(\phi_1 a)^* = \phi_1 a^*$ and $\phi_1 a^* = \phi_1 b^* \mathcal{L}^* \phi_1 b$.
Hence, $\phi_1(a)\mathcal{L}^*\phi_1(b)$ by transitivity of $\mathcal{L}^*$.

Now, we establish that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a)$, $(\forall a \in S)$
Since S_1 is regular, $\mathcal{L}(S_1) = \mathcal{L}^*(S_1)$.
So that for $a \in S$, $\phi_1 a \in S_1$, $(\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a) \in \mathcal{L}_{\phi_1 a}$ since the idempotent must be unique.

Therefore given a type$A$ monoid S, there is an inverse semigroup S_1 and an embedding $\phi_1: S \to S_1$ such that $\phi_1 a^* = (\phi_1 a)^* = (\phi_1 a)^{-1}(\phi_1 a)$.

$\mathcal{R}^*$ is dual to $\mathcal{L}^*$. Thus, given a type$A$ monoid S, there is an inverse semigroup S_2 and an embedding $\phi_2: S \to S_2$ such that $\phi_2 a^\dagger = (\phi_2 a)^\dagger = (\phi_2 a)(\phi_2 a)^{-1}$.

Corollary 2.3: If $(a, b) \in \mathcal{R}^*(S)$, then $[\phi_1(a), \phi_1(b)] \in \mathcal{R}^*[\phi_1(S)]$

Definition: Let H be a subset of a semigroup S. The *upper saturation* $H\omega$ of H in S is defined by: $H\omega = \{s \in S : (\exists h \in H)\ h \le s\}$.

Proposition 2.4: ϕ_1 preserves subsemigroups and upper saturations
Proof:
Let H be a subsemigroup of $(S,\cdot)$ and $a, b \in H$. Then $a \cdot b \in H$, and since $\phi_1(a), \phi_1(b) \in \phi_1(H)$, then with $\phi_1(a \cdot b) = \phi_1 a \cdot \phi_1 b \in \phi_1 H$, $\phi_1 H$ is a subsemigroup of S_1.
Let $H\omega$ be the upper saturation of H and assuming $s \in H\omega$. This implies that $\exists h \in H$ such that $h = se$ $[e \in E(S)]$. Therefore $\phi_1 h = \phi_1 se = \phi_1 s \phi_1 e$. $\phi_1 h \in \phi_1 H$, consequently $\phi_1 h \le \phi_1 s$. Thus, $\phi_1 s \in \phi_1 H\omega$ and $\phi_1 H\omega$ is the upper saturation of $\phi_1 H$ in $\phi_1 S$.

Lemma 2.5: For an inverse semigroup S_1, $\Gamma: a \mapsto \lambda_a$ is an isomorphism from S_1 onto $\Gamma(S_1)$.
Proof:
For $a, b \in S_1$, $\Gamma(ab) = \lambda_{ab} = \lambda_a \lambda_b = \Gamma(a)\Gamma(b)$. Therefore, Γ is a homomorphism.
Let $\lambda_a = \lambda_b$.
$a = aa^{-1}a = \lambda_a a^{-1} a = \lambda_b a^{-1} a = ba^{-1}a \le b$.
Similarly, $b \le a$, and therefore $a = b$.
Thus, Γ is injective. It is also an onto map since $\forall \lambda_a \in \Gamma(S_1)$, $\exists a \in S$ with $a \mapsto \lambda_a$.

Lemma 2.6: For a type$A$ semigroup S, $\Gamma: a \mapsto \lambda_a$ is an isomorphism from S onto $\Gamma(S)$.
For $a, b \in S_1$, $\Gamma(ab) = \lambda_{ab} = \lambda_a \lambda_b = \Gamma(a)\Gamma(b)$. Therefore, Γ is a homomorphism.
$a = aa^* = \lambda_a a^* = \lambda_b a^* = ba^* \le b$. Similarly, $b \le a$, and therefore $a = b$.
Thus, Γ is injective. It is also an onto map since $\forall \lambda_a \in \Gamma(S)$, $\exists a \in S$ with $a \mapsto \lambda_a$.

Corollary 2.7: For an inverse semigroup S_1, $\Delta_{S_1}: a \mapsto \rho_a$ is an isomorphism from S_1 onto $\Delta(S_1)$. Similarly, for a type$A$ semigroup S, $\Delta_S: a \mapsto \rho_a$ is an isomorphism from S onto $\Delta(S)$.

Proposition 2.8: Given a type$A$ monoid S, there are inverse semigroups S_1, S_2, and embeddings $\psi_1: \Lambda(S) \to \Lambda(S_1)$, $\psi_2: P(S) \to P(S_2)$ such that $\psi_1(\lambda^*) = [\psi_1(\lambda)]^* = [\psi_1(\lambda)]^{-1}\psi_1(\lambda)$, $\psi_2(\rho^\dagger) = [\psi_2(\rho)]^\dagger = \psi_2(\rho)[\psi_2(\rho)]^{-1}$.

Proof:

Let us denote the symmetric inverse semigroup on $\Lambda(S)$ by $\mathfrak{T}_{\Lambda(S)}$. For each $\lambda \in \Lambda(S)$, we define a map θ_λ: $\lambda^*\Lambda(S) \to \lambda\Lambda(S)$ by $\theta_\lambda(\lambda^*\lambda_1) = \lambda\lambda_1$, $\lambda_1 \in \Lambda(S)$.

We show that θ_λ is one-one.
Let $\theta_\lambda(\lambda^*\lambda_1) = \theta_\lambda(\lambda^*\lambda_2)$, $\lambda_1, \lambda_2 \in \Lambda(S)$. This implies that $\lambda\lambda_1 = \lambda\lambda_2$.
Since $\lambda\mathcal{L}^*\lambda^*$, $\lambda\lambda_1 = \lambda\lambda_2 \Leftrightarrow \lambda^*\lambda_1 = \lambda^*\lambda_2$.
Thus, θ_λ is one-one and $\theta_\lambda \in \mathfrak{T}_{\Lambda(S)}$. So that we take $\Lambda(S_1)$ to be $\mathfrak{T}_{\Lambda(S)}$

Evidently, θ_λ is surjective since $\forall\lambda\lambda_1 \in \lambda\Lambda(S)$, $\lambda\lambda_1 = \lambda\lambda^*\lambda_1 = \theta_\lambda\lambda^*\lambda_1$. Implying that every $\lambda\lambda_1 \in \lambda\Lambda(S)$ has a pre-image $\lambda^*\lambda_1$ in $\lambda^*\Lambda(S)$.

Hence, $\forall\lambda \in \Lambda(S)$, there is a bijection θ_λ: $\lambda^*\Lambda(S) \to \lambda\Lambda(S)$ defined by $\theta_\lambda(\lambda^*\lambda_1) = \lambda\lambda_1$, $\lambda_1 \in \Lambda(S)$, which maps λ^* to λ.

Now, define the map $\psi_1: \Lambda(S) \to \Lambda(S_1)$ by $\psi_1(\lambda) = \theta_\lambda$.
We show that ψ_1 is a homomorphism.
For $\lambda, l \in \Lambda(S)$, the domain of $\theta_l\theta_\lambda$ is $\theta_\lambda^{-1}[\lambda\Lambda(S) \cap l^*\Lambda(S)]$
This implies that $dom\ \theta_l\theta_\lambda = \theta_\lambda^{-1}[l^*\lambda\Lambda(S)]$ since for a type A semigroup , $eS \cap aS = eaS$, $(\forall a \in S)$ $(\exists e \in E)$
$\theta_\lambda^{-1}[l^*\lambda\Lambda(S)] = \theta_\lambda^{-1}[\lambda(l^*\lambda)^*\Lambda(S)] = \theta_\lambda^{-1}\theta_\lambda[(l^*\lambda)^*\Lambda(S)] = (l^*\lambda)^*\Lambda(S)$
Since $l\lambda\mathcal{L}^*(l\lambda)^* = (l^*\lambda)^*$, $dom\ \theta_{l\lambda} = (l^*\lambda)^*\Lambda(S)$.
Thus, $dom\ \theta_l\theta_\lambda = dom\ \theta_{l\lambda}$.
Moreover, for $(l^*\lambda)^*\lambda_1 \in dom\ \theta_{l\lambda}$
$\theta_{l\lambda}[(l^*\lambda)^*\lambda_1] = l\lambda\lambda_1 = ll^*\lambda\lambda_1 = \theta_l[l^*\lambda\lambda_1] = \theta_l[\lambda(l^*\lambda)^*\lambda_1] = \theta_l\theta_\lambda[(l^*\lambda)^*\lambda_1]$
Hence, $\theta_l\theta_\lambda = \theta_{l\lambda}$.
Therefore, $\forall\lambda, l \in \Lambda(S)$, $\psi_1(l\lambda) = \theta_{l\lambda} = \theta_l\theta_\lambda = \psi_1(l)\psi_1(\lambda)$
Thus, ψ_1 is a homomorphism.

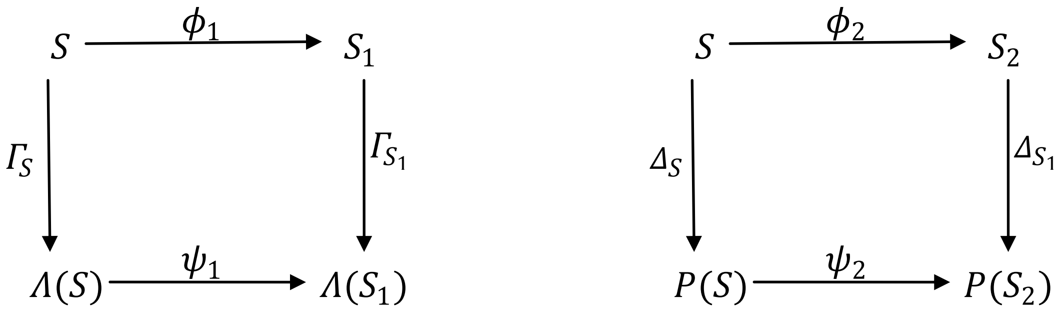
Let $\psi_1(l) = \psi_1(\lambda)$. Then, $\theta_l = \theta_\lambda$. That is, $dom\ \theta_l = dom\ \theta_\lambda = \lambda'\Lambda(S)$ (say).
Therefore, $\lambda = \theta_\lambda(\lambda') = \theta_l(\lambda') = l$.
Thus, ψ_1 is injective and hence an embedding.

$\mathcal{L}(\Lambda(S_1)) = \mathcal{L}^*(\Lambda(S_1))$ since $\Lambda(S_1)$ is regular. Therefore, for each $\psi_1(\lambda) \in \Lambda(S_1)$, $[\psi_1(\lambda)]^* = [\psi_1(\lambda)]^{-1}\psi_1(\lambda)$ $\in \mathcal{L}_{\psi_1(\lambda)} = \mathcal{L}^*{}_{\psi_1(\lambda)}$ since the idempotents are unique.
Now, we just need to show that $\psi_1(\lambda^*) = [\psi_1(\lambda)]^*$
For $\lambda, \lambda', \lambda'' \in \Lambda(S)$, let $\psi_1(\lambda)\psi_1(\lambda') = \psi_1(\lambda)\psi_1(\lambda'') \Leftrightarrow \psi_1(\lambda\lambda') = \psi_1(\lambda\lambda'') \Leftrightarrow \lambda\lambda' = \lambda\lambda''$

$$\Leftrightarrow \lambda^*\lambda' = \lambda^*\lambda'' \Leftrightarrow \psi_1(\lambda^*)\psi_1(\lambda') = \psi_1(\lambda^*)\psi_1(\lambda'')$$

So that $\psi_1(\lambda)\mathcal{L}^*\psi_1(\lambda^*)$.
Since ψ_1 is a homomorphism and λ^* an idempotent in $\Lambda(S)$, $\psi_1(\lambda^*)$ is an idempotent in $\Lambda(S_1)$ and since idempotent in $\mathcal{L}_{\psi_1(\lambda)}$ must be unique, $\psi_1(\lambda^*) = [\psi_1(\lambda)]^*$.
By dual argument, it follows that $\psi_2: P(S) \to P(S_2)$ is an embedding such that $\psi_2(\rho^\dagger) = [\psi_2(\rho)]^\dagger = \psi_2(\rho)[\psi_2(\rho)]^{-1}$

Proposition 2.9: Each of the diagrams



(i) (ii)

commutes.

Proof:

We defined the map $\phi_1: S \to S_1$ by $\phi_1(a) = \eta_a$. The rest are –

$\Gamma_S: a \to \lambda_a$, $\Gamma_{S_1}: \eta_a \to \lambda_{\eta_a}$ and $\psi_1: \lambda_a \to \theta_{\lambda_a}$

Thus, $\psi_1 \Gamma_S(a) = \theta_{\lambda_a}$ and $\Gamma_{S_1}\phi_1(a) = \lambda_{\eta_a}$

So, for any $x \in S$, $\theta_{\lambda_a}(x) = \lambda_a(x) = ax$

and $\lambda_{\eta_a}(x) = ax = \lambda_a(x) = \theta_{\lambda_a}(x)$

Therefore, $\psi_1 \Gamma_S = \Gamma_{S_1}\phi_1$. Hence, diagram (i) commutes and dually, diagram (ii) commutes. Here is the end of the proof of theorem 2.1

Proposition 2.10: $\psi_{1\circ}\psi_1^{-1}$ is an idempotent-separating congruence on $\Gamma(S)$

Proof:

$\psi_{1\circ}\psi_1^{-1} = \{(\lambda_a, \lambda_b) \in \Gamma(S) \times \Gamma(S): \theta_{\lambda_a} = \theta_{\lambda_b}\}$

$\psi_{1\circ}\psi_1^{-1}$ is clearly an equivalence.

Now, let $(\lambda_a, \lambda_b), (\lambda'_a, \lambda'_b) \in \psi_{1\circ}\psi_1^{-1}$

This implies that $\theta_{\lambda_a} = \theta_{\lambda_b}$, $\theta_{\lambda'_a} = \theta_{\lambda'_b}$

$$\theta_{\lambda_a\lambda'_a} = \psi_1(\lambda_a\lambda'_a) = \psi_1(\lambda_a)\psi_1(\lambda'_a) = \theta_{\lambda_a}\theta_{\lambda'_a} = \theta_{\lambda_b}\theta_{\lambda'_b} = \psi_1(\lambda_b)\psi_1(\lambda'_b)$$
$$= \psi_1(\lambda_b\lambda'_b) = \theta_{\lambda_b\lambda'_b}$$

$\theta_{\lambda_a}\theta_{\lambda'_a} = \psi_1(\lambda_a)\psi_1(\lambda'_a) = \psi_1(\lambda_a\lambda'_a) = \theta_{\lambda_a\lambda'_a}$

and

$\theta_{\lambda_b}\theta_{\lambda'_b} = \psi_1(\lambda_b)\psi_1(\lambda'_b) = \psi_1(\lambda_b\lambda'_b) = \theta_{\lambda_b\lambda'_b}$

$\theta_{\lambda_a\lambda'_a} = \theta_{\lambda_b\lambda'_b}$ since $\theta_{\lambda_a} = \theta_{\lambda_b}$, $\theta_{\lambda'_a} = \theta_{\lambda'_b}$

Therefore, $\psi_{1\circ}\psi_1^{-1}$ is a congruence.

Further, since ψ_1 and Γ_S are one-one, $\theta_{\lambda_a} = \theta_{\lambda_b} \Rightarrow a = b$. So that

$$\psi_{1\circ}\psi_1^{-1} = \{(\lambda_a, \lambda_b) \in \Gamma(S) \times \Gamma(S): a = b\}$$

Now, let $(\lambda_e, \lambda_f) \in \psi_{1\circ}\psi_1^{-1} \cap E_{\Lambda(S)} \times E_{\Lambda(S)}$. Then, $e = f$ and therefore $\lambda_e = \lambda_f$

Thus, $\psi_{1\circ}\psi_1^{-1}$ is idempotent-seperating.

ψ_1 is equally idempotent-seperating since if $\lambda_e, \lambda_f \in E_{\Gamma(S)}$ such that $\psi_1(\lambda_e) = \psi_1(\lambda_f)$.

Then, $\theta_{\lambda_e} = \theta_{\lambda_f} \Rightarrow e = f \Rightarrow \lambda_e = \lambda_f$.

A semigroup homomorphism $\rho: S \to T$ is said to be a $good$ homomorphism if for all $a, b \in S$, $a\, L^*(S) b$ implies $a\rho\, L^*(T) b\rho$ and that $a\, R^*(S) b$ implies $a\rho\, R^*(T) b\rho$.

Proposition 2.11: ψ_1 is a good homomorphism

Proof:

Suppose $\lambda, \lambdabar \in \Gamma(S)$ such that $(\lambda, \lambdabar) \in \mathcal{L}^*[\Gamma(S)]$. Then $\lambda \mathcal{L}^* \lambda^* \mathcal{L}^* \lambdabar$, which implies that the maps $\theta_\lambda: \lambda^* \Lambda(S) \to \lambda\Lambda(S)$ and $\theta_{\lambdabar}: \lambda^*\Lambda(S) \to \lambdabar\Lambda(S)$ have the same domain.

We note that θ_λ and $\theta_{\lambdabar}$ are elements of the symmetric inverse semigroup $\mathfrak{I}_{\Lambda(S)}$ on $\Lambda(S)$. The compositions $\theta_\lambda^{-1}\theta_\lambda$ and $\theta_{\lambdabar}^{-1}\theta_{\lambdabar}$ are identity maps on $\lambda^*\Lambda(S)$

We note that $\theta_\lambda^{-1}\theta_\lambda$ is read as θ_λ then θ_λ^{-1}. $[\theta_\lambda^{-1}\theta_\lambda = 1_{dom\, \theta_\lambda}]$

Therefore, $\theta_{\lambda}{}^{-1}\theta_{\lambda} = 1_{\lambda^*\Lambda(S)} = \theta_{\bar{\lambda}}{}^{-1}\theta_{\bar{\lambda}}$
Thus, $\theta_{\lambda} = \theta_{\lambda}\theta_{\lambda}{}^{-1}\theta_{\lambda} = \theta_{\lambda}\theta_{\bar{\lambda}}{}^{-1}\theta_{\bar{\lambda}}$
And $\theta_{\bar{\lambda}} = \theta_{\bar{\lambda}}\theta_{\bar{\lambda}}{}^{-1}\theta_{\bar{\lambda}} = \theta_{\bar{\lambda}}\theta_{\lambda}{}^{-1}\theta_{\lambda}$
Therefore, $\theta_{\lambda}\mathcal{L}[\mathfrak{T}_{\Lambda(S)}]\theta_{\bar{\lambda}}$.
Since $\mathfrak{T}_{\Lambda(S)}$ is an oversemigroup of $\Gamma(S_1)$, $\theta_{\lambda}\mathcal{L}^*[\Gamma(S_1)]\theta_{\bar{\lambda}}$. That is $\psi_1(\lambda)\mathcal{L}^*[\Gamma(S_1)]\psi_1(\bar{\lambda})$.
Recall that a semigroup homomorphism $\psi: S \to T$ is called a $*$-homomorphism if for any elements $a, b \in S$, $a\mathcal{L}^*(S)b$ if and only if $\psi(a)\mathcal{L}^*(T)\psi(b)$ [and $a\mathcal{R}^*(S)b$ if and only if $\psi(a)\mathcal{R}^*(T)\psi(b)$]

A semigroup homomorphism $\varphi: S \to T$ is said to be a *-*homomorphism* if for all $a, b \in S$, $a\, L^*(S)b$ if and only if $a\varphi\, L^*(T)b\varphi$ and $a\, R^*(S)b$ if and only if $a\varphi\, R^*(T)b\varphi$.

Proposition 2.12: $\psi_1: \Gamma(S) \to \Gamma(S_1)$ is a $*$-homomorphism

Proof:

One half of this proof has been answered by the fact that ψ_1 is good. That is $\lambda\mathcal{L}^*[\Gamma(S)]\bar{\lambda} \Rightarrow \psi_1(\lambda)\mathcal{L}^*[\Gamma(S_1)]\psi_1(\bar{\lambda})$ $[\forall \lambda, \bar{\lambda} \in \Gamma(S)]$
Conversely, let $\psi_1(\lambda)\mathcal{L}^*[\Gamma(S_1)]\psi_1(\bar{\lambda})$
We wish to show that $\lambda\mathcal{L}^*[\Gamma(S)]\bar{\lambda}$.
We know that for $\lambda, \bar{\lambda} \in \Gamma(S)$, $\lambda^*\mathcal{L}^*[\Gamma(S)]\lambda$ and $\bar{\lambda}^*\mathcal{L}^*[\Gamma(S)]\bar{\lambda}$, and since ψ_1 is good, we have
$\psi_1(\lambda^*)\mathcal{L}^*[\Gamma(S_1)]\psi_1(\lambda)$ and $\psi_1(\bar{\lambda}^*)\mathcal{L}^*[\Gamma(S_1)]\psi_1(\bar{\lambda})$
So that our assumption – $\psi_1(\lambda)\mathcal{L}^*[\Gamma(S_1)]\psi_1(\bar{\lambda})$ – yields $\psi_1(\lambda^*)\mathcal{L}^*[\Gamma(S_1)]\psi_1(\bar{\lambda}^*)$ by transitivity.
$\psi_1(\lambda^*)$ and $\psi_1(\bar{\lambda}^*)$ are idempotents and are right identities to each other by their $\mathcal{L}^*$-relationship.
So, we have $\psi_1(\lambda^*)\psi_1(\bar{\lambda}^*) = \psi_1(\lambda^*)$ and $\psi_1(\bar{\lambda}^*)\psi_1(\lambda^*) = \psi_1(\bar{\lambda}^*)$
That is $\psi_1(\lambda^*\bar{\lambda}^*) = \psi_1(\lambda^*)$ and $\psi_1(\bar{\lambda}^*\lambda^*) = \psi_1(\bar{\lambda}^*)$
$\lambda^*\bar{\lambda}^*$ and $\bar{\lambda}^*\lambda^*$ are idempotents and since ψ_1 is idempotent-seperating,
$\lambda^*\bar{\lambda}^* = \lambda^*$ and $\bar{\lambda}^*\lambda^* = \bar{\lambda}^*$ so that $\lambda^* = \bar{\lambda}^*$ implying that $\lambda^*\mathcal{L}^*[\Gamma(S)]\bar{\lambda}^*$
and therefore, $\lambda\mathcal{L}^*[\Gamma(S)]\lambda^*\mathcal{L}^*[\Gamma(S)]\bar{\lambda}^*\mathcal{L}^*[\Gamma(S)]\bar{\lambda}$
so that, $\lambda\mathcal{L}^*[\Gamma(S)]\bar{\lambda}$.

Proposition 2.13: If δ is a congruence on $\Gamma(S)$, then $\psi_1(\delta)$ is a congruence on $\Gamma(S_1)$

Proof:

Suppose δ is a left compatible relation on $\Gamma(S)$ and let $(\theta_{\lambda}, \theta_{\bar{\lambda}}) \in \psi_1(\delta)$ and $\theta_{\lambda'} \in \Gamma(S_1)$.
Since ψ_1 is a bijection, $\psi_1{}^{-1}(\theta_{\lambda}) = \lambda$. So that $(\lambda, \bar{\lambda}) \in \delta$ and $\lambda' \in \Gamma(S)$, and since δ is left compatible, $(\lambda'\lambda, \lambda'\bar{\lambda}) \in \delta$. So that

$$[\psi_1(\lambda'\lambda), \psi_1(\lambda'\bar{\lambda})] = [\psi_1(\lambda')\psi_1(\lambda), \psi_1(\lambda')\psi_1(\bar{\lambda})] = (\theta_{\lambda'}\theta_{\lambda},\ \theta_{\lambda'}\theta_{\bar{\lambda}}) \in \psi_1(\delta).$$

By similar argument, right compatibility follows.
Now, let δ be an equivalence relation on $\Gamma(S)$ and assuming $\theta_{\lambda} \in \Gamma(S_1)$. Then, again, $\psi_1{}^{-1}(\theta_{\lambda}) = \lambda \in \Gamma(S)$ and since δ is reflexive, $(\lambda, \lambda) \in \delta$.
Therefore, $\psi_1(\lambda, \lambda) = [\psi_1(\lambda), \psi_1(\lambda)] = (\theta_{\lambda}, \theta_{\lambda}) \in \psi_1(\delta)$. Hence, $\psi_1(\delta)$ is reflexive.
Suppose $(\theta_{\lambda}, \theta_{\bar{\lambda}}) \in \psi_1(\delta)$. Then, $\psi_1{}^{-1}(\theta_{\lambda}, \theta_{\bar{\lambda}}) = [\psi_1{}^{-1}(\theta_{\lambda}),\ \psi_1{}^{-1}(\theta_{\bar{\lambda}})] \in \delta$ and since δ is symmetric, $[\psi_1{}^{-1}(\theta_{\bar{\lambda}}),\ \psi_1{}^{-1}(\theta_{\lambda})] \in \delta$
Therefore, $(\theta_{\bar{\lambda}}, \theta_{\lambda}) \in \psi_1(\delta)$. Thus, $\psi_1(\delta)$ is symmetric.
Now, let $(\theta_{\lambda}, \theta_{\lambda'}) \in \psi_1(\delta)$ and $(\theta_{\lambda'}, \theta_{\bar{\lambda}}) \in \psi_1(\delta)$. This implies that $(\lambda, \lambda') \in \delta$ and $(\lambda', \bar{\lambda}) \in \delta$ and since δ is transitive, $(\lambda, \bar{\lambda}) \in \delta$. So that, $\psi_1(\lambda, \bar{\lambda}) = (\theta_{\lambda}, \theta_{\bar{\lambda}}) \in \psi_1(\delta)$ and therefore $\psi_1(\delta)$ is transitive.

Definition:

- An ideal F of a semilattice E is called a P-*ideal* if the intersection of F with any other principal ideal of the semigroup is a principal ideal.
- A semigroup homomorphism $\psi: S \to T$ is called a P-*homomorphism* if $\langle\psi(E_S)\rangle$ is a P-ideal of E_T.

Theorem 2.14: If ϕ_1 and Γ_{S_1} are P-homomorphisms, then so is the composition $\psi_1\Gamma_S$.

Proof:

Note: To avoid use of too many subscripts in this proof, we replace ϕ_1 with ϕ.
Now, suppose $\theta_e \in E_{\Lambda(S_1)}$, with the assumption that ϕ and Γ_{S_1} are P-homomorphisms. We wish to show that $\langle\Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e$ is a principal ideal of $E_{\Lambda(S_1)}$.
With $\theta_e \in E_{\Lambda(S_1)}$, and since Γ_{S_1} is a P-homomorphism, $\exists \theta_{e'} \in E_{\Lambda(S_1)}$ such that

$$\langle \Gamma_{S_1}(E_{S_1})\rangle \cap E_{\Lambda(S_1)}\theta_e = E_{\Lambda(S_1)}\theta_{e'} \text{ ....(i)}$$

Suppose also $\phi(f) \in E_{S_1}$ such that $\theta_{e'} \leq \Gamma_{S_1}\phi(f)$.
Then since ϕ is a P-homomorphism, $\exists \phi(f') \in E_{S_1}$ such that

$$\langle \phi(E_S)\rangle \cap E_{S_1}\phi(f) = E_{S_1}\phi(f') \text{.................................(ii)}$$

Suppose again that $g \in E_S$ such that $\phi(f') \leq \phi(g)$.
$\Gamma_{S_1}\phi(g) \in E_{\Lambda(S_1)}$, therefore $\theta_e\Gamma_{S_1}\phi(g) = \Gamma_{S_1}\phi(g)\theta_e \in E_{\Lambda(S_1)}\theta_e$
and $\theta_e\Gamma_{S_1}\phi(g) \in \langle \Gamma_{S_1}\phi(E_S)\rangle$
Thus, $\theta_e\Gamma_{S_1}\phi(g) \in \langle \Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e$
We note here that $E_{\Lambda(S_1)}\theta_e\Gamma_{S_1}\phi(g)$ is a principal ideal of $E_{\Lambda(S_1)}$
Now, we wish to show that

$$\langle \Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e = E_{\Lambda(S_1)}\theta_e\Gamma_{S_1}\phi(g)$$

Assuming $\theta_n \in E_{\Lambda(S_1)}\theta_e\Gamma_{S_1}\phi(g)$
Then, $\theta_n \leq \theta_e\Gamma_{S_1}\phi(g) \in \langle \Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e$
Therefore, $E_{\Lambda(S_1)}\theta_e\Gamma_{S_1}\phi(g) \subseteq \langle \Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e$
Conversely,
$\forall\theta_m \in \langle \Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e$, we are sure $\theta_m \in E_{\Lambda(S_1)}\theta_e$
So that $\theta_m \leq \theta_e$ ....(iii)
and we are sure $\theta_m \in \langle \Gamma_{S_1}\phi(E_S)\rangle$
So that with $g' \in E_S$, $\theta_m \in \Gamma_{S_1}\phi(g')$ ....(iv)
And therefore,
$\theta_m \in \langle \Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e \subseteq \langle \Gamma_{S_1}(E_{S_1})\rangle \cap E_{\Lambda(S_1)}\theta_e = E_{\Lambda(S_1)}\theta_{e'}$ [from (i)]
Thus, $\theta_m \leq \theta_{e'} \leq \Gamma_{S_1}\phi(f)$.................. (v)
And from (iv) and (v), and since θ_m is an idempotent
$\theta_m.\theta_m \leq \Gamma_{S_1}\phi(f).\Gamma_{S_1}\phi(g')$
That is $\theta_m \leq \Gamma_{S_1}\phi(f)\phi(g')$ .... (vi)
$\phi(f)\phi(g') = \phi(g')\phi(f) \in E_{S_1}\phi(f)$ and $\phi(f)\phi(g') \leq \phi(g') \in \langle \phi(E_S)\rangle$
So that $\phi(f)\phi(g') \in \langle \phi(E_S)\rangle \cap E_{S_1}\phi(f) = E_{S_1}\phi(f')$ [from(ii)]
Therefore, $\phi(f)\phi(g') \leq \phi(f) \leq \phi(g)$ [from the definition of g above]
$\theta_m \leq \Gamma_{S_1}\phi(f)\phi(g') \leq \Gamma_{S_1}\phi(g)$ and taking this to (iii), we have
$\theta_m = \theta_m.\theta_m \leq \theta_e\Gamma_{S_1}\phi(g)$
Therefore, $\theta_m \in E_{S_1}\theta_e\Gamma_{S_1}\phi(g)$
Hence, $\langle \Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e \subseteq E_{\Lambda(S_1)}\theta_e\Gamma_{S_1}\phi(g)$
So that $\langle \Gamma_{S_1}\phi(E_S)\rangle \cap E_{\Lambda(S_1)}\theta_e = E_{\Lambda(S_1)}\theta_e\Gamma_{S_1}\phi(g)$
Now, we have shown that the intersection of $\langle \Gamma_{S_1}\phi(E_S)\rangle$ with any principal ideal $E_{\Lambda(S_1)}\theta_e$ of $E_{\Lambda(S_1)}$ is a principal ideal. Thus, $\langle \Gamma_{S_1}\phi(E_S)\rangle$ is a principal ideal of $E_{\Lambda(S_1)}$ and therefore the composition $\Gamma_{S_1}\phi$ is a P-homomorphism. Finally, commutativity of our diagram guarantees that $\psi_1\Gamma_S$ is equally a P-homomorphism.

III. CONCLUSION

In this piece of work, we looked into monoid embedding of type A semigroup into an inverse semigroup obtained by Fountain (1979). We extend this result to translational hulls and, in effect, gave a faithful representation of translational hull of type A semigroup. We showed that the embedding is structure – preserving. A couple of accompanying results were also obtained.

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