Eulerian Integral Associated with Product of Two Multivariable Gimel-Functions, and Special Functions of Several Variables II

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ABSTRACT
Recently, Raina and Srivastava [6] and Srivastava and Hussain [11] have provided closed-form expressions for a number of a Eulerian integral about the multivariable H-functions. The present paper is evaluated a new Eulerian integral associated with the product of two multivariable Gimel-functions defined by Ayant [2], a generalized Lauricella function, a multivariable I-function defined by Prasad and multivariable Aleph-function defined by Ayant [1] with general arguments . Finally we shall give few remarks.

Keywords: Eulerian integral, multivariable Gimel-function, generalized Lauricella function of several variables, expansion of multivariable I-function expansion of multivariable aleph-function, generalized hypergeometric function, class of polynomials.

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1. Introduction

The well-known Eulerian Beta integral

$$\int_a^b (z-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(Re(\alpha)>0, Re(\beta)>0, b>a) \tag{1.1}$$

is a basic result for evaluation of numerous other potentially useful integrals involving various special functions and polynomials. The authors Raina and Srivastava [6], Saigo and Saxena [7], Srivastava and Hussain [11], Srivastava and Garg [10] and other have studied the Eulerian integral. In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable Gimel-functions defined by Ayant [2], the expansion of special functions of several variables with general arguments. These functions is an extension of the multivariable H-function defined by Srivastava and Panda [13,14].

We will use the contracted form about the multivariable aleph-fonction $\aleph(z_1,\cdots,z_u)$ by :

$$\aleph(z_1,\cdots,z_u) = \aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\ ;\ \mathrm{C} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_u & \mathrm{B};\ \mathrm{D} \end{array}\right) = \frac{1}{(2\pi\omega)^u}\int_{L_1}\cdots\int_{L_u}\psi'(s_1,\cdots,s_u)\prod_{k=1}^{u}\theta'_k(s_k)z_k^{s_k}\mathrm{d}s_1\cdots\mathrm{d}s_u \tag{1.2}$$

with $\omega=\sqrt{-1}$

See Ayant [1], concerning the definition of the following quantities $V, W, \psi'(s_1,\cdots,s_r), A, B, C, D$ and $\theta_k(s_k)$.

The complex numbers z_i are not zero. Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.
We can obtain the series representation and behaviour for small values for the function $\aleph(z_1,\cdots,z_v)$ defined and represented by (1.6). The series representation may be given as follows, which is valid under the following conditions :

$$\aleph(z_1,\cdots,z_u) = \sum_{G_i=1}^{m_i}\sum_{g_i=1}^{\infty}\psi'_1\frac{\prod_{i=1}^{u}\theta'_i z_i'^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{u}g_i}}{\prod_{i=1}^{u}\epsilon^{(i)}_{G_i}g_i!} \tag{1.1}$$

where $\phi'_1 = \psi'(\eta_{G_1,g_1},\cdots,\eta_{G_u,g_u})$ and $\phi'_i = \theta'_i(\eta_{G_i,g_i})$ and $\eta_{G_i,g_i} = \dfrac{d^{(i)}_{G^{(i)}}+g_i}{\delta^{(i)}_{g^{(i)}}}, i=1,\cdots,u$ and

$\delta_i^{(h)}[d_i^{(j)}+u] \neq \delta_i^{(j)}[d_i^{(h)}+\mu]$ for $j\neq h, j,h=1,\cdots,M_i\,, u,\mu=0,1,2,\cdots$

The poles are simples

The multivariable I-function defined by Prasad [4] is an extension of the multivariable H-function defined by Srivastava and Panda [13,14]. We will use the contracted form.

The I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_v) = I^{0,n_2;0,n_3;\cdots;0,n_v:m^{(1)},n^{(1)};\cdots;m^{(v)},n^{(v)}}_{p_2,q_2,p_3,q_3;\cdots;p_v,q_v:p^{(1)},q^{(1)};\cdots;p^{(v)},q^{(v)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_v & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{vj};\alpha^{(1)}_{vj},\cdots,\alpha^{(u)}_{vj})_{1,p_v}:(a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(v)}_j,\alpha^{(v)}_j)_{1,p^{(v)}} \\ \\ (\mathrm{b}_{uj};\beta^{(1)}_{vj},\cdots,\beta^{(v)}_{vj})_{1,q_v}:(b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(v)}_j,\beta^{(v)}_j)_{1,q^{(v)}} \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^v}\int_{L'_1}\cdots\int_{L'_v}\psi''(s_1,\cdots,s_v)\prod_{i=1}^{v}\theta''_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_v \tag{1.2}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [4]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}}\alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}}\beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)} + \left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_v}\alpha_{vk}^{(i)} - \sum_{k=n_v+1}^{p_v}\alpha_{vk}^{(i)}\right) - \left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)} + \sum_{k=1}^{q_3}\beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_v}\beta_{vk}^{(i)}\right) \tag{1.3}$$

where $i=1,\cdots,v$

The complex numbers z_i are not zero. We can obtain the series representation and behaviour for small values for the function $I(z_1,\cdots,z_v)$ defined and represented by (1.3). The series representation may be given as follows ,which is valid under the following conditions :

$$I(z_1,\cdots,z_v) = \sum_{H_i=1}^{m^{(i)}}\sum_{h_i=1}^{\infty}\psi''_1\frac{\prod_{i=1}^v\theta''_i z_i^{\eta_{H_i,h_i}}(-)^{\sum_{i=1}^v h_i}}{\prod_{i=1}^v\epsilon^{(i)}_{H_i}h_i!} \tag{1.4}$$

where $\beta_i^{(h)}[b_i^{(j)}+v] \neq \beta_i^{(j)}[b_i^{(h)}+\mu]$ for $j \neq h, j,h = 1,\cdots,M_i\,, v,\mu = 0,1,2,\cdots$ and $z_i \neq 0$

$\phi_1'' = \psi_1''(\eta_{H_1,h_1},\cdots,\eta_{H_u;h_u})$ and $\phi_i' = \theta_i'(\eta_{H_i,h_i})$ and $\eta_{H_i,h_i} = \dfrac{b_{H^{(i)}}^{(i)}+\beta_i}{b_{h^{(i)}}^{(i)}}, i = 1,\cdots,v$, the poles are simple

We consider a generalized transcendental function called Gimel function of several complex variables. We use the contracted form.

$$\gimel(z_1,\cdots,z_r) = \gimel_{X;p_{i_r},q_{i_r},\tau_{i_r}:R_r:Y}^{U;0,n_r:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\mathbf{A}:\ A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B}:\ B \end{array}\right) = \frac{1}{(2\pi\omega)^r}\int_{L_1''}\cdots\int_{L_r''}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (1.5)$$

with $\omega = \sqrt{-1}$

The following quantities $\mathbb{A},\mathbf{A},A,\mathbb{B},\mathbf{B},B,X,Y,U,V,\psi(s_1,\cdots,s_r)$ and $\theta_k(s_k)(k=1,\cdots,r)$ are defined by Ayant [2].

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,)\,, max(\,|z_1|,\cdots,|z_r|\,) \to 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,)\,, min(\,|z_1|,\cdots,|z_r|\,) \to \infty$ where $i = 1,\cdots,r$:

$$\alpha_i = \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] \text{ and } \beta_i = \max_{1\leqslant j\leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$$

2 Required result

Lemma

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$\times F_D^{(k)}\left[\alpha,-\sigma_1,\cdots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right] \quad (2.1)$$

where $a,b \in \mathbb{R}(a<b), \alpha,\beta,f_i,g_i,\sigma_i \in \mathbb{C}, (i=1,\cdots,k)$; $min(Re(\alpha),Re(\beta)) > 0$ and

$$\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\} < 1$$

$F_D^{(k)}$ is a Lauricella's function of k-variables, see Srivastava and Manocha ([12], page 60)

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$ by means of the formula :

$$(1-z)^{-\alpha} = \sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}z^r(|z|<1) \quad (2.2)$$

integrating term by term with the help of the integral given by Saigo and Saxena [7, p. 93, eq.(3.2)] and applying the definition of the Lauricella function $F_D^{(k)}$ [9, p. 454].

3. Eulerian integral

We shall note in this section

$$U = 0, n_2; 0, n_3; \cdots ; 0, n_{r-1}; 0, n'_2; 0, n'_3; \cdots ; 0, n'_{s-1}; \underbrace{0, 0; \cdots ; 0, 0}_{l+k} \tag{3.1}$$

$$V = m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \cdots ; m^{(r)}, n^{(r)}; m'^{(1)}, n'^{(1)}; m'^{(2)}, n'^{(2)}; \cdots ; m'^{(s)}, n'^{(s)};$$

$$\underbrace{1, 0; \cdots ; 1, 0}_{k}, \underbrace{1, 0; \cdots ; 1, 0}_{l} \tag{3.2}$$

$$X = p_{i_2}, q_{i_2}, \tau_{i_2}; R_2; \cdots ; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}} : R_{r-1}; \underbrace{0; \cdots ; 0}_{s+l+k} \tag{3.3}$$

$$Y = p'_{i'^{(1)}}, q'_{i'^{(1)}}, \tau'_{i'^{(1)}}; R'^{(1)}; \cdots ; p'_{i^{(s)}}, q'_{i'^{(s)}}; \tau'_{i'^{(s)}}; R'^{(s)}; \underbrace{0, 1; \cdots ; 0, 1}_{l}, \underbrace{0, 1; \cdots ; 0, 1}_{k} \tag{3.4}$$

$$\mathbb{A} = [(a_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}; [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1,p_{i_3}}; \cdots ; [(\mathrm{a}_{(r-1)j}; \alpha_{(r-1)j}^{(1)}, \cdots , \alpha_{(r-1)j}^{(r-1)}; A_{(r-1)j})]_{1,n_{r-1}},$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}}; \alpha_{(r-1)ji_{r-1}}^{(1)}, \cdots , \alpha_{(s-1)ji_{r-1}}^{(r-1)}; A_{(r-1)ji_{r-1}})]_{n_{r-1}+1,p_{i_{r-1}}};$$

$$(a'_{2j}; \alpha_{2j}'^{(1)}, \alpha_{2j}'^{(2)}; A'_{2j})]_{1,n'_2}, [\tau'_{i'_2}(a'_{2ji'_2}; \alpha_{2ji'_2}'^{(1)}, \alpha_{2ji'_2}'^{(2)}; A'_{2ji'_2})]_{n'_2+1,p'_{i'_2}}; [(a'_{3j}; \alpha_{3j}'^{(1)}, \alpha_{3j}'^{(2)}, \alpha_{3j}'^{(3)}; A'_{3j})]_{1,n'_3},$$

$$[\tau'_{i'_3}(a'_{3ji'_3}; \alpha_{3ji'_3}'^{(1)}, \alpha_{3ji'_3}'^{(2)}, \alpha_{3ji'_3}'^{(3)}; A'_{3ji'_3})]_{n'_3+1,p'_{i'_3}}; \cdots ; [(\mathrm{a'}_{(s-1)j}; \alpha_{(s-1)j}'^{(1)}, \cdots , \alpha_{(s-1)j}'^{(s-1)}; A'_{(s-1)j})]_{1,n'_{s-1}},$$

$$,[\tau'_{i'_{s-1}}(a'_{(s-1)ji'_{s-1}}; \alpha_{(s-1)ji'_{s-1}}'^{(1)}, \cdots , \alpha_{(s-1)ji'_{s-1}}'^{(s-1)}; A'_{(s-1)ji'_{s-1}})]_{n'_{s-1}+1,p'_{i'_{s-1}}}; \tag{3.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj}; \alpha_{rj}^{(1)}, \cdots , \alpha_{rj}^{(r)}, \underbrace{0, \cdots , 0}_{s}, \underbrace{0, \cdots , 0}_{l}, \underbrace{0, \cdots , 0}_{k}; A_{rj})]_{1,n_r},$$

$$[\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \cdots , \alpha_{rji_r}^{(r)}, \underbrace{0, \cdots , 0}_{s}, \underbrace{0, \cdots , 0}_{l}, \underbrace{0, \cdots , 0}_{k}; A_{rji_r}]_{1,n_r},$$

$$[(\mathrm{a}'_{sj}; \underbrace{0, \cdots , 0}_{r}, \alpha_{sj}'^{(1)}, \cdots , \alpha_{sj}'^{(s)}, \underbrace{0, \cdots , 0}_{l}, \underbrace{0, \cdots , 0}_{k}; A'_{sj})]_{1,n'_s},$$

$$[\tau'_{i'_s}(a'_{sji'_s}; \underbrace{0, \cdots , 0}_{r}, \alpha_{sji'_s}'^{(1)}, \cdots , \alpha_{sji'_s}'^{(s)}, \underbrace{0, \cdots , 0}_{l}, \underbrace{0, \cdots , 0}_{k}; A'_{sji'_s})]_{n'_s+1,p'_{i'_s}} :$$

$$\mathrm{A} = [(\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})]_{1,n^{(1)}}, [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}; C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,p_i^{(1)}}; \cdots ;$$

$$[(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})]_{1,m^{(r)}}, [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}; C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}}$$

$$[(c_j^{\prime(s)},\gamma_j^{\prime(s)};C_j^{\prime(s)})]_{1,m^{\prime(s)}},[\tau'_{i^{\prime(s)}}(c_{ji^{\prime(s)}}^{\prime(s)},\gamma_{ji^{\prime(s)}}^{\prime(s)};C_{ji^{\prime(s)}}^{\prime(s)})]_{m^{\prime(s)}+1,p_i^{\prime(s)}},\underbrace{(1,0;1),\cdots,(1,0;1)}_{l},\underbrace{(1,0;1),\cdots,(1,0;1)}_{k} \quad (3.6)$$

$$A^*=(1-\alpha-\sum_{i=1}^{v}\eta_{H_i,h_i}a'_i-\sum_{i=1}^{u}\eta_{G_i,g_i}a_i;\mu_1,\cdots,\mu_r,\mu'_1,\cdots,\mu'_s,\underbrace{1,\cdots,1}_{k},\upsilon_1,\cdots,\upsilon_l;1),$$

$$(1-\beta-\sum_{i=1}^{v}\eta_{H_i,h_i}b'_i-\sum_{i=1}^{u}\eta_{G_i,g_i}b_i;\rho_1,\cdots,\rho_r,\rho'_1,\cdots,\rho'_s,\underbrace{0,\cdots,0}_{l},\tau_1,\cdots,\tau_l;1),,$$

$$[1+\sigma_j-\sum_{i=1}^{v}\eta_{H_i,h_i}\lambda_j^{\prime\prime(i)}-\sum_{i=1}^{u}\lambda_j^{\prime\prime\prime(i)}\eta_{G_i,g_i};\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j^{\prime(1)}\cdots,\lambda_s^{\prime(s)},\underbrace{0,\cdots,1\cdots,0}_{\substack{k\\ \mathrm{j}}},\zeta'_j,\cdots,\zeta_j^{(l)};1]_{1,k}$$

$$(1-A_j;\underbrace{0,\cdots,0}_{r},\underbrace{0,\cdots,0}_{s},\underbrace{0,\cdots,0}_{k},\underbrace{1,\cdots,1}_{l};1)_{1,P} \quad (3.7)$$

$$\mathbb{B}=[(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}};[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})]_{1,m_{r-1}},$$

$$;\ (b'_{2j};\beta_{2j}^{\prime(1)},\beta_{2j}^{\prime(2)};B'_{2j})]_{1,m'_2},[\tau'_{i'_2}(b'_{2ji'_2};\beta_{2ji'_2}^{\prime(1)},\beta_{2ji'_2}^{\prime(2)};B'_{2ji'_2})]_{m'_2+1,q'_{i'_2}};[(b'_{3j};\beta_{3j}^{\prime(1)},\beta_{3j}^{\prime(2)},\beta_{3j}^{\prime(3)};B'_{3j})]_{1,m'_3},$$

$$[\tau'_{i'_3}(b'_{3ji'_3};\beta_{3ji'_3}^{\prime(1)},\beta_{3ji'_3}^{\prime(2)},\beta_{3ji'_3}^{\prime(3)};b'_{3ji'_3})]_{m'_3+1,q'_{i'_3}};\cdots;[(\mathrm{b'}_{(s-1)j};\beta_{(s-1)j}^{\prime(1)},\cdots,\beta_{(s-1)j}^{\prime(s-1)};B'_{(s-1)j})]_{1,n'_{s-1}},$$

$$[\tau'_{i'_{s-1}}(b'_{(s-1)ji'_{s-1}};\beta_{(s-1)ji'_{s-1}}^{\prime(1)},\cdots,\beta_{(s-1)ji'_{s-1}}^{\prime(s-1)};B'_{(s-1)ji'_{s-1}})]_{m'_{s-1}+1,q'_{i'_{s-1}}}; \quad (3.8)$$

$$\mathbf{B}=[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)},\underbrace{0,\cdots,0}_{s},\underbrace{0,\cdots,0}_{l},\underbrace{0,\cdots,0}_{k};B_{rji_r}]_{m_r+1,q_r},$$

$$[\tau'_{i'_s}(b_{sji'_s};\underbrace{0,\cdots,0}_{r},\beta_{sji'_s}^{\prime(1)},\cdots,\beta_{sji'_s}^{\prime(s)},\underbrace{0,\cdots,0}_{l},\underbrace{0,\cdots,0}_{k};B'_{sji'_s})]_{m'_s+1,q'_{i'_s}}: \quad (3.9)$$

$$B^*=(1-\alpha-\beta-\sum_{i=1}^{v}\eta_{H_i,h_i}(a'_i+b'_i)-\sum_{i=1}^{u}(a_i+b_i)\eta_{G_i,g_i};\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu'_1+\rho'_1,\cdots,\mu'_r+\rho'_r,$$

$$\underbrace{1,\cdots,1}_{k},\upsilon_1+\tau_1,\cdots,\upsilon_l+\tau_l;1),\ \ [1-B_j;\underbrace{0,\cdots,0}_{r},\underbrace{0,\cdots,0}_{s},\underbrace{0,\cdots,0}_{k},\underbrace{1,\cdots,1}_{l};1]_{1,Q}$$

$$[1+\sigma_j-\sum_{i=1}^{u}\eta_{H_i,h_i}\lambda_j^{\prime\prime\prime(i)}-\sum_{i=1}^{v}\lambda_j^{\prime\prime(i)}\eta_{G_i,g_i};\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j^{\prime(1)}\cdots,\lambda_s^{\prime(s)},\underbrace{0\cdots,0}_{k},\zeta'_j,\cdots,\zeta_j^{(l)};1]_{1,k} \quad (3.10)$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,q_i^{(1)}};\cdots;$$

$$[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}},$$

$$[(\mathrm{d}'^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)]_{1,m'^{(1)}},[\tau'_{i'^{(1)}}(d'^{(1)}_{ji'^{(1)}},\delta'^{(1)}_{ji'^{(1)}};D'^{(1)}_{ji'^{(1)}})]_{m'^{(1)}+1,q'^{(1)}_i};\cdots;$$

$$[(d'^{(s)}_j,\delta'^{(s)}_j;D'^{(s)}_j)]_{1,m'^{(s)}},[\tau'_{i'^{(s)}}(d'^{(s)}_{ji'^{(s)}},\delta'^{(s)}_{ji'^{(s)}};D'^{(s)}_{ji'^{(s)}}]_{m'^{(s)}+1,q'^{(s)}_i},\underbrace{(0,1;1),\cdots,(0,1;1)}_{l},\underbrace{(0,1;1),\cdots,(0;1,1)}_{k} \tag{3.11}$$

$$P_1=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{h}(af_j+g_j)^{\sigma_j}\right\}\frac{\prod_{j=1}^{q}\Gamma(B_j)}{\prod_{j=1}^{p}\Gamma(A_j)} \tag{3.12}$$

$$B_{u,v}=(b-a)^{\sum_{i=1}^{v}(a_i+b_i)\eta_{G_i,g_i}+\sum_{i=1}^{u}\eta_{H_i,h_i}(a'_i+b'_i)}\left\{\prod_{j=1}^{k}(af_j+g_j)^{-\sum_{i=1}^{v}\lambda'''^{(i)}_j\eta_{G_i,g_i}-\sum_{i=1}^{u}\lambda''^{(i)}_j\eta_{H_i,h_i}}\right\} \tag{3.13}$$

We have the following Eulerian integral

Theorem

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\aleph\left(\begin{array}{c}\mathrm{x}_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{x}_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'''^{(u)}_j}\end{array}\right)$$

$$I\left(\begin{array}{c}\mathrm{z}''_1(t-a)^{a'_1}(b-t)^{b'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda''^{(1)}_j}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v(t-a)^{a'_v}(b-t)^{b'_v}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda''^{(v)}_j}\end{array}\right)$$

$$\beth\left(\begin{array}{c}\mathrm{z}_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(1)}_j}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda^{(r)}_j}\end{array}\right)$$

$$\beth\left(\begin{array}{c}\mathrm{z}'_1(t-a)^{\mu'_1}(b-t)^{\rho'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(1)}_j}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_s(t-a)^{\mu'_s}(b-t)^{\rho'_s}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda'^{(s)}_j}\end{array}\right)$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{i=1}^{l}z''''_i(t-a)^{\upsilon_i}(b-t)^{\tau_i}\prod_{j=1}^{k}(f_jt+g_j)^{-\varsigma^{(i)}_j}\right]\mathrm{d}t=$$

$$P_1\sum_{H_i=1}^{m^{(i)}}\sum_{h_i=1}^{\infty}\phi_1''\frac{\prod_{i=1}^{v}\phi_i''z_i''^{\eta_{H_i,h_i}}(-)^{\sum_{i=1}^{v}h_i}}{\prod_{i=1}^{v}\epsilon_{H_i}^{(i)}h_i!}\sum_{G_i=1}^{m_i}\sum_{g_i=1}^{\infty}\phi_1'\frac{\prod_{i=1}^{u}\phi_i'z_i'^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{u}g_i}}{\prod_{i=1}^{u}\epsilon_{G_i}^{(i)}g_i!}\prod_{k=1}^{u}x_k^{R_k}B_{u,v}$$

$$\gimel_{X;p_{i_r}+p'_{i_s}+P+k+2:q_{i_r}+q'_{i'_s}+Q+k+1,\tau_{i_r},\tau_{i'_s}:R_r,R_{s'}:Y}^{U;0,n_r+n'_s+P+k+2:V}\left(\begin{array}{c|c}\frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \\ \cdots & \\ \cdots & \mathbb{A};\mathrm{A}^*,\mathbf{A}:A \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z'_1(b-a)^{\mu'_1+\rho'_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(1)}_j}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z'_s(b-a)^{\mu'_s+\rho'_s}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda'^{(s)}_j}} & \cdot \\ \frac{(b-a)f_1}{(af_1+g_1)} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{(b-a)f_k}{(af_k+g_k)} & \cdot \\ \frac{z''''_1(b-a)^{\upsilon_1+\tau_1}}{\prod_{j=1}^{k}(af_1+g_1)^{\zeta'_j}} & \cdot \\ \cdots & \mathbb{B};\mathbf{B},\mathrm{B}^*:B \\ \cdots & \\ \frac{z''''_l(b-a)^{\upsilon_l+\tau_l}}{\prod_{j=1}^{k}(af_j+g_j)^{\zeta_j^{(l)}}} & \end{array}\right) \quad (3.14)$$

Provided that

(A) $a,b\in\mathbb{R}(a<b);\mu_i,\mu'_u,\rho_i,\rho'_u,\lambda_j^{(i)},\lambda'^{(u)}_j,h_v\in\mathbb{R}^+$, $f_i,g_j,\tau_v,\sigma_j,\lambda_v\in\mathbb{C}$ $(i=1,\cdots,r;j=1,\cdots;k;$

$u=1,\cdots,s;v=1,\cdots,l),a_i,b_i,\lambda''^{(i)}_j,\zeta''^{(i)}_j\in\mathbb{R}^+,(i=1,\cdots,u;j=1,\cdots,k)$

$a'_i,b'_i,\lambda'''^{(i)}_j,\zeta'''^{(i)}_j\in\mathbb{R}^+,(i=1,\cdots,v;j=1,\cdots,k)$

(B) $\max_{1\leqslant j\leqslant k}\left\{\left|\frac{(b-a)f_i}{af_i+g_i}\right|\right\}<1$

(C) $Re\left(\alpha+\sum_{i=1}^{u}a_i\eta_{G_i,g_i}+\sum_{i=1}^{v}a'_i\eta_{H_i,h_i}\right)+\sum_{i=1}^{r}\mu_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]+\sum_{i=1}^{s}\mu'_i\min_{1\leqslant j\leqslant m'^{(i)}}Re\left[D'^{(i)}_j\left(\frac{d'^{(i)}_j}{\delta'^{(i)}_j}\right)\right]>0$

(C) $Re\left(\beta+\sum_{i=1}^{u}b_i\eta_{G_i,g_i}+\sum_{i=1}^{v}b'_i\eta_{H_i,h_i}\right)+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]+\sum_{i=1}^{s}\rho'_i\min_{1\leqslant j\leqslant m'^{(i)}}Re\left[D'^{(i)}_j\left(\frac{d'^{(i)}_j}{\delta'^{(i)}_j}\right)\right]>0$

(D) $Re\left(\alpha+\sum_{i=1}^{u}\eta_{G_i,g_i}a_i+\sum_{i=1}^{v}\eta_{H_i,h_i}a'_i+\sum_{i=1}^{r}\mu_is_i+\sum_{i=1}^{s}t_i\mu'_i\right)>0$

$$Re\left(\beta+\sum_{i=1}^{u}\eta_{G_i,g_i}b_i+\sum_{i=1}^{v}\eta_{H_i,h_i}b'_i+\sum_{i=1}^{r}v_is_i+\sum_{i=1}^{s}t_i\rho'_i\right)>0$$

$$Re\left(\lambda_j+\sum_{i=1}^{u}\eta_{G_i,g_i}\lambda_j'''^{(i)}+\sum_{i=1}^{v}\eta_{H_i,h_i}\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\zeta_j^{(i)}+\sum_{i=1}^{s}t_i\zeta_j'^{(i)}\right)>0(j=1,\cdots,l);$$

$$Re\left(-\sigma_j+\sum_{i=1}^{u}\eta_{G_i,g_i}\lambda'''^{(i)}+\sum_{i=1}^{u}\eta_{H_i,h_i}\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\lambda_j^{(i)}+\sum_{i=1}^{s}t_i\lambda_j'^{(i)}\right)>0(j=1,\cdots,k);$$

(E) Let $\frac{1}{2}\pi B_i^{(k)}=\frac{1}{2}\pi A_i^{(k)}-\mu_i-\rho_i-\sum_{l=1}^{k}\lambda_j^{(i)}-\sum_{l=1}^{l}\zeta_j^{(i)}>0\quad(i=1,\cdots,r)$

Let $\frac{1}{2}\pi B_i'^{(k)}=\frac{1}{2}\pi A_i'^{(k)}-\mu'_i-\rho'_i-\sum_{l=1}^{k}\lambda_j'^{(i)}-\sum_{l=1}^{l}\zeta_j'^{(i)}>0\quad(i=1,\cdots,s)$

(F) $\left|arg\left(z_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}B_i^{(k)}$; $\left|arg\left(z_i\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|<\frac{1}{2}B_i'^{(k)}$

$A_i^{(k)}$ is defined by Ayant [2]

(G) $P\leqslant Q+1$. The equality holds, when , in addition,

either $P>Q$ and $\left|z_i''''\left(\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right)\right|^{\frac{1}{Q-P}}<1$ or $P\leqslant Q$ and $\max_{1\leqslant i\leqslant k}\left[\left|\left(z_i''''\prod_{j=1}^{k}(f_jt+g_j)^{-\zeta_j^{(i)}}\right)\right|\right]<1$

Proof

First expressing the the I-function defined by Prasad. [4] in series with the help of (1.4) and the multivariable Aleph in series with the help of (1.1) we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the multivariable Gimel-function of r-variables and s-variables by the Mellin-Barnes contour integral, see Ayant [2], the generalized hypergeometric function ${}_PF_Q(.)$ in Mellin-Barnes contour integral with the help of (2.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Now collect the power of (f_jt+g_j) with $j=1,\cdots,k$ and use the equations (2.2) and we obtain $k-$Mellin-Barnes contour integral. Changing the order of integrations and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and interpreting $(r+s+k+l)-$Mellin-barnes contour integral to modified multivariable Gimel-function defined by Ayant [2] , we obtain the desired result (4.1).

Remarks 1:

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable Gimel-functions defined by Ayant [2], multivariable I-function defined by Prathima et al. [5] and multivariable Aleph-function defined by Ayant. [1] .

We have similar integrals concerning other multivariable special functions.

By the following similar procedure, we can to obtain a product of any finite number of multivariable Gimel-functions class of polynomials [8] and class of Srivastava and Garg polynomials [10].

Remark 2:

We obtain the same formulae concerning the multivariable Aleph- function defined by Ayant [1], the multivariable I-function defined by Prathima et al. [5], the multivariable I-function defined by Prasad [4] and the multivariable H-function defined by Srivastava and Panda [13,14].

4. Conclusion

The importance of our all the results lies in their manifold generality. Firstly, in view of the Eulerian integrals with general arguments utilized in this study, we can obtain a large variety of single Eulerian finites integrals. Secondly by specializing the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Thirdly, by specialising the various parameters as well as variables of class of the multivariable function multivariable A-function, we can get a large number of special function of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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