On Transformation Involving Basic I-Function of Two Variables

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ABSTRACT
In this paper, fractional order q-integrals and q-derivatives involving a basic analogue of I-function of two variables have been obtained. At the end of this paper, we give an application concerning the basic analogue of I-function of two variables and one variable and q-extension of the Leibniz rule for the fractional q-derivative for a product of two basic functions.

Key vords : Fractional q-integral, q-derivative operators, basic I-function of two variables, q-Leibniz rule.

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1. Introduction.

The fractional q-calculus is the q-extension of the ordinary fractional calculus. The subject deals with the investigations of q-integral and q-derivatives of arbitrary order, and has gained importance due to its various applications in the areas like ordinary fractional calculus, solutions of the q-difference (differential) and q-integral equations, q-transform analysis see ([1] and [9]). Motivated by these avenues of applications, a number of workers have made use of these operators to evaluate fractional q-calculus formulae for various special function, basic analogue of Fox's H-function, general class of q-polynomials etc. One may refer to the recent paper [4]-[6], [11] and [12]-[17] on the subject.

In this paper, we have established three theorems involving the fractional q-integral and q-derivative operator, which generalizes the Riemann-Liouville and Weyl fractional q-integral operators. We shall see an application of q-Leibniz formula.

In the theory of q-series, for real or complex a and $|q|<1$, the q-shifted factorial is defined as :

$$(a;q)_n=\prod_{i=1}^{n-1}(1-aq^i)=\frac{(a;q)_\infty}{(aq^n;q)_\infty}\qquad (n\in\mathbb{N}) \tag{1.1}$$

so that $(a;q)_0=1$,

or equivalently

$$(a,q)_n=\frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}\qquad (a\neq 0,-1,-2,\cdots). \tag{1.2}$$

The q-gamma function [4] is given by

$$\Gamma_q(\alpha)=\frac{(q;q)_\infty(1-q)^{\alpha-1}}{(q^\alpha q)_\infty}=\frac{[1-q]_{\alpha-1}}{(1-q)^{\alpha-1}}=\frac{(q;q)_{\alpha-1}}{(1-q)^{\alpha-1}}(\alpha\neq 0,-1,-2,\cdots). \tag{1.3}$$

The q-analogue of the familiar Riemann-Liouville fractional integral operator of a function $f(x)$ due to Al-Salam [3], is given by

$$I_q^\mu\{f(x)\}=\frac{1}{\Gamma_q(\mu)}\int_0^x(x-tq)_{\mu-1}f(t)d_qt\qquad (Re(\alpha)>0,|q|<1). \tag{1.4}$$

also

$$[x-y]_v=x^v\prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{n+v}}\right] \tag{1.5}$$

Also the basic integral, see Gasper and Rahman [4] are given by

$$\int_0^x f(t)d_qt = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k) \tag{1.6}$$

The equation (1.4) in conjonction with (1.6) yield the following series representation of the Riemann-Liouville fractional integral operator

$$I_q^{\mu} f(x) = \frac{x^{\mu}(1-q)}{\Gamma_q(\alpha)}\sum_{k=0}^{\infty} q^k[1-q^{k+1}]_{\mu-1} f(xq^k) \tag{1.7}$$

In particular, for $f(x) = x^{\lambda-1}$, the above

$$I_q^{\mu}\left(x^{\lambda-1}\right) = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)} x^{\lambda+\mu-1} \tag{1.8}$$

2. Basic I-function of two variables

Recently I-function of two variables has been introduced and studied by Kumari et al. [8], it's an extension of the H-function of two variables due to Gupta and Mittal. [7,10]. In this paper we introduce a basic of I-function of two variables.

We note

$$G(q^a) = \left[\prod_{n=0}^{\infty}(1-q^{a+n})\right]^{-1} = \frac{1}{(q^a;q)_{\infty}} \tag{2.1}$$

We have

$$I(z_1,z_2;q) = I_{p_1,q_1;p_2,q_2;p_3,q_3}^{0,n_1;m_2,n_2,m_3,n_3}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array} ;q \left| \begin{array}{c} \{(a_i;\alpha_i,A_i;\mathbf{A}_i)\}_{1,p_1},\{(e_i;E_i;\mathbf{E}_i)\}_{1,p_2},\{(g_i;G_i;\mathbf{G}_i)\}_{1,p_3} \\ \cdot \\ \cdot \\ \{(b_i;\beta_i,B_i;\mathbf{B}_i)\}_{1,q_1},\{(f_i;F_i;\mathbf{F}_i)\}_{1,q_2},\{(h_i;H_i;\mathbf{H}_i)\}_{1,q_3} \end{array}\right.\right)$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,t;q)\theta_1(s;q)\theta_2(t;q)x^s y^t d_qs d_qt \tag{2.2}$$

where

$$\phi(s,t;q) = \frac{\prod_{i=1}^{n_1} G(q^{1-a_i+\alpha_i\mathbf{A}_i s+A_i\mathbf{A}_i t})}{\prod_{i=1}^{q_1} G(q^{1-b_i+\beta_i\mathbf{B}_i s+B_j\mathbf{B}_i t})\prod_{i=n_1+1}^{p_1} G(q^{a_i-\alpha_i\mathbf{A}_i s-A_i\mathbf{A}_i t})} \tag{2.3}$$

$$\theta_1(s;q) = \frac{\prod_{i=1}^{m_2} G(q^{f_i-F_i\mathbf{F}_i s})\prod_{i=1}^{n_2} G(q^{1-e_i+E_i\mathbf{E}_i s})}{\prod_{i=m_2+1}^{q_3} G(q^{1-f_i+F_i\mathbf{F}_i s})\prod_{i=n_2+1}^{p_2} G(q^{e_i-E_i\mathbf{E}_i s})G(q^{1-s})\sin\pi s}$$

$$\theta_2(t;q) = \frac{\prod_{i=1}^{m_3} G(q^{h_i-H_i\mathbf{H}_i t})\prod_{i=1}^{n_3} G(q^{1-g_i+G_i\mathbf{G}_i t})}{\prod_{i=m_3+1}^{q_3} G(q^{1-h_i+H_i\mathbf{H}_i t})\prod_{i=n_3+1}^{p_3} G(q^{1-G_i\mathbf{G}_i t})G(q^{1-t})\sin\pi t} \tag{2.5}$$

where x and y are not zero and an empty product is interpreted as unity. Also $m_i, n_i, p_i, q_i (i=1,2,3)$ are all positive integers such that $0 \leqslant m_i \leqslant q_i; 0 \leqslant n_i \leqslant p_i (i=1,2,3)$. The letters $\alpha, \beta, A, B, E, F, G, H$ and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and are all positive numbers and the letters a, b, e, f, g, h are complex numbers. The contour L_1 is in the s-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of , $G(q^{f_j-F_j\mathbf{F}s})(j=1,\cdots,m_2)$ are to the right and all

the poles of $G(q^{1-a_j+\alpha_j\mathbf{A}_j s+A_j\mathbf{A}_j t})(j=1,\cdots,n_1)$, $G(q^{1-e_j+E_j\mathbf{E}_j s})$,$(j=1,\cdots,n_2)$ lie to the left of L_1. The contour L_2 is in the t-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $G(q^{h_j-H_j\mathbf{H}_j s})(j=1,\cdots,m_3)$ are to the right and all the poles of $G(q^{1-a_j+\alpha_j\mathbf{A}_j s+A_j\mathbf{A}_j t})(j=1,\cdots,n_1)$, $G(q^{1-g_j+G_j\mathbf{G}_j t})(j=1,\cdots,n_3)$ lie to the left of L_2. For large values of $|s|$ and $|t|$ the integrals converge if $Re(slog(z_1)-log\sin\pi s)<0$ and $Re(tlog(z_2)-log\sin\pi t)<0$
The poles of the integrand are assumed to be simple.

we shall note

$$A=\{(a_i;\alpha_i,A_i;\mathbf{A}_i)\}_{1,p_1};A'=\{(e_i;E_i;\mathbf{E}_i)\}_{1,p_2},\{(g_i;G_i;\mathbf{G}_i)\}_{1,p_3} \tag{2.6}$$

$$B=\{(b_i;\beta_i,B_i;\mathbf{B}_i)\}_{1,q_1};B'=\{(f_i;F_i;\mathbf{F}_i)\}_{1,q_2},\{(h_i;H_i;\mathbf{H}_i)\}_{1,q_4} \tag{2.7}$$

3. Main result

We have the following results

Theorem 1.

For Re $(\mu)>0$, ρ and σ being any positive integers, the Riemann Liouville fractional q-integral of a product of two basic function exists an under

$$I_q^{\mu}\left\{x^{\lambda-1}I^{0,n_1;m_2,n_2,m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} \mathrm{z}_1x^{\rho}\\ \cdot\\ \cdot\\ \mathrm{z}_2x^{\sigma}\end{array};\mathrm{q}\;\middle|\;\begin{array}{c}\mathrm{A}\;;\;\mathrm{A'}\\ \cdot\\ \cdot\\ \mathrm{B;B'}\end{array}\right)\right\}=(1-q)^{\mu}x^{\lambda+\mu-1}$$

$$I^{0,n_1+1;m_2,n_2,m_3,n_3}_{p_1+1,q_1+1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} \mathrm{z}_1x^{\rho}\\ \cdot\\ \cdot\\ \mathrm{z}_2x^{\sigma}\end{array};\mathrm{q}\;\middle|\;\begin{array}{c}(1\text{-}\lambda;\rho,\sigma;1),A;A'\\ \cdot\\ \cdot\\ \mathrm{B},(1\text{-}\lambda-\mu;\rho,\sigma;1);B'\end{array}\right) \tag{3.1}$$

where $Re(slog(z_1)-log\sin\pi s)<0$ and $Re(tlog(z_2)-log\sin\pi t)<0$

Proof
To prove the above theorem, we consider the left hand side of equation (3.1) (say L) and make use of the definitions (1.4) and (2.2), we obtain

$$L=I_q^{\mu}\left\{x^{\lambda-1}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\pi^2\phi(s,t;q)\theta_1(s;q)\theta_2(t;q)x^{\rho s}y^{t\sigma}d_qsd_qt\right\} \tag{3.2}$$

interchanging the order of integrations which is justified under the conditions mentioned above, we obtain

$$L=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\pi^2\phi(s,t;q)\theta_1(s;q)\theta_2(t;q)x^sy^tI_q^{\mu}\{x^{\rho s+\sigma t+\lambda-1}\}d_qsd_qt \tag{3.3}$$

We use the formula (1.8) and we obtain

$$L=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\pi^2\phi(s,t;q)\theta_1(s;q)\theta_2(t;q)x^sy^t(1-q)^{\mu}\frac{(q^{\rho s+\sigma t+\lambda+\mu};q)_\infty}{(q^{\rho s+\sigma t+\lambda};q)_\infty}x^{\rho s+\sigma t+\lambda+\mu-1}\;d_qsd_qt \tag{3.3}$$

Now, interpreting the q-Mellin-Barnes double integrals contour in terms of the basic I-function of two variables, we get the desired result (3.1).
If replace μ by $-\mu$ and use the fractional q derivative operator defined as :

$$I_q^{-\mu} f(x) = D_{x,q}^{\mu} f(x) = I_q^{\mu}\{f(x)\} = \frac{1}{\Gamma_q(-\mu)} \int_0^x (x-yq)_{-\mu-1} f(t) d_q y \tag{3.4}$$

where $Re(\mu) < 0$ and we have the following result

Theorem 2.

For Re $(\mu) < 0$, ρ and σ being any positive integers, the Riemann Liouville fractional q-integral of a product of two basic function exists and we obtain

$$D_{x,q}^{\mu}\left\{ x^{\lambda-1} I^{0,n_1;m_2,n_2,m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} z_1 x^{\rho} \\ \cdot \\ \cdot \\ z_2 x^{\sigma} \end{array} ; q \,\middle|\, \begin{array}{c} A ; A' \\ \cdot \\ \cdot \\ B;B' \end{array} \right) \right\} = (1-q)^{-\mu} x^{\lambda-\mu-1}$$

$$I^{0,n_1+1;m_2,n_2,m_3,n_3}_{p_1+1,q_1+1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} z_1 x^{\rho} \\ \cdot \\ \cdot \\ z_2 x^{\sigma} \end{array} ; q \,\middle|\, \begin{array}{c} (1\text{-}\lambda;\rho,\sigma;1), A; A' \\ \cdot \\ \cdot \\ B,(1\text{-}\lambda+\mu;\rho,\sigma;1); B' \end{array} \right) \tag{3.5}$$

where $Re(slog(z_1) - log\sin\pi s) < 0$ and $Re(tlog(z_2) - log\sin\pi t) < 0$

The proof of the theorem 2 is similar that theorem1.

4. Leibniz's application.

We have the q-extension of the Leibniz rule for the fractional q-derivative for a product of two basic functions in terms of a series involving the fractional q-derivatives of the function in the following manner :

Lemma 1

$$D_{x,q}^{\alpha}\{U(x)V(x)\} = \sum_{n=0}^{\infty} \frac{(-)^n q^{\frac{n(n+1)}{2}} [q^{-\mu};q]_n}{(q;q;)_n} D_{x,q}^{\mu-n}\{U(xq^n)\} D_{x,q}^{n}\{W(x)\} \tag{4.1}$$

We have the formula

Theorem 3.

For Re $(\mu) < 0$, ρ and σ being any positive integers, the Riemann Liouville fractional q-integral of a product of two basic function exists and we obtain

$$I^{0,n_1+1;m_2,n_2,m_3,n_3}_{p_1+1,q_1+1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} z_1 x^{\rho} \\ \cdot \\ \cdot \\ z_2 x^{\sigma} \end{array} ; q \,\middle|\, \begin{array}{c} (1\text{-}\lambda;\rho,\sigma;1), A; A' \\ \cdot \\ \cdot \\ B,(1\text{-}\lambda-\mu;\rho,\sigma;1); B' \end{array} \right) =$$

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n\lambda+\frac{n(n-1)}{2}} [q^{-\mu};q]_n}{(q;q;)_n (q;q)_{n-\mu}} I^{0,n_1+1;m_2,n_2,m_3,n_3}_{p_1+1,q_1+1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} z_1 x^{\rho} \\ \cdot \\ \cdot \\ z_2 x^{\sigma} \end{array} ; q \,\middle|\, \begin{array}{c} (0;\rho,\sigma;1), A; A' \\ \cdot \\ \cdot \\ B,(n;\rho,\sigma;1); B' \end{array} \right) \tag{4.2}$$

where $Re(slog(z_1) - log\sin\pi s) < 0$ and $Re(tlog(z_2) - log\sin\pi t) < 0$

Proof

On taking in the q-Leibniz rule $U(x)=x^{\lambda-1}$ and $V(x)=I^{0,n_1;m_2,n_2,m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} z_1x^{\rho} \\ . \\ . \\ z_2x^{\sigma}\end{array} ; q \left| \begin{array}{c} A;\ A' \\ . \\ . \\ B;\ B' \end{array}\right.\right)$

We get (see the above Lemma).

$$D^{\mu}_{x,q}\left\{x^{\lambda-1}I^{0,n_1;m_2,n_2,m_3,n_3}_{p_1,q_1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} z_1x^{\rho} \\ . \\ . \\ z_2x^{\sigma}\end{array} ; q \left| \begin{array}{c} A\ ;\ A' \\ . \\ . \\ B;B' \end{array}\right.\right)\right\}=\sum_{n=0}^{\infty}\frac{(-)^n q^{\frac{n(n+1)}{2}}[q^{-\mu};q]_n}{(q;q;)_n(q;q)_{n-\mu}}$$

$$D^{\mu-n}_{x,q}\ (xq^n)^{\lambda-1}D^n_{x,q}\{I(z_1x^{\rho},z_2x^{\sigma})\} \tag{4.3}$$

Using the theorem 1 and let $\lambda=1$, we obtain

$$D^{\mu}_{x,q}\{I(z_1x^{\rho},z_2x^{\sigma})\}=(1-q)^{-\mu}x^{-\mu}I^{0,n_1+1;m_2,n_2,m_3,n_3}_{p_1+1,q_1+1;p_2,q_2;p_3,q_3}\left(\begin{array}{c} z_1x^{\rho} \\ . \\ . \\ z_2x^{\sigma}\end{array} ; q \left| \begin{array}{c} (0;\rho,\sigma;1),A;A' \\ . \\ . \\ B,(\mu;\rho,\sigma;1);B' \end{array}\right.\right) \tag{4.4}$$

Now using the equations (4.4) and (1.8), we obtain the theorem 3 after several algebraic manipulations.

We have the following relation concerning the basic analogue I-function of one variable.

Corollary 1

For Re $(\mu)<0$, ρ and σ being any positive integers, the Riemann Liouville fractional q-integral of a product of two basic function of one variable exists and we obtain

$$I^{0,n_1+1}_{p_1+1,q_1+1}\left(\begin{array}{c} zx^{\rho}\end{array} ; q \left| \begin{array}{c} (1\text{-}\lambda;\rho,\sigma;1),(a_i;\alpha_i;\mathbf{A}_i)_{1,p_1} \\ . \\ . \\ (b_i;\beta_i;\mathbf{B}_i)_{1,q_1},(1-\lambda-\mu;\rho,\sigma;1) \end{array}\right.\right)=$$

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n\lambda+\frac{n(n-1)}{2}}[q^{-\mu};q]_n}{(q;q;)_n(q;q)_{n-\mu}}I^{0,n_1+1}_{p_1+1,q_1+1}\left(\begin{array}{c} z\ x^{\rho}\end{array} ; q \left| \begin{array}{c} (0;\rho,\sigma;1),(a_i;\alpha_i;\mathbf{A}_i)_{1,p_1} \\ . \\ . \\ (b_i;\beta_i;\mathbf{B}_i)_{1,q_1},(n;\rho,\sigma;1) \end{array}\right.\right) \tag{4.5}$$

where $Re(slog(z)-log\sin\pi s)<0$

Remarks

We obtain the same formulaes with special functions of several variables.
If the I-function of two variables reduces in H-function of two variables, we obtain the results of Yadav et al. [17].

5. Conclusion.

The importance of our all the results lies in their manifold generality. By specialising the various parameters as well as variables in the basic I-function of two variables, we obtain a large number of results involving remarkably wide variety of useful basic functions (or product of such basic functions) which are expressible in terms of basic H-function [10] Basic Meijer's G-function, Basic E-function, basic hypergeometric function of one and two variables and simple],r special basic functions of one and two variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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