Topology

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Abstract

In this project we are going to explore the "Countable Topological Spaces" in greater details. We shall try to understand how these axioms are affected to spaces, by taking open and closed sets. A clear idea of the separation properties typical of the spaces we are studying helps us understand what kind of proof techniques to use. When working with T_1- Spaces, we use that points are closed. In particular, the inverse image of (via a continuous function) of a points in a T_1- Spaces is a closed sets. Finally we conclude that an Axiom of Countability is a properties of certain mathematical objects that requires the existence of a countable set with certain properties, while without it such sets might not exists.

Keywords:
Countable spaces, T_1 – spaces, Regular spaces

Section 1.1 First and Second Countable Topological Spaces:

Definition 1.1.1

A topological space (X, τ) is said to have a **countable local basis (or countable basis)** at a point $x \in X$ if there exists a countable collection say $\mathfrak{B}_x$ of open sets containing x such that for each open set U containing x there exists $V \in \mathfrak{B}_x$ with $V \subseteq U$.

Definition 1.1.2

A topological space (X, τ) is said to be **first countable** or said to satisfy the **first countability axiom** if for each $x \in X$ there exists a countable local base at x.

Definition 1.1.3

If a topological space (X, τ) has a countable basis B then we say that (X, τ) is a **second countable topological spaces** or it satisfies the **second countability axiom.**

Definition 1.1.4

A topological space (X, τ) is said to be a **separable topological space** if there exists a countable subset say A of X such that $\overline{A} = X$.

Definition 1.1.5

A topological space (X, τ) is said to be a **Lindelof space** if for any collection A of open sets such that $X = \bigcup_{A\in A} A$, there exists a countable subcollection say $B \subseteq A$ such that $X = \bigcup_{B\in B} B$. That is, a topological space (X, τ) is said to be a Lindelof space if and only if every open cover of X has a countable subcover for X.

Theorem 1.1.6

If (X, τ) is a second countable topological space then (X, τ) is a Lindelof space.

Proof.

Let $\mathfrak{B} = \{B_1, B_2, B_3, \ldots\}$ be a countable basis for (X, τ) and A be an open cover for X.
Assume that, $X \neq \phi$, $A \neq \phi$, for each $A \in A$ and $B \neq \phi$, for each $B \in \mathfrak{B}$.
Let $A \in A$ and $x \in A$.
Now $x \in A$, A is an open set then there exists $B \in \mathfrak{B}$ such that

$$x \in B \subseteq A. \qquad \rightarrow (1)$$

For each $n \in \mathbb{N}$, let $F_n = \{A \in A : B_n \subseteq A\}$.

$\therefore F_n = \phi$, for some $n \in \mathbb{N}$.

$\therefore (1) \Longrightarrow \{n \in \mathbb{N}: F_n \neq \phi\}$ is a nonempty set.

Let $\{n \in \mathbb{N}: F_n \neq \phi\} = \{n_1, n_2, \ldots, n_k \ldots\}$

Take $A_{nk} \in F_{nk}$.

$\therefore B_{nk} \subseteq A_{nk} \in A$.

To prove that: $\bigcup_{k=1}^{\infty} A_{nk} = X$.

So, let $x \in X$ and A is an open cover for X then $x \in A$ for some A.

Let $x \in A$, B is a basis for (X, τ) then there exists $k \in \mathbb{N}$ such that $x \in B_{nk} \subseteq A$.

$\therefore A \in F_{nk}$. Also $A_{nk} \in F_{nk}$.

By definition of F_{nk}, $B_{nk} \subseteq A_{nk}$.

Hence $x \in X$ then $x \in A_{nk}$, for some $k \in \mathbb{N}$.

$\therefore X \subseteq \bigcup_{k=1}^{\infty} A_{nk}$.

$\{A_{nk}\}^{\infty}_{k=1}$ is a countable subcover for A.

$\therefore$ Every open cover A of X has a countable subcover.

Hence (X, τ) is a Lindelof space.

Theorem 1.1.7

Every second countable topological space (X, τ) is a separable space.

Proof.

Given that (X, τ) is a second countable topological space then there exists a countable basis say $\mathfrak{B} = \{B_1, B_2, \ldots\}$ for (X, τ) and $\mathfrak{B}$ is a countably infinite set.

For some $n \in \mathbb{N}$, $\mathfrak{B} = \{B_1, B_2, \ldots B_n\}$ or $\mathfrak{B} \neq \phi$ or $\mathfrak{B}$ is a countably infinite set.

If $X \neq \phi$ then $\mathfrak{B} \neq \phi$.

If for some $k \in \mathbb{N}$, $B_k = \phi$, then $\mathfrak{B}^1 = \{B_1, B_2, \ldots B_{k-1}, B_{k+1} \ldots\}$ is also a basis for (X, τ).

Assume that each $B_n \neq \phi$ for all n.

Since $B_n \neq \phi$, for each $n \in \mathbb{N}$,

Let $x_n \in B_n$ and $A = \{x_1, x_2, x_3, \ldots\}$ and A is a finite set.

Now to prove that: $\bar{A} = X$.

Take an $x \in X$ and an open set U containing x.

Now $\mathfrak{B}$ is a basis for (X, τ), U is an open set containing x then there exists $\mathfrak{B}_n \in B$ such that $x \in B_n$ and $B_n \subseteq U$. Also $x_n \in B_n$.

$\therefore x_n \in U \cap A$.

$\Longrightarrow U \cap A \neq \phi$.

we have proved that $U \cap A \neq \phi$ for each open set U containing x.

For $x \in X$ and $x \in \bar{A}$ and hence $\bar{A} = X$.

$\therefore (X, \tau)$ has a countable dense subset.

$\therefore (X, \tau)$ is a separable space.

Section 1.2 Properties of First Countable Topological Spaces:

Theorem 1.2.1

If (X, τ) is a first countable topological space then for each $x \in X$ there exists a countable local base say $\{V_n(x)\}_{n=1}$ such that $V_{n+1}(x) \subseteq V_n(x)$.

Proof:

Let $x \in X$.

Now (X, τ) is a first countable topological space then there exists a countable local base say $\{U_n\}^{\infty}_{n=1}$ at x.

Let $V_n(x) = U_1 \cap U_2 \cap \ldots\ldots \cap U_n$ then $\{V_n(x)\}^{\infty}_{n=1}$ is a collection of open sets such that $V_{n+1}(x) \subseteq V_n(x)$ for all $n \in \mathbb{N}$.

It is enough to prove that $\{V_n(x)\}^{\infty}_{n=1}$ is a local base at x.

Let V be an open set containing x.

Now $\{U_n\}^{\infty}_{n=1}$ is a local base at x and V is an open set containing x then there exists $n_0 \in N$ such that $U_{n0} \subseteq V$.

By definition of $V_n(x)'s$ we have $V_{n0}(x) \subseteq U_{n0}$.
For each open set V containing x then there exists $n_0 \in N$ such that $V_{n0}(x) \subseteq V$.
$\therefore \{V_n(x)\}$ is a local base at x satisfying $V_{n+1}(x) \subseteq V_n(x)$ for all $n \in \mathbb{N}$.

Theorem 1.2.2

Let (X, τ) be a first countable topological space and A be a nonempty subset of X. Then for each $x \in X$, $x \in \bar{A}$ if and only if there exists a sequence $\{x_n\}_{n=1}$ in A such that $x_n \to x$ as $n \to \infty$.

Proof.

Let us assume that $x \in \bar{A}$.
Now (X, τ) is a first countable topological space then there exists a countable local base say $\mathfrak{B} = \{V_n\}^{\infty}_{n=1}$ such that $V_{n+1} \subseteq V_n$, for all $n \in \mathbb{N}$.
Hence $x \in A$ then $A \cap V_n \neq \phi$, for each $n \in N$.
Let $x_n \in A \cap V_n$.

Claim:

$x_n \to x$ as $n \to \infty$
Let U be an open set containing x then there exists $n_0 \in N$ such that $x \in V_{n0} \subseteq U$.
Hence $x_n \in V_n \subseteq V_{n0} \subseteq U$ for all $n \geq n_0$.
ie., $x_n \in U$ for all $n \geq n0$:
$\therefore x_n \to x$ as $n \to \infty$.
Conversely, suppose there exists a sequence $\{x_n\}^{\infty}_{n=1}$ in A such that $x_n \to x$.
Then for each open set U containing x there exists a positive integer n_0 such that $x_n \in U$ for all $n \geq n_0$.
In particular, $x_{n0} \in U \cap A$.
Hence for each open set U containing x, $U \cap A \neq \phi$
$\therefore x \in \bar{A}$. (by theorem 1.1.7)

Section 1.3 Regular and Normal Topological Spaces:

Definition 1.3.1

A topological space (X, τ) is called a **T_1 space** if for each $x \in X$, the singleton set $\{x\}$ is a closed set in (X, τ).

Definition 1.3.2

A T_1-topological space (X, τ) is called a **regular space** if for each $x \in X$ and for each closed subset A of X with $x \notin A$, there exist open sets U, V in X satisfying the following:
(i) $x \in U$, $A \subseteq V$, (ii) $U \cap V = \Phi$.

Definition 1.3.3

A topological space (X, τ) is said to be a **normal space** if and only if it satisfies:
(i) (X, τ) is a T_1-space,
(ii) A, B closed sets in X, $A \cap B = \Phi$ implies there exist open sets U, V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \Phi$.

Theorem 1.3.4

Every metric space (X, d) is a normal space, That is if τ_d is the topology induced by the metric then the topological space (X, τ_d) is a normal space.

Proof.

Let A, B be disjoint closed subsets of X.
Then for each $a \in A$, $a \notin B = \bar{B}$ implies $d(a, B) = \inf\{d(a; b) : b \in B\} > 0$.
If $r_a = d(a, B) > 0$ then $B(a, r_a) \cap B = \emptyset$ (if there exists $b_0 \in B$ such that $d(b_0, a) < r_a$, then $r_a = d(a, B) < d(a, b_0) \nleq r_a$ a contradiction).

Similarly for each $b \in B$ there exists $r_b > 0$ such that $B(b, r_b) \cap A = \Phi$.
Let $U = \bigcup_{a \in A} B(a, r_a/3)$, $V = \bigcup_{b \in B} B(a, r_b/3)$.
It is easy to prove that $U \cap V = \emptyset$.
Hence if A, B are disjoint closed subsets of X then there exist open sets U, V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.
$\Longrightarrow$ (X, τ_d) is a normal space.

Theorem 1.3.5

A T_1-topological space (X, τ) is regular if and only if whenever x is a point of X and U is an open set containing x then there exists an open set V containing x such that $\bar{V} \subseteq U$.

Proof.

Assume that (X, τ) is a regular topological space, $x \in X$ and U is an open set containing x.
Now $x \in U$ implies $x \notin A = U^c = X \setminus U$, the complement of the open set U.
Now A is a closed set and $x \notin A$.
Hence X is a regular space then there exist open sets V and W of X such that $x \in V$, $A = U^c \subseteq W$ and $V \cap W = \emptyset$.
Now $V \cap W = \emptyset$ then $V \subseteq W^c \subseteq U$ (we have $U^c \subseteq W$), $V \subseteq W^c$ implies $\bar{V} \subseteq \bar{W}^c = W^c$
$\Longrightarrow \bar{V} \subseteq U$.
Hence for $x \in X$ and for each open set U containing x, there exists an open set V containing x such that $\bar{V} \subseteq U$.
Now let us assume that the above statement is satisfied.
Our aim to prove that (X, τ) is a regular space.
Take a closed set A of X and a point $x \in X \setminus A$.
Now A is a closed subset of X implies $U = X \setminus A$ is an open set containing x.
Hence by our assumption there exists an open set V containing x such that $\bar{V} \subseteq U = A^c$.
Now $\bar{V} \subseteq A^c$ then $A \subseteq (\bar{V})^c = X \setminus V$.
Then V and $(\bar{V})^c = W$ are open sets satisfying $x \in V$, $A \subseteq W$ and $V \cap W = V \cap (\bar{V})^c \subseteq V \cap V^c = \emptyset$.
$\therefore V \subseteq \bar{V}$ then $(\bar{V})^c \subseteq V^c$.
Hence by definition (X, τ) is a regular space.

Theorem 1.3.6

A T_1-topological space is a normal space if and only if whenever A is a closed subset of X and U is an open set containing A, then there exists an open set V containing A such that $\bar{V} \subseteq U$.

Proof.

Assume that (X, τ) is a normal topological space.
Take a closed set A and an open set U in X such that $A \subseteq U$.
Now $A \subseteq U$ then $U^c \subseteq A^c$.
Here A, $U^c = B$ are closed sets such that $A \cap B = A \cap U^c \subseteq U \cap U^c = \emptyset$.
ie., A, B are disjoint closed subsets of the normal space (X, τ).
Hence there exist open sets U, W in X such that $A \subseteq V$, $B = U^c \subseteq W$ and $V \cap W = \emptyset$.
Further $\bar{V} \subseteq W^c$.
Now $\bar{V} \subseteq W^c \subseteq U$.
Hence whenever A is a closed set and U is an open set containing A then there exists an open set V such that $A \subseteq V$, $\bar{V} \subseteq U$.
Now let us assume that the above statement is satisfied.
our aim is to prove that (X, τ) is a normal space.
Let A, B be two disjoint closed subsets of X.
Now $A \cap B = \emptyset$ then $A \subseteq B^c = U$.
ie., U is an open set containing the closed set A.
Hence by our assumption there exists an open set V such that $A \subseteq V$, $\bar{V} \subseteq U$.
Now $\bar{V} \subseteq U$ then $U^c \subseteq (\bar{V})^c \Longrightarrow B \subseteq (\bar{V})^c$.

Further $V \cap (\bar{V})^c \subseteq V \cap V^c = \emptyset$.
ie., whenever A, B are closed subsets of X, then there exist open sets V and $(\bar{V})^c = W$ such that $A \subseteq V$, $B \subseteq W$ and $V \cap W = \emptyset$.
$\therefore (X, \tau)$ is a normal space.

Theorem 1.3.7

Every compact Hausdorff topological space (X, τ) is a regular space.

Proof.

Let A be a closed subset of X and $x \in X \setminus A$, then for each $y \in A$, $x \neq y$.
Hence X is a Hausdorff space then there exist open sets U_y, V_y in X satisfying $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$.
W.K.T. closed subset of a compact space is compact.
Here $A \subseteq \bigcup_{y \in A} V_y$.
ie., $\{V_y : y \in A\}$ is an open cover for the compact space A.
$\therefore$ There exists $n \in \mathbb{N}$ and $y_1, y_2 \ldots y_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} V_{yi}$.
Let $U = \bigcap_{i=1}^{n} U_{yi}$ and $V = \bigcup_{i=1}^{n} V_{yi}$.
Then U, V are open sets in X satisfying $x \in U$, $A \subseteq V$ and $U \cap V \subseteq U \cap (V_{y1} \cup V_{y2} \cup \ldots \cup V_{yn}) = (U \cap V_{y1}) \cup (U \cap V_{y2}) \cup \ldots \cup (U \cap V_{yn}) \subseteq (U_{y1} \cap Vy_1) \cup (U_{y2} \cap V_{y2}) \cup \ldots \cup (U_{yn} \cap V_{yn}) = \emptyset$.
$\therefore (X, \tau)$ is a regular space.

Theorem 1.3.8

Every compact Hausdorff space (X, τ) is a normal space.

Proof.

Let A, B be disjoint closed sets in X.
Then for each $x \in A$, $x \notin B$.
Now (X, τ) is a regular space then there exist open sets U_x, V_x satisfying: $x \in U_x$; $B \subseteq V_x$ and $U_x \cap V_x = \Phi$.
Now $\{U_x : x \in A\}$ is an open cover for A then there exists $n \in \mathbb{N}$, $x_1, x_2, \ldots x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} U_{xi}$.
Let $U = U_{x1} \cup U_{x2} \cup \ldots \cup U_{xn}$ and $V = V_{x1} \cap V_{x2} \cap \ldots \cap V_{xn}$.
Then U, V are open sets in X satisfying $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.
$\therefore (X, \tau)$ is a normal space.

Theorem 1.3.9

Let (X, τ) be a normal space and A, B be disjoint non empty closed subsets of X. Then for a, b $\in \mathbb{R}$, $a < b$ there exists a continuous function $f : X \to [a, b]$ such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.

Proof.

Define $g : [0, 1] \to [a, b]$ as $g(t) = a + (b - a)t$ then g is continuous.
Now by Urysohn Lemma theorem, there is a continuous function $f_1 : X \to [0, 1]$ such that $f_1(x) = 0$, for all $x \in A$ and $f_1(x) = 1$ for all $x \in B$.
The function $f = g_o f_1 : X \to [a, b]$ is a continuous function and further $f(x) = g(f_1(x)) = g(0) = a$ for all $x \in A$ and $f(x) = g(f_1(x)) = g(1) = b$ for all $x \in B$.

CONCLUSION

These are ordered roughly chronologically (although this is obscured by the fact that the most recent editions or versions are cited). I have included only those texts that I have looked at myself, that are at least at the level of the more elementary chapters here, and that offer significant individuality of treatment. The above mentioned are many other textbooks in topology.

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