The Marcus Topology Coincides with the Quotient of Slapal's Topology

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ABSTRACT. In this paper we study properties of Marcus topology and Slapal's topology and the relation between them.

Key words: Marcus topology, Slapal's topology, quotient topology, 8-adjacent, 4-adjacent, $T_{1/2}$-topology.

1. INTRODUCTION

An important problem of digital topology is to provide the digital plane Z^2 with a convenient structure for the study of geometric and topological properties of digital images. A basic criterion for such a convenience is the validity of an analogy of the Jordan curve theorem. It was in 1990 that a topology on Z^2 convenient for the study of digital images was introduced by Khalimsky. A drawback of the Khalimsky topology is that the Jordan curves with respect to it can never turn at an acute angle. To overcome this deficiency, another topology on Z^2 was introduced by Slapal.

Marcus topology is the basic topology on Z^2

Notation 1.1. [5] Let $z = (x, y) \in Z^2$. Put

$$H_2(z) = \{(x-1, y), (x+1, y)\}$$

$$V_2(z) = \{(x, y-1), (x, y+1)\}$$

$$D_4(z) = H_2(z) \cup \{(x-1, y-1), (x+1, y-1)\}$$

$$U_4(z) = H_2(z) \cup \{(x-1, y+1), (x+1, y+1)\}$$

$$L_4(z) = V_2(z) \cup \{(x-1, y-1), (x-1, y+1)\}$$

$$R_4(z) = V_2(z) \cup \{(x+1, y-1), (x+1, y+1)\}$$

Then we put

$$A_4(z) = H_2(z) \cup V_2(z)$$
$$A_8(z) = H_2(z) \cup L_4(z) \cup R_4(z)$$
$$= V_2(z) \cup D_4(z) \cup U_4(z)$$

and $A'_4(z) = A_8(z) - A_4(z)$.

$A_4(z)$ and $A_8(z)$ are said to be 4-adjacent and 8-adjacent to z respectively.

$$H_2(z), V_2(z), D_4(z), V_4(z), L_4(z), R_4(z)$$

and $A'_4(z)$ are called horizontally 2-adjacent, vertically 2-adjacent, down 4-adjacent, up 4-adjacent, left 4-adjacent, right 4-adjacent and diagonally 4-adjacent to z respectively.

Definition 1.2. Let u be the Alexandroff $T_{1/2}$-topology on Z^2 as follows.

For any $z = (x, y) \in Z^2$

$$u(z) = \begin{cases} \{z\} \cup A_4(z) & \text{if } x + y \text{ is even} \\ \{z\} & \text{otherwise.} \end{cases}$$

(Z^2, u) is called the Marcus topological space.

Definition 1.3. [4] Let w be the Alexandroff $T_{1/2}$ topology on Z^2 defined as follows.

For any point $z = (x, y) \in Z^2$

$$w(z) = \begin{cases} \{z\} \cup A_8(z) & \text{if } x = 4k,\ y = 4l,\ k, l \in Z \\ \{z\} \cup A'_4(z) & \text{if } x = 2 + 4k,\ y = 2 + 4l,\ k, l \in Z \\ \{z\} \cup D_4(z) & \text{if } x = 2 + 4k,\ y = 1 + 4l,\ k, l \in Z \\ \{z\} \cup U_4(z) & \text{if } x = 2 + 4k,\ y = 3 + 4l,\ k, l \in Z \\ \{z\} \cup L_4(z) & \text{if } x = 1 + 4k,\ y = 2 + 4l,\ k, l \in Z \\ \{z\} \cup R_4(z) & \text{if } x = 3 + 4k,\ y = 2 + 4k,\ k, l \in Z \\ \{z\} \cup H_2(z) & \text{if } x = 2 + 4k,\ y = 4l,\ k, l \in Z \\ \{z\} \cup V_2(z) & \text{if } x = 4k,\ y = 2 + 4l,\ k, l \in Z \\ \{z\} & \text{otherwise} \end{cases}$$

Theorem 1.4. *The Marcus topology u coincides with the quotient topology of w generated by g.*

Proof. We will show that for any points $z_1, z_2 \in Z^2$, $z_1 \in u(z_2)$ if and only if there are points $a \in g^{-1}(z_1)$ and $b \in g^{-1}(z_2)$ such that $a \in w(b)$. This is true if $z_1 = z_2$. Therefore suppose $z_1 \neq z_2$.

Let $z_1 \in t(z_2)$. Then z_2 is an open set in (Z^2, u) hence $z_2 = (x, y)$ where one of the numbers x, y is even and the other is odd, and $z_1 \in A_4(z_2) - \{z_2\}$. It follows that $z_2 = (k+l+1, l-k)$ and $g^{-1}(z_2) = \{(4k+2, 4l+2)\}$ for some $k, l \in Z$, $k+l$ odd, if x is even while $g^{-1}(z_2) = A_{12}(4k+2, 4l+2)$ for some $k, l \in Z$, $k+l$ even, if x is odd. We have $(4k+2, 4l+2) \in g^{-1}(z_2)$ (in both the cases when x is even or odd) and one of the following four cases occurs.

(1) $z_1 = (k+l+2, l-k)$, hence $g^{-1}(z_1) = A_8(4k+4, 4l+4)$, $(4k+3, 4l+3) \in g^{-1}(z_1)$ and we have $(4k+3, 4l+3) \in w\{4k+2, 4l+2\}$
(2) $z_1 = (k+l, l-k)$, hence $g^{-1}(z_1) = A_8(4k, 4l)$, $(4k+1, 4l+1) \in g^{-1}(z_1)$ and we have $(4k+1, 4l+1) \in w\{4k+2, 4l+2\}$
(3) $z_1 = (k+l+1, l-k+1)$, hence $g^{-1}(z_1) = A_8(4k, 4l+4)$, $(4k-1, 4l+3) \in g^{-1}(z_1)$ and we have $(4k-1, 4l+3) \in w\{4k+2, 4l+2\}$
(4) $z_1 = (k+l+1, l-k-1)$, hence $g^{-1}(z_1) = A_8(4k+4, 4l)$, $(4k+3, 4l+1) \in g^{-1}(z_1)$ and we have $(4k+3, 4l+1) \in w\{(4k+2, 4l+2)\}$

We have shown that whenever $z_1 \in u(z_2)$ there are points $a \in g^{-1}(z_1)$ and $b \in g^{-1}(z_2)$ such that $a \in w(b)$.

Conversely suppose that there are points $a \in g^{-1}(z_1)$ and $b \in g^{-1}(z_2)$ such that $a \in w(b)$. Then $g^{-1}(z_1)$ is not open in (Z^2, w). Therefore $g^{-1}(z_1) = A_8(4k, 4l)$ which means that $z_1 = (k+l, l-k)$ for some $k, l \in Z$. Further we have one of the following four cases,

1. $z_2 = (k+l+1, l-k)$ because $g^{-1}(z_2) = \{(4k+2, 4l+2)\}$ or $g^{-1}(z_2) = A_{12}(4k+2, 4l+2)$, $a = (4k+1, 4l+1) \in g^{-1}(z_1)$ and $b = (4k+2, 4l+2) \in g^{-1}(z_2)$, so we have $z_1 \in u(z_2)$
2. $z_2 = (k+l-1, l-k) = ((k-1)+(l-1)+1, (l-1)-(k-1))$ because then $g^{-1}(z_2) = \{(4(k-1)+2, 4(l-1)+2)\} = \{4k-2, 4l-2\}$ or $g^{-1}(z_2) = A_{12}(4k-2, 4l-2)$, $a = (4k-1, 4l-1) \in g^{-1}(z_1)$ and $b = (4k-2, 4l-2) \in g^{-1}(z_2)$. Then we have $z_1 \in u\{z_2\}$

3. $z_2 = (k+l, l-k+1) = (1+(k-1)+l, l-(k-1))$ because then $g^{-1}(z_2) = \{(4(k-1)+2, 4l+2)\} = \{(4k-2, 4l-2)\}$ or $g^{-1}(z_2) = A_{12}(4k-2, 4l)$, $a = (4k-1, 4l+1) \in g^{-1}(z_1)$ and $b = (4k-2, 4l+2) \in g^{-1}(z_2)$. Then also $z_1 \in u\{z_2\}$
4. $z_2 = (k+l, l-k-1) = (k+(l-1)+l, (l-1)-k)$ because then $g^{-1}(z_2) = \{4k+2, 4(l-1)+2\} = \{(4k+2, 4l-2)\}$ or $g^{-1}(z_2) = A_{12}(4k+2, 4l-2)$, $a = (4k+1, 4l-1) \in g^{-1}(z_1)$ and $b = (4k+2, 4l-2) \in g^{-1}(z_2)$. Then also $z_1 \in u\{z_2\}$.

We have shown that $a \in g^{-1}(z_1)$, $b \in g^{-1}(z_2)$ and $a \in w(b)$ imply $z_1 \in u\{z_2\}$. By lemma **??**, u is the quotient topology of w generated by g □

Result 1.5. *1. Marcus topological space can be obtained from the Khalimsky topological space. It is homeomorphic to the subspace of the Khalimsky topological space given by the pure points and it is got by putting*

$$\phi(x, y) = (x-y+1, x+y+1) \text{ for all } (x, y) \in Z^2.$$

2. Both the Marcus topological space (Z^2, u) and Khalimsky topological space (Z^2, v) the following two conditions are satisfied for any pair of different points $(z_1, z_2) \in Z^2$.

(a) If (z_1, z_2) is connected, then z_1 and z_2 are 8-adjacent.

(b) If z_1 and z_2 are 4-adjacent, then $\{z_1, z_2\}$ is connected.

(c) The topology w satisfies only the condition (a), makes it less convenient for applications in digital topology.

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