On Unified Eulerian Type Integrals Involving a Multivariable Gimel-Function with General Arguments

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ABSTRACT
The aim of the present paper is to evaluate an unified Eulerian type integral. This integral involves the product of function $R_n^{(\alpha,\beta)}$ with multivariable gimel-function. Further, the argument of this integral are quite general in nature. Next, we give three special cases. A number of other integrals can also be obtained as special cases of our integral thus unifying several simpler integrals lying scattered in the literature.

Keywords:Multivariable Gimel-function , general sequence of functions $R_n^{(\alpha,\beta)}$, Eulerian integral.

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1.Introduction.

We define a generalized transcendental function of several complex variables noted ℷ.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right.$$

$$[(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})]_{1,n_r},$$
$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad\cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{n_r+1,p_r}:\ [(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,p_i^{(1)}}$$
$$[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_r}:\ [(d_j^{(1)}),\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,q_i^{(1)}}$$

$$\left.\begin{array}{l};\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}}\\ ;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,n^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{n^{(r)}+1,q_i^{(r)}}\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \quad (1.2)$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.3)$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r).$

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k).$

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k).$

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k).$

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r).$

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r).$

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r).$

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r).$

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k (k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r), \Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \quad (1.4)$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D^{(i)}_j \left(\frac{d^{(i)}_j}{\delta^{(i)}_j} \right) \right] \text{ and } \beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C^{(i)}_j \left(\frac{c^{(i)}_j - 1}{\gamma^{(i)}_j} \right) \right]$$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [2].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [10,11].

Agarwal and Chaubey ([1], p. 1155) have introduced and studied a sequence of functions which will be defined and represented in the following slightly modified manner

$$R_n^{(\alpha,\beta)}[x;a,b,c,d;p,q;\gamma,\delta;\lambda,K;(x)] = \frac{(ax^p+b)^{-\alpha}(cx^q+d)^{-\beta}}{K_n'\omega(x)} T_{\lambda,K}^n\left[(ax^p+b)^{\alpha+\gamma n}(cx^q+d)^{\beta+\delta n}\omega(x)\right] \quad (1.5)$$

where the differential operator $T_{\lambda,K}^n = [x^K(\lambda + xD_x)]^n, D_x = \frac{d}{dx}, \{K_n'\}_{n=0}^{\infty}$ is a sequence of constants, and $\omega(x)$ is independent of n and differentiable an arbtrary number of times.

If we set $\omega(x) = e^{-sx^r}$ in (1.5), then the explicit series form of this general sequence of functions by Tariq ([12], p.169, Eq. (8)) is

$$R_n^{(\alpha,\beta)}[x;a,b,c,d;p,q;\gamma,\delta;\lambda,K;e^{-sx^r}] = \frac{b^{\gamma n}x^{Kn}(cx^q+d)^{\delta n}K^n e^{sx^r}}{K_n'}$$

$$\sum_{m,v,u,t,e} \frac{(-)^{t+m}(-v)_u(-t)_e(\alpha)_t s^m(-\alpha-\gamma n)_e}{m!u!v!t!e!}\frac{(-\beta-\delta n)}{(1-\alpha-t)_e}\left(\frac{pe+rm+\lambda+qu}{K}\right)_n\left(\frac{cx^q}{cx^q+d}\right)^v\left(\frac{ax^p}{b}\right)^t x^{rm} \quad (1.6)$$

where $\sum_{m,v,u,t,e} = \sum_{m=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{e=0}^{t}$

and the infinite serie on the right hand side of (1.6) is absolutely convergent. Now, we give an explicit series form of a special case of $R_n^{(\alpha,\beta)}(x)$ for $s=0$ in the following form, which will be required in the derivation of the main integral.

$$R_n^{(\alpha,\beta)}[x;a,b,c,d;p,q;\gamma,\delta;1] = \sum_{v,u,t,e}\theta_1(v,u,t,e)x^{R'}\left(1+\frac{cx^q}{d}\right)^{-v+n\delta} \quad (1.7)$$

where

$$\theta_1(v,u,t,e) = \frac{b^{n\gamma}K^n d^{n\delta-v}(-)^l(-v)_u(-t)_e(\alpha)_t(a/b)^{pt}(-\alpha-\gamma n)_e}{K_n' v!u!t!e!(1-\alpha-t)_e}(-\beta-n\delta)_v\left(\frac{be+\lambda+qu}{K}\right)_n \quad (1.8)$$

$R' = Kn+qv+pt$ and

$$\sum_{v,u,t,e} = \sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{e=0}^{t} \quad (1.9)$$

It may be pointed out that $R_n^{(\alpha,\beta)}(x)$ unifies and extends a large number of named classical polynomials and other polynomials studied by several research workers.

2. Required result.

The following integral due to Gupta ([4], p.303, Eq. 8) in a slightly modified form will also be required in the sequel

Lemma

$$\int_{\lambda_1}^{\mu_1}(x-\lambda_1)^{\rho_1-1}(\mu_1-x)^{\sigma_1-1}\left[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{\upsilon}+q_1]^{\rho}\right]^{-\omega_1}\mathrm{d}x =$$

$$= (\mu_1-\lambda_1)^{\rho_1+\sigma_1-1}\Gamma(\sigma_1)\sum_{s,k,l=0}^{\infty}\frac{(-)^{k+l}A^sB^kp_1^lq_1^{\rho k-l}(\mu_1-\lambda_1)^{\mu s+\upsilon l}}{s!k!l!}\frac{\Gamma(\rho_1+\mu s+\upsilon l)\Gamma(\omega_1+k+s)\Gamma(-\rho k+l)}{\Gamma(\omega_1)\Gamma(-\rho k)\Gamma(\rho_1+\sigma_1+\mu s+\upsilon l)} \quad (2.1)$$

provided

$|A(\mu_1-\lambda_1)^{\mu} - B[p_1(\mu_1-\lambda_1)^{\upsilon}+q_1]^{\rho}| < 1$

$|B[p_1(\mu_1-\lambda_1)^{\upsilon}+q_1]^{\rho}|<|A(\mu_1-\lambda_1)^{\mu}|:\left|\frac{p_1(\mu_1-\lambda_1)^{\upsilon}}{q_1}\right|<1$ and

$min(\rho_1,\sigma_1)>0;A,B,\omega_1\mu,\upsilon,\rho\in\mathbb{R}^+$

3. Main integral.

In this section, we shall evaluate a unified Eulerian integral.

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})]_{1,n_{r-1}},$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})]_{n_{r-1}+1,p_{i_{r-1}}}\quad(3.1)$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)},0,0,0;A_{rj})]_{1,n_r},[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)},0,0,0;A_{rji_r})]_{\mathfrak{n}+1,p_{i_r}}\quad(3.2)$$

$$A_1=(1-\rho_1-\rho_2R';\rho_1',\cdots,\rho_r',\rho_2q,\mu,\upsilon;1),(1-\omega_1-\omega_2R-k;\omega_1',\cdots,\omega_r',\omega_2q,1,0;1),$$

$$(1-\sigma_1-\sigma_2R',\sigma_1',\cdots,\sigma_r',\sigma_2q,0,0;1)\quad(3.3)$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,p_i^{(1)}};\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}}\quad(3.4)$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})]_{1,q_{i_{r-1}}}\quad(3.5)$$

$$\mathbf{B}=[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)},0,0,0;B_{rji_r})]_{1,q_{i_r}}\quad(3.6)$$

$$B_1=(1-\rho_1-\sigma_1-(\rho_2+\sigma_2)R';\rho_1'+\sigma_1',\cdots,\rho_r'+\sigma_r',(\rho_2+\sigma_2)q,\mu,\upsilon;1),$$

$$(1-\omega_1-\omega_2R-k;\omega_1',\cdots,\omega_r',\omega_2q,0,0;1)\quad(3.7)$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)}]_{m^{(1)}+1,q_i^{(1)}};\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}}\quad(3.8)$$

$$U=0,n_2;0,n_3;\cdots;0,n_r;V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}\quad(3.9)$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}\quad(3.10)$$

Theorem

$$\int_{\lambda_1}^{\mu_1}(x-\lambda_1)^{\rho_1-1}(\mu_1-x)^{\sigma_1-1}\left[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{\upsilon}+q_1]^{\rho}\right]^{-\omega_1}$$

$$R_n^{(\alpha,\beta)}\Big(y(x-\lambda_1)^{\rho_2-1}(\mu_1-x)^{\sigma_2-1}\left[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{\upsilon}+q_1]^{\rho}\right]^{-\omega_2}\Big)$$

$$\gimel\begin{pmatrix} z_1(x-\lambda_1)^{\rho_1'}(\mu_1-x)^{\sigma_1'}\left[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{\upsilon}+q]^{\rho}\right]^{-\omega_1'} \\ \cdot \\ \cdot \\ \cdot \\ z_r(x-\lambda_1)^{\rho_r'}(\mu_1-x)^{\sigma_r'}\left[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{\upsilon}+q]^{\rho}\right]^{-\omega_r'} \end{pmatrix}\mathrm{d}x=$$

$$(\mu_1-\lambda_1)^{\rho_1+\sigma_1-1}\sum_{v,u,t,e}\sum_{k=0}^{\infty}\frac{(\mu_1-\lambda_1)^{(\rho_2+\sigma_2)R'}y^{R\prime}}{\Gamma(\upsilon-n\delta)}\phi_1(v,u,t,e)\frac{(-Bq_1^{\rho})^k}{k!\Gamma(-\rho k)}$$

$$\gimel^{U;0,n_r+3:V;1,1;1,0;1,1}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y;1,1;0,1;1,1}\left(\begin{matrix} z_1(\mu_1-\lambda_1)^{\rho_1'+\sigma_1'} \\ \cdot \\ \cdot \\ \cdot \\ z_r(\mu_r-\lambda_r)^{\rho_r'+\sigma_r'} \\ \frac{c}{d}y^q(\mu_1-\lambda_1)^{(\rho_2+\sigma_2)q} \\ \text{-}A(\mu_1-\lambda_1)^{\mu} \\ \frac{p_1}{q_1}(u_1-\lambda_1)^{\upsilon} \end{matrix}\ \middle|\ \begin{matrix} \mathbb{A};\mathrm{A}_1,\mathbf{A}:A;(1-\upsilon+n\delta,1;1);-;(1+\rho k,1;1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mathbb{B};\mathbf{B},\mathrm{B}_1:B;(0,1;1);(0,1;1);(0,1;1) \end{matrix}\right) \quad (3.12)$$

provided

$$\rho_2,\sigma_2,\omega_2,\rho_i',\sigma_i',\omega_i'>0(i=1,\cdots,r)\;;Re(\rho_1+\rho_2R')+\sum_{i=1}^{r}\rho_i'\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$Re(\sigma_1+\sigma_2R')+\sum_{i=1}^{r}\sigma_i'\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$\left|arg\left(z_i(x-\lambda_1)^{\rho_i'}(\mu_1-x)^{\sigma_i'}\left[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{\upsilon}+q]^{\rho}\right]^{-\omega_i'}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

Expressing the sequence of functions in series with the help of (1.7) and the multivariable Gimel-function in terms of Mellin-Barnes multiple integrals contour with the help of (1.1). Then interchanging the order of summations and integrations which is permissible under the stated conditions. We obtain (say I)

$$I=\sum_{v,u,t,e}\theta_1(v,u,t,e)y^{R'}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\int_{\lambda_1}^{\mu_1}(x-\lambda_1)^{\rho_1+\rho_2R'+\sum_{i=1}^{r}\rho_i's_i-1}(\mu_1-x)^{\sigma_1+\sigma_2R'+\sum_{i=1}^{r}\sigma_i's_i-1}$$

$$\left[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{\upsilon}+q_1]^{\rho}\right]^{-\omega_1-\omega_2R'-\sum_{i=1}^{r}\omega_i's_i-1}\Big]$$

$$\left(1+\frac{c}{d}y^q(x-\lambda_1)^{\rho}(\mu_1-x)^{\sigma}\left[1-A(x-\lambda_1)^{\mu}+B\left[p_1(x-\lambda_1)^{\upsilon}+q_1\right]^{\rho}\right]^{-\omega}2^q\right)^{-\upsilon+n\delta}\mathrm{d}x\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (3.12)$$

Now we express the binomial term in (3.12) in its contour integral form ([9], p.18, Eq.(2.6.4)) and change the order of integration (which is permissible under the conditions stated with the main result), we have

$$I = \sum_{v,u,t,e} \frac{\theta_1(v,u,t,e)y^{R'}}{\Gamma(v-n\delta)} \frac{1}{(2\pi\omega)^{r+1}} \int_{L_1} \cdots \int_{L_r} \int_{L_{r+1}} \psi(s_1,\cdots,s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k}$$

$$\left(\frac{c}{d}y^q\right)^{s_{r+1}} \Gamma(-s_{r+1})\Gamma(v-n\delta+s_{r+1})$$

$$\int_{\lambda_1}^{\mu_1} (x-\lambda_1)^{\rho_1+\rho_2 R'+\sum_{i=1}^r \rho_i' s_i+\rho_2 q s_{r+1}-1} (\mu_1-x)^{\sigma_1+\sigma_2 R'+\sum_{i=1}^r \sigma_i' s_i+\sigma_2 q s_{r+1}-1}$$

$$[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{v}+q_1]^{\rho}]^{-\omega_1-\omega_2 R'-\sum_{i=1}^r \omega_i' s_i-\omega_2 q s_{r+1}-1} \mathrm{d}x \bigg] \mathrm{d}s_1\cdots\mathrm{d}s_{r+1} \tag{3.13}$$

Finally, we evaluate the x-integral with the help of lemma and interpreting the resulting Mellin-Barnes multiple integrals contour with the help of (1.1) in term of multivariable Gimel-function of $(r+3)$-variables, we get the desired result (3.11).

4.Special cases.

In this section we shall see two particular cases studied by Gupta and Agrawal [5].

1) The sequence of functions $R_n^{(\alpha,\beta)}$ reduces to the polynomial set $S_n^{\alpha,\beta,\tau}$ see Raizada [8] for more details, the multivariable Gimel-function reduces to Appell's function ([9], p.89, Eq. 56.4.4)). Taking$\sigma_2=\omega_2=\rho_1'=\rho_2'=\omega_1'=\omega_2'=0, z_1=z_2=-1$ therein, we have the following integrals

Corollary 1.

$$\int_{\lambda_1}^{\mu_1} (x-\lambda_1)^{\rho_1-1}(\mu_1-x)^{\sigma_1-1}[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{v}+q_1]^{\rho}]^{-\omega_1} S_n^{\alpha,\beta,\tau}[y(x-\lambda_1)^{\rho_2}]$$

$$F_1[a_1,c_1,c_1';b_1;(\mu_1-x)^{\sigma_1'},(\mu_1-x)^{\sigma_2'}]\mathrm{d}x = (\mu_1-\lambda_1)^{\rho_1+\sigma_1-1}\frac{\Gamma(\sigma_1}{\Gamma(\omega_1)}\sum_{v,u,t,e}\sum_{k=0}^{\infty}\zeta_1(v,u,t,e)$$

$$F^{3:1;1;1;1;1}_{2:0;0;0;0;0}\left(\begin{array}{c}(\text{u-}\lambda_1)^{\sigma_1'}\\ (\text{u-}\lambda_1)^{\sigma_2'}\\ \tau y^q(\mu_1-\lambda_1)^{\rho_2 q}\\ \text{A}(\mu_1-\lambda_1)^{\mu}\\ \text{-}\frac{p_1}{q_1}(\mu_1-\lambda_1)^{v}\end{array}\left|\begin{array}{c}(\rho_1+\rho_2R';0,0,\rho_2 q,\mu,v),(\sigma_1;\sigma_1',\sigma_2';0,0,0):(c_1;1);(c_1';1);(v-n\delta;1);(\omega_1+k;1);(-\rho k;1)\\ \cdot\\ \cdot\\ \cdot\\ (\rho_1+\sigma_1+\rho_2R';\sigma_1',\sigma_2',q,\mu,v),(b_1;1,1,0,0,0)\end{array}\right.\right) \tag{4.1}$$

provided that

$$\left|(\mu_1-x)^{\sigma_1'}\right|<1, \left|(\mu_1-x)^{\sigma_2'}\right|<1,$$

where

$$\zeta' v,u,t,e) = \frac{\Gamma(\rho_1+\rho_2R')[y(\mu_1-\lambda_1)^{\rho_2}]^{R'}}{\Gamma(\rho_1+\sigma_1+\rho_2R')}\frac{\Gamma(\omega_1+k)(-Bq_1^{\rho})^k}{k!}\frac{b^{n\gamma}K^n(-)^{t+v}(-v)_u(-t)_e(\alpha)_t\tau^v(a/b)^t(-\alpha-n\gamma)_e}{v!u!t!e!(1-\alpha-t)_e}$$

the validity conditions (3.11) of are verified.

2) In the first integral, if we reduce multivariable Gimel-function to the product of two fox H-function and then reduce one H-function to ${}_2F_1$ ([9], p.19, Eq. (2.6.8)) and the second H-function to generalized Wright's bessel function ([9], p.19, Eq.(2.6.10)) and taking $A=\rho_2=\rho_1'=\omega_1'=\rho_2'=\sigma_2'=0$, we obtain the following integral

Corollary 2.

$$\int_{\lambda_1}^{\mu_1}(x-\lambda_1)^{\rho_1-1}(\mu_1-x)^{\sigma_1-1}\left[1-A(x-\lambda_1)^{\mu}+B[p_1(x-\lambda_1)^{\upsilon}+q_1]^{\rho}\right]^{-\omega_1}$$

$$R_n^{(\alpha,\beta)}\left(y(\mu_1-x)^{\sigma_2}[1+B[p_1(x-\lambda_1)^{\upsilon}+q_1]^{\rho}]^{-\omega_2}\,{}_2F_1(c_1,c_1';d_1;-z_1(\mu_1-x)^{\sigma_1'}\right)$$

$$J_{\lambda'}^{\upsilon'}(z_2[1+B[p_1(x-\lambda_1)^{\upsilon}+q_1]^{\rho}]^{-\omega_2'})\mathrm{d}x=(\mu_1-\lambda_1)^{\rho_1+\sigma_1-1}\frac{\Gamma(\rho_1)}{\Gamma(1+\lambda')}\sum_{\upsilon,u,t,e}\sum_{k=0}^{\infty}\zeta_2(\upsilon,u,t,e)$$

$$F_{2:1;1;0;0}^{2:2;0;1;2}\left(\begin{array}{c}-z_1(u-\lambda_1)^{\sigma_1'}\\ -z_2\\ -\frac{c}{d}y^q(\mu_1-\lambda_1)^{\sigma_2 q}\\ -\frac{p_1}{q_1}(\mu_1-\lambda_1)^{\upsilon}\end{array}\left|\begin{array}{c}(\omega_1+\omega_2R'+k;0,\omega_2',\omega_2q,0),(\sigma_1+\sigma_2R';\sigma_1',0,\sigma_2q,0):(c_1;1);(c_1';1);-;(\upsilon-n\delta;1);(-\rho k;1),(\rho_1;1)\\ \cdot\\ \cdot\\ (\omega_1+\omega_2R';0,\omega_2',\omega_2q,0),(\rho_1+\sigma_1+\sigma_2R';\sigma_1',0,\sigma_2q,\upsilon):(d_1;1);(1+\lambda',\upsilon')\end{array}\right.\right)\quad(4.2)$$

the validity conditions of (3.11) are verified.

where

$$\zeta_2(\upsilon,u,t,e)=\frac{\Gamma(\omega_1+\omega_2R'+k)\Gamma(\sigma_1+\sigma_2R')[y(\mu_1-\lambda_1)^{\rho_2}]^{R'}}{\Gamma(\omega_1+\omega_2R')\Gamma(\rho_1+\sigma_1+\rho_2R')}\theta_1(\upsilon,u,t,e)\frac{(-Bq_1^{\rho})^k}{k!}$$

Remark :
We obtain the same Eulerian integrals with the functions defined in section I.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in this Eulerian integral, we can obtain a large simpler single finite integrals, Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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