Geraghty Type Contraction and Common Coupled Fixed Point Theorems in Bipolar Metric Spaces with Applications to Homotopy

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Abstract

In this paper, we establish the existence of common coupled fixed point results of two covariant mappings in a complete bipolar metric spaces under Geraghty type contraction by using weakly compatible mappings with an example. We have also provided an applications to Homotopy theory.

Keywords - *Bipolar metric space, weakly-compatible mappings, common coupled fixed point.*

I. INTRODUCTION

This work is motivated by the recent work on extension of Banach contraction principle on Bipolar metric spaces, which has been done by Mutlu and Gürdal [1]. Also they investigated some fixed point and coupled fixed point results on this spaces (see [1], [2]). Later, we proved some fixed point theorems in our earlier papers (see [3], [4]). Subsequently, many authors established coupled fixed point theorems in different spaces (see [5]-[8]).

In 1982, Sessa [9] initiated to studied common fixed point theorems for weakly commuting pair of mappings. Afterward, in 1986, Jungck [10] introduced concept of weakened weakly commuting mappings to compatible mappings in metric spaces and established compatible pair of mappings commute on the sets of coincidence point of the involved mappings. In 1998, the weak compatibility notion initiated by Jungck and Rhoades [11], if they commute at their coincidence points and proved that compatible mappings are weakly compatible but the converse does not hold. In 1973, Geraghty ([12]) introduced a generalization of Banach contraction principle ([13]) in which the contraction constant was replaced by a function having some specified properties. Later, many authors refereed it to Geraghty type fixed point results and extended different types of distance spaces (see [14],[15]).

The aim of this paper is to initiate the study of a common coupled fixed point results for two covariant mappings under Geraghty type contractive conditions in bipolar metric spaces. We also given illustrate the validity of the hypotheses of our results.

Definition 1.1: ([1]) Let A, B be two non-empty sets. Suppose that d: A×B → [0, ∞) be a mapping satisfying the below properties:

(B_1) If d (a, b) = 0, if and only if a=b for all (a, b) ∈ A×B,

(B_2) If d (a, b) =d (b, a), for all a, b ∈ A∩B

(B_3) If $d(a_1, b_2) \leq d(a_1, b_1) + d(a_2, b_1) + d(a_2, b_2)$ for all $a_1, a_2 \in A$, and $b_1, b_2 \in B$.

Then the mapping d is termed as Bipolar-metric of the pair (A, B) and the triple (A, B, d) is termed as Bipolar-metric space.

Definition 1.2: ([1]) Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets and a function as F: $A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a covariant map. If F $(A_1) \subseteq A_2$ and F $(B_1) \subseteq B_2$, and denote this with F: $(A_1, B_1) \rightrightarrows (A_2, B_2)$. And the mapping F: $A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a contravariant map. If F $(A_1) \subseteq B_2$, and F $(B_1) \subseteq A_2$, and write F: $(A_1, B_1) \rightleftarrows (A_2, B_2)$. In particular, if d_1 and d_2 are bipolar metric on (A_1, B_1) and (A_2, B_2) respectively, we sometimes use the notation F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and F: $(A_1, B_1, d_1) \rightleftarrows (A_2, B_2, d_2)$.

Definition 1.3: *([1])* Assume (A, B, d) as a bipolar metric space. A point v ∈A ∪B is termed as a left point if v ∈ A, a right point if v ∈ B and a central point if both. Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left sequence and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence. A sequence $\{v_n\}$ is considered convergent to a point v, if and only if $\{v_n\}$ is the left sequence, v is the right point and $\lim_{n\to\infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$. A bi-sequence $(\{a_n\}, \{b_n\})$ on (A, B, d) is a sequence on the set A ×B. If the sequence $\{a_n\}$ and $\{b_n\}$ are convergent, then the bi-sequence $(\{a_n\}, \{b_n\})$ is said to be convergent. $(\{a_n\}, \{b_n\})$ is Cauchy sequence, if $\lim_{n\to\infty} d(a_n, b_n) = 0$. In a bipolar metric space, every convergent Cauchy bi-sequence is bi-convergent. A bipolar metric space is called complete, if every Cauchy bi-sequence is convergent hence bi-convergent.

Definition 1.4: *([1])* Let (A_1, B_1, d_1) *and* (A_2, B_2, d_2) be bipolar metric spaces.
(i) A map F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ is called left-continuous at a point $a_0 \in A_1$, if for every $\epsilon>0$, there is a $\delta>0$ such that $d_1(a_0, b) < \delta$ implies that $d_2(F(a_0), F(b)) < \epsilon$ for all b∈ B_1.
(ii) A map F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ is called right-continuous at a point $b_0 \in B_1$, if for every $\epsilon>0$, there is a $\delta>0$ such that $d_1(a, b_0) < \delta$ implies $d_2(F(a), F(b_0)) < \epsilon$ for all a ∈ A_1.
(iii) A map F is considered continuous, if it left continuous at each point a∈ $A_1$and righty continuous at each point b∈ B_1.
(iv) A contravariant map F: $(A_1, B_1, d_1) \rightleftarrows (A_2, B_2, d_2)$ is continuous if and only if F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ it is continuous as a covariant map.
It is observed from the definition (1.3) that a contravariant or a covariant map F from(A_1, B_1, d_1) to (A_2, B_2, d_2) is continuous if and only if $(u_n) \to$ v on (A_1, B_1, d_1) implies $F((u_n)) \to F(v)$ on (A_2, B_2, d_2).

II. MAIN RESULTS

In this section, we give some common coupled fixed point theorems for two covariant mappings satisfying Geraghty type contractive conditions using weakly compatible property in bipolar metric spaces.

Definition 2.1*:* Let (A, B, d) be a bipolar metric space, F: $(A^2, B^2) \rightrightarrows (A, B)$ and S:(A, B) $\rightrightarrows (A, B)$be two covariant mappings.

(i) If F(a, b)=a and F(b, a)=b for (a, b)∈ $A^2 \cup B^2$ then (a, b) is called a coupled fixed point of F.
(ii) If F(a, b)=Sa and F(b, a)=Sb for (a, b)∈ $A^2 \cup B^2$ then (a, b) is called a coupled coincidence point of F and S.
(iii) If F(a, b)=Sa=a and F(b, a)=Sb=b for (a, b)∈ $A^2 \cup B^2$ then (a, b) is called a common coupled point of F and S.

Definition 2.2*:* Let (A, B, d) be a bipolar metric space, F: $(A^2, B^2) \rightrightarrows (A, B)$ and S:(A, B) $\rightrightarrows (A, B)$be two covariant mappings. Then

(i) (F, S) is called weakly compatible if S(F(a, b))=F(Sa, Sb) and S(F(b, a))=F(Sb, Sa) whenever for all (a, b)∈ $A^2 \cup B^2$such that F(a, b)=Sa and F(b, a)=Sb.
(ii) (F, S) is called compatible if $\lim_{n\to\infty} d(F(Sa_n, Sb_n), SF(p_n, q_n)) = \lim_{n\to\infty} d(SF(a_n, b_n), F(Sp_n, Sq_n)) = 0$ and $\lim_{n\to\infty} d(F(Sb_n, Sa_n), SF(q_n, p_n)) = \lim_{n\to\infty} d(SF(b_n, a_n), F(Sq_n, Sp_n)) = 0$ for bi-sequences $(\{a_n\}, \{p_n\})$ and $(\{b_n\}, \{q_n\})$ in (A, B) such that
$\lim_{n\to\infty} F(a_n, b_n) = \lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Sp_n = \lim_{n\to\infty} F(p_n, q_n)$ and
$\lim_{n\to\infty} F(b_n, a_n) = \lim_{n\to\infty} Sb_n = \lim_{n\to\infty} Sq_n = \lim_{n\to\infty} F(q_n, p_n)$

Lemma 2.3: Let (A, B, d) be a bipolar metric space, F: $(A^2, B^2) \rightrightarrows (A, B)$ and S:(A, B) $\rightrightarrows (A, B)$be two covariant mappings. If (F, S) is compatible then (F, S) is ω-compatible.
Proof. Let F(a, b)=Sa, F(b, a)=Sb and F(p, q)=Sp, F(q, p)=Sq for some a, b ∈A, p, q∈B. Consider the constant sequences $a_n \cong p$, $b_n \cong q$ and $p_n \cong a$, $q_n \cong b$ for all n∈N.
It is obvious that$F(a_n, b_n) = Sa_n \to Sp$, $F(b_n, a_n) = Sb_n \to Sq$ as n→ ∞ and$F(p_n, q_n) = Sp_n \to Sa$, $F(q_n, p_n) = Sq_n \to Sb$ as n→ ∞. Since (F, S) is compatible, $d(F(Sa_n, Sb_n), SF(p_n, q_n)) \to 0$,

$d(SF(a_n, b_n), F(Sp_n, Sq_n)) \to 0$ and $d(F(Sb_n, Sa_n), SF(q_n, p_n)) \to 0$, $d(SF(b_n, a_n), F(Sq_n, Sp_n)) \to 0$.
Thus SF(p, q)= F(Sa, Sb), F(Sp, Sq)=SF(a, b) and SF(q, p)= F(Sb, Sa), F(Sq, Sp)=SF(b, a).
On the other hand
d(a, p)=d$\left(\lim_{n\to\infty} p_n, \lim_{n\to\infty} a_n\right)$=$\lim_{n\to\infty} d(a_n, p_n)$=0
and
d(b, q)=d$\left(\lim_{n\to\infty} q_n, \lim_{n\to\infty} b_n\right)$=$\lim_{n\to\infty} d(b_n, q_n)$=0. Hence (F, S) is weakly compatible. But converse is need not be true. For example, let A=(0, ∞) and B=[-1, 1], define d:A×B→[0, ∞) as d(a, b)=$|a^2 - b^2|$ for all (a, b)$\in A \times B$.
Then obviously, (A, B, d) is bipolar metric space.
Define two covariant mappings F:$A^2 \cup B^2$ →A∪B and S: A ∪B→ A ∪B as follows;

$$F(a, b)= \begin{cases} 1-2a+2b, & a \in \left[\frac{1}{2}, 2\right], \ b \in \left[\frac{1}{2}, 5\right] \\ \frac{1}{2}, & a \in (2,\infty), b \in (-1, 2] \end{cases}, \qquad Sa= \begin{cases} a, & a \in \left[\frac{1}{2}, 2\right] \\ \frac{1}{2}, & a \in (2, \infty) \end{cases}$$

and

$$F(p, q)= \begin{cases} 1-p+q, & p \in \left[-1, \frac{1}{2}\right), \ q \in \left[\frac{1}{2}, 1\right] \\ \frac{1}{2}, & p \in \left[\frac{1}{2}, \infty\right), q \in [0, \frac{1}{2}] \end{cases}, \qquad Sp= \begin{cases} \frac{p+1}{2}, & p \in [-1, 0] \\ \frac{1}{2}, & p \in [\frac{1}{2}, \infty) \end{cases}$$

Now we define the bi-sequences (a_n, p_n) and (b_n, q_n) as a_n=1+$\frac{1}{n}$, b_n=1+$\frac{4}{3n}$ and p_n=1-$\frac{1}{n}$, q_n=1-$\frac{1}{2n}$,then
$F(a_n, b_n)$= 1+$\frac{2}{3n} \to 1$ as n→ ∞, Sa_n=1+$\frac{1}{n} \to 1$ as n→ ∞, $F(b_n, a_n)$=$\frac{1}{2}$, Sb_n=$\frac{1}{2}$
and $F(p_n, q_n)$= 1+$\frac{}{2n} \to 1$ as n→ ∞, Sp_n=1-$\frac{1}{2n} \to 1$ as n→ ∞, $F(q_n, p_n)$=$\frac{1}{2}$, Sq_n=$\frac{1}{2}$.
But $\lim_{n\to\infty} d(F(Sa_n, Sb_n), SF(p_n, q_n))$=$\lim_{n\to\infty} d(F\left(1+\frac{1}{n}, \frac{1}{2}\right), S(1+\frac{1}{2n}))$=$\lim_{n\to\infty} d(\frac{-2}{n}, \frac{1}{2})$=$\frac{1}{4} \neq 0$,
$\lim_{n\to\infty} d(SF(a_n, b_n), F(Sp_n, Sq_n))$=$\lim_{n\to\infty} d\left(S\left(1+\frac{2}{3n}\right), F(1-\frac{1}{2n}, \frac{1}{2})\right)$=$\lim_{n\to\infty} d(\frac{1}{2}, \frac{1}{2})$=0,
also, we have $\lim_{n\to\infty} d(F(Sb_n, Sa_n), SF(q_n, p_n))$=$\lim_{n\to\infty} d(F\left(\frac{1}{2}, 1+\frac{1}{n}\right), S(\frac{1}{2}))$=$\lim_{n\to\infty} d(2+\frac{2}{n}, \frac{1}{2})$=$\frac{15}{4} \neq 0$,
$\lim_{n\to\infty} d(SF(b_n, a_n), F(Sq_n, Sp_n))$=$\lim_{n\to\infty} d\left(S\left(\frac{1}{2}\right), F(\frac{1}{2}, 1-\frac{1}{2n})\right)$=$\lim_{n\to\infty} d(\frac{1}{2}, \frac{1}{2})$=0.
Thus the pair (F, S) is not compatible. Also, the coupled coincidence point of F and S is $(\frac{2}{3}, \frac{1}{2})$. It is namely that a=p=$\frac{2}{3}$ and b=q=$\frac{1}{2}$, F(a, b)=1-2a+2b=1-$\frac{4}{3}$+1=$\frac{2}{3}$=Sa , F(b, a)=Sb=$\frac{1}{2}$ and F(p, q)=1-p+q=1-$\frac{2}{3}+\frac{1}{2}=\frac{5}{6}=\frac{\frac{2}{3}+1}{2}$ =Sp ,
F(q, p)=Sq=$\frac{1}{2}$. Then we get SF(a, b)=S($\frac{2}{3}$)= $\frac{2}{3}$=F(Sa, Sb), meanwhile SF(b, a)=S($\frac{1}{2}$)= $\frac{1}{2}$=F(Sb, Sa). Which implies that (F, S) is weakly compatible but not compatible.

Let Θ= $\{\theta: \theta: [0, \infty) \times: [0, \infty) \to$[0, 1)} be the class of functions, which satisfies the following conditions.

(i) θ(a, b)= θ(b, a) for all a, b $\in [0, \infty)$

(ii) For any two sequences $\{a_n\}$ and $\{b_n\}$ of non-negative real numbers θ $(a_n, b_n) \to 1 \Rightarrow a_n, b_n \to 0$.

Theorem 2.4: Let (A, B, d) be a complete bipolar metric space, F: (A^2, B^2) ⇉(A, B) and S:(A, B) ⇉ (A, B)be two covariant mappings satisfying the following conditions

(ψ_0) d$(F(a,b), F(p,q)) \leq \theta(d(Sa, Sp), d(Sb, Sq))$ max{d(Sa, Sp), d(Sb, Sq)}
Where $\theta \in$ Θand a, b$\in A$, p, q$\in B$.

(ψ_1) F($A^2 \cup B^2$)⊆S(A∪B).

(ψ_2) The pair (F, S) is compatible.

(ψ_4) S is continuous.

Then the mappings F: $A^2 \cup B^2 \to A \cup B$ and S: A∪B→ A∪B have unique common fixed point.

Proof. Let a_0, b_0 ∈A and p_0, q_0 ∈B and from (ψ_1), we construct the bi-sequences ($\{a_n\}$,$\{p_n\}$), ($\{b_n\}$,$\{q_n\}$), ($\{\omega_n\}$,$\{\chi_n\}$) and ($\{\xi_n\}$,$\{\kappa_n\}$) in (A, B) as

F(a_n, b_n)=Sa_{n+1}=ω_n, F(p_n, q_n)=Sp_{n+1}=χ_n,
F(b_n, a_n)=Sb_{n+1}=ξ_n, F(q_n, p_n)=Sq_{n+1}=κ_n, for n=0, 1,2, 3…

Now from (ψ_0), we have
d(ω_n, χ_{n+1})=d(F(a_n, b_n), F(p_{n+1}, q_{n+1}))
$\leq \theta(d(Sa_n, Sp_{n+1}), d(Sb_n, Sq_{n+1}))$ max{d(Sa_n, Sp_{n+1}), d(Sb_n, Sq_{n+1})}
< max{d(Sa_n, Sp_{n+1}), d(Sb_n, Sq_{n+1})}
= max{ d(ω_{n-1}, χ_n), d(ξ_{n-1}, κ_n) }   (1)
and
d(ξ_n, κ_{n+1})=d(F(b_n, a_n), F(q_{n+1}, p_{n+1}))
$\leq \theta\left(d(Sb_n, Sq_{n+1}), d(Sa_n, Sp_{n+1})\right)$ max{ d(Sb_n, Sq_{n+1}), $d(Sa_n, Sp_{n+1})$}

$< \max\{ d(Sb_n, Sq_{n+1}), d(Sa_n, Sp_{n+1})\}$
$= \max\{ d(\xi_{n-1}, \kappa_n), d(\omega_{n-1}, \chi_n)\}$ (2)

Combining (1) and (2), we get
$\frac{\max\{d(\omega_n,\chi_{n+1}), d(\xi_n,\kappa_{n+1})\}}{\max\{d(\omega_{n-1},\chi_n), d(\xi_{n-1},\kappa_n)\}} \leq \theta(d(\omega_{n-1},\chi_n), d(\xi_{n-1},\kappa_n))<1.$ (3)

Letting $n\to\infty$, it follows that $\theta(d(\omega_{n-1},\chi_n), d(\xi_{n-1},\kappa_n))\to 1.$ (4)

By the property of $\theta\in\Theta$, we obtain $d(\omega_{n-1},\chi_n)\to 0$ and $d(\xi_{n-1},\kappa_n)\to 0$ as $n\to\infty$.

Therefore, $\max\{ d(\omega_{n-1}, \chi_n), d(\xi_{n-1}, \kappa_n) \}\to 0$ as $n\to\infty$. (5)

On the other hand

$d(\omega_{n+1}, \chi_n)=d(F(a_{n+1}, b_{n+1}), F(p_n, q_n))$
$\leq \theta(d(Sa_{n+1},Sp_n), d(Sb_{n+1},Sq_n)) \max\{d(Sa_{n+1}, Sp_n), d(Sb_{n+1}, Sq_n)\}$
$< \max\{d(Sa_{n+1}, Sp_n), d(Sb_{n+1}, Sq_n)\}$
$= \max\{ d(\omega_n, \chi_{n-1}), d(\xi_n, \kappa_{n-1}) \}$ (6)

and

$d(\xi_{n+1}, \kappa_n)=d(F(b_{n+1}, a_{n+1}), F(q_n, p_n))$
$\leq \theta(d(Sb_{n+1},Sq_n), d(Sa_{n+1},Sp_n)) \max\{ d(Sb_{n+1}, Sq_n), d(Sa_{n+1},Sp_n)\}$
$< \max\{ d(Sb_{n+1}, Sq_n), d(Sa_{n+1},Sp_n)\}$
$= \max\{ d(\xi_n, \kappa_{n-1}), d(\omega_n, \chi_{n-1})\}$ (7)

Combining (6) and (7), we get
$\frac{\max\{d(\omega_{n+1},\chi_n), d(\xi_{n+1},\kappa_n)\}}{\max\{d(\omega_n,\chi_{n-1}), d(\xi_n,\kappa_{n-1})\}} \leq \theta(d(\omega_n,\chi_{n-1}), d(\xi_n,\kappa_{n-1}))<1.$ (8)

Letting $n\to\infty$, it follows that $\theta(d(\omega_n,\chi_{n-1}), d(\xi_n,\kappa_{n-1}))\to 1.$ (9)

By the property of $\theta\in\Theta$, we obtain $d(\omega_n,\chi_{n-1})\to 0$ and $d(\xi_n,\kappa_{n-1})\to 0$ as $n\to\infty$.

Therefore, $\max\{ d(\omega_n, \chi_{n-1}), d(\xi_n, \kappa_{n-1}) \}\to 0$ as $n\to\infty$. (10)

Moreover,

$d(\omega_n, \chi_n)=d(F(a_n, b_n), F(p_n, q_n))$
$\leq \theta(d(Sa_n,Sp_n), d(Sb_n,Sq_n)) \max\{d(Sa_n, Sp_n), d(Sb_n, Sq_n)\}$
$< \max\{d(Sa_n, Sp_n), d(Sb_n, Sq_n)\}$
$= \max\{ d(\omega_{n-1}, \chi_{n-1}), d(\xi_{n-1}, \kappa_{n-1}) \}$ (11)

and

$d(\xi_n, \kappa_n)=d(F(b_n, a_n), F(q_n, p_n))$
$\leq \theta(d(Sb_n,Sq_n), d(Sa_n,Sp_n)) \max\{ d(Sb_n, Sq_n), d(Sa_n,Sp_n)\}$
$< \max\{ d(Sb_n, Sq_n), d(Sa_n,Sp_n)\}$
$= \max\{ d(\xi_n, \kappa_{n-1}), d(\omega_n, \chi_{n-1})\}$ (12)

Combining (11) and (12), we get
$\frac{\max\{d(\omega_n,\chi_n), d(\xi_n,\kappa_n)\}}{\max\{d(\omega_{n-1},\chi_{n-1}), d(\xi_{n-1},\kappa_{n-1})\}} \leq \theta(d(\omega_{n-1},\chi_{n-1}), d(\xi_{n-1},\kappa_{n-1}))<1.$ (13)

Letting $n\to\infty$, it follows that $\theta(d(\omega_{n-1},\chi_{n-1}), d(\xi_{n-1},\kappa_{n-1}))\to 1.$ (14)

By the property of $\theta\in\Theta$, we obtain $d(\omega_{n-1},\chi_{n-1})\to 0$ and $d(\xi_{n-1},\kappa_{n-1})\to 0$ as $n\to\infty$.

Therefore, $\max\{ d(\omega_{n-1}, \chi_{n-1}), d(\xi_{n-1}, \kappa_{n-1}) \}\to 0$ as $n\to\infty$. (15)

Now we shall show $(\{\omega_n\},\{\chi_n\})$ and $(\{\xi_n\},\{\kappa_n\})$ are Cauchy bi-sequences in (A, B).Suppose to the contrary that $(\{\omega_n\},\{\chi_n\})$ and $(\{\xi_n\},\{\kappa_n\})$ are not Cauchy bi-sequences. Then there exists $\epsilon>0$, for which we can find sub sequences $\{\omega_{n_k}\}$, $\{\omega_{m_k}\}$ of $\{\omega_n\}$, $\{\chi_{n_k}\}$, $\{\chi_{m_k}\}$ of $\{\chi_n\}$ and $\{\xi_{n_k}\}$, $\{\xi_{m_k}\}$ of $\{\xi_n\}$, $\{\kappa_{n_k}\}$, $\{\kappa_{m_k}\}$ of $\{\kappa_n\}$ with $n_k>m_k\geq k$ such that

$R_k=\max\{ d(\omega_{n_k},\chi_{m_k}), d(\xi_{n_k},\kappa_{m_k}) \}\geq \epsilon$
$\max\{ d(\omega_{n_{k-1}},\chi_{m_k}), d(\xi_{n_{k-1}},\kappa_{m_k}) \}< \epsilon$ (16)

and

$r_k=\max\{ d(\omega_{m_k},\chi_{n_k}), d(\xi_{m_k},\kappa_{n_k}) \}\geq \epsilon$
$\max\{ d(\omega_{m_{k-1}},\chi_{n_k}), d(\xi_{m_{k-1}},\kappa_{n_k}) \}< \epsilon.$ (17)

By view of (16) and triangle inequality, we get

$\epsilon \leq R_k=\max\{ d(\omega_{n_k},\chi_{m_k}), d(\xi_{n_k},\kappa_{m_k}) \}$
$< \max\{ d(\omega_{n_k},\chi_{n_{k-1}}), d(\xi_{n_k},\kappa_{n_{k-1}}) \}+ \max\{ d(\omega_{n_{k-1}},\chi_{n_{k-1}}), d(\xi_{n_{k-1}},\kappa_{n_{k-1}}) \}$
$+ \max\{ d(\omega_{n_{k-1}},\chi_{m_k}), d(\xi_{n_{k-1}},\kappa_{m_k}) \}$
$< \max\{ d(\omega_{n_k},\chi_{n_{k-1}}), d(\xi_{n_k},\kappa_{n_{k-1}}) \}+ \max\{ d(\omega_{n_{k-1}},\chi_{n_{k-1}}), d(\xi_{n_{k-1}},\kappa_{n_{k-1}}) \}+\epsilon$

Letting $k\to\infty$, we obtain

$R_k=\max\{ d(\omega_{n_k},\chi_{m_k}), d(\xi_{n_k},\kappa_{m_k}) \}\to \epsilon$ (18)

Again by means of triangle inequality, we have

$R_k=\max\{ d(\omega_{n_k},\chi_{m_k}), d(\xi_{n_k},\kappa_{m_k}) \}$

$< \max\{ d(\omega_{n_k},\chi_{n_{k+1}}), d(\xi_{n_k},\kappa_{n_{k+1}}) \}+ \max\{ d(\omega_{n_{k+1}},\chi_{n_{k+1}}), d(\xi_{n_{k+1}},\kappa_{n_{k+1}}) \}$
$+ \max\{ d(\omega_{n_{k+1}},\chi_{m_k}), d(\xi_{n_{k+1}},\kappa_{m_k}) \}$
$< \max\{ d(\omega_{n_k},\chi_{n_{k+1}}), d(\xi_{n_k},\kappa_{n_{k+1}}) \}+ \max\{ d(\omega_{n_{k+1}},\chi_{n_{k+1}}), d(\xi_{n_{k+1}},\kappa_{n_{k+1}}) \}$
$+\theta\big(d(\omega_{n_k},\chi_{m_{k-1}}), d(\xi_{n_k},\kappa_{m_{k-1}})\big) \max\{ d(\omega_{n_k},\chi_{m_{k-1}}), d(\xi_{n_k},\kappa_{m_{k-1}}) \}.$

Letting n→ ∞, it yields that $\theta\big(d(\omega_{n_k},\chi_{m_{k-1}}), d(\xi_{n_k},\kappa_{m_{k-1}})\big) \to 1$. Now by means of property of θ. It follows $d(\omega_{n_k},\chi_{m_{k-1}}) \to 0$ and $d(\xi_{n_k},\kappa_{m_{k-1}}) \to 0$ as k→ ∞, which implies that

$\lim_{k\to\infty} R_k= \lim_{k\to\infty} \max\{ d(\omega_{n_k},\chi_{m_k}), d(\xi_{n_k},\kappa_{m_k}) \}=0.$ (19)

Similarly, we can prove

$\lim_{k\to\infty} r_k= \lim_{k\to\infty} \max\{ d(\omega_{m_k},\chi_{n_k}), d(\xi_{m_k},\kappa_{n_k}) \}=0.$ (20)

Which are contracts with (16) and (17). Thus $(\{\omega_n\},\{\chi_n\})$ and $(\{\xi_n\},\{\kappa_n\})$ are Cauchy bi-sequences in (A, B). Therefore, $\lim_{n\to\infty} (\omega_n, \chi_m)=\lim_{n\to\infty} (\xi_n, \kappa_m)=0$. Since (A, B, d) is complete, there exist u, v∈ A and w, z∈ B with

$\lim_{n\to\infty} \omega_n=\lim_{n\to\infty} F(a_n, b_n)=\lim_{n\to\infty} Sa_{n+1}=w,$ $\lim_{n\to\infty} \xi_n=\lim_{n\to\infty} F(b_n,a_n)=\lim_{n\to\infty} Sb_{n+1}=z,$

$\lim_{n\to\infty} \chi_{n+1}=\lim_{n\to\infty} F(p_n, q_n)=\lim_{n\to\infty} Sp_{n+1}=u,$ $\lim_{n\to\infty} \kappa_{n+1}=\lim_{n\to\infty} F(q_n,p_n)=\lim_{n\to\infty} Sq_{n+1}=v.$

Since S:A∪B→ A∪B is continuous,

$\lim_{n\to\infty} S^2a_{n+1} =Sw,$ $\lim_{n\to\infty} S^2b_{n+1} =Sz,$ $\lim_{n\to\infty} SF(a_n,b_n) =Sw,$ $\lim_{n\to\infty} SF(b_n,a_n) =Sz$

and

$\lim_{n\to\infty} S^2p_{n+1} =Su,$ $\lim_{n\to\infty} S^2q_{n+1} =Sv,$ $\lim_{n\to\infty} SF(p_n,q_n) =Su,$ $\lim_{n\to\infty} SF(q_n,p_n) =Sv.$

Since (F, S) is compatible, we have

$F(Sa_n,Sb_n) \to Sw,$ $F(Sb_n,Sa_n) \to Sz$ and $F(Sp_n,Sq_n) \to Su,$ $F(Sq_n,Sp_n) \to Sv.$

Putting (a, b)= (Sa_n,Sb_n), (p, q)= (Sp_n,Sq_n) in the inequality (ψ_0), we have

$d(F(Sa_n,Sb_n), F(Sp_n,Sq_n))\leq \theta(d(SSa_n, SSp_n), d(SSb_n, SSq_n)) \max\{ d(SSa_n, SSp_n), d(SSb_n, SSq_n)\}$
$< \max\{ d(SSa_n, SSp_n), d(SSb_n, SSq_n)\}.$

Letting n→ ∞, we obtain d(Sw, Su)< max{ d(Sw, Su), d(Sz, Sv) } (21)

and putting (a, b)= (Sb_n,Sa_n), (p, q)= (Sq_n,Sp_n) in the inequality (ψ_0), we have

$d(F(Sb_n,Sa_n), F(Sq_n,Sp_n))\leq \theta(d(SSb_n, SSq_n), d(SSa_n, SSp_n)) \max\{ d(SSb_n, SSq_n), d(SSa_n, SSp_n)\}$
$< \max\{ d(SSb_n, SSq_n), d(SSa_n, SSp_n)\}$

Letting n→ ∞, we obtain d(Sz, Sv)< max{ d(Sz, Sv) , d(Sw, Su) }. (22)

Combining with inequalies (21) and (22), we get

max{ d(Sw, Su) , d(Sz, Sv) }< max{ d(Sw, Su) , d(Sz, Sv) }. It is contraction. Therefore, Sw=Su and Sz=Sv.

Now we shall show Sw=F(w, z), Sz=F(z, w) and Su=F(u, v), Sv=F(v, u).

From (ψ_0), we have

$d(F(u, v), Sw)\leq d(F(u, v), Sp_{n+1})+ d(Sa_{n+1}, Sp_{n+1})+ d(Sa_{n+1}, Sw)$
$\leq d(F(u, v), F(p_n, q_n))+ d(Sa_{n+1}, Sp_{n+1})+ d(Sa_{n+1}, Sw)$
$\leq \theta(d(Su, Sp_n), d(Sv, Sq_n)) \max \{ d(Su, Sp_n), d(Sv, Sq_n)\}+ d(Sa_{n+1}, Sp_{n+1})+ d(Sa_{n+1}, Sw)$
$< \max \{ d(Su, Sp_n), d(Sv, Sq_n)\}+ d(Sa_{n+1}, Sp_{n+1})+ d(Sa_{n+1}, Sw)\to 0$ as n→ ∞.

Therefore, d(F(u, v), Sw)=0 implies F(u, v)=Sw.

Similarly, we can show F(v, u)=Sz, F(w, z)=Su and F(z, w)=Sv.

Analogously, we can also obtain that F(u, v)=Sw=Su=F(w, z) and F(v, u)=Sz=Sv=F(z, w).

Now we prove Su=u, Sv=v and Sw=w, Sz=z.

Now consider,

$d(Su, \chi_n)=d(F(u, v), F(p_n, q_n))\leq \theta(d(Su, Sp_n), d(Sv, Sq_n))\max\{ d(Su, Sp_n), d(Sv, Sq_n)\}$
$< \max\{ d(Su, Sp_n), d(Sv, Sq_n)\}.$

Letting n → ∞, we get d(Su, u)<max{d(Su, u), d(Sv, v)} (23)

and similarly, we shall show d(Sv, v)<max{d(Sv, v), d(Su, u)} (24)

Therefore, from (23) and (24), we get max{d(Su, u), d(Sv, v)}< max{d(Su, u), d(Sv, v)}, which implies that Su=u and Sv=v. Similarly, we can prove Sw=w and Sz=z.

Therefore,

F(w, z)=w=Sw=Su=u=F(u, v) and F(z, w)=z=Sz=Sv=v=F(v, u), therefore, (u, v) $A^2 \cap B^2$ is coupled fixed point of covariant mappings F and S.

Now we prove uniqueness, we begin by taking $(u^*, v^*) \in A^2 \cup B^2$ be another coupled fixed point of F and S. If $(u^*, v^*) \in A^2$, then we have

$d(u, u^*)=d(F(u, v), F(u^*, v^*))$
$\leq \theta(d(Su, Su^*), d(Sv, Sv^*)) \max\{ d(Su, Su^*), d(Sv, Sv^*) \}$
$< \max\{ d(Su, Su^*), d(Sv, Sv^*) \}$

$= \max\{ d(u, u^*), d(v, v^*) \}$ (25)

and

$d(v, v^*)=d(F(v, u), F(v^*, u^*))$

$\leq \theta(d(Sv, Sv^*), d(Su, Su^*)) \max\{d(Sv, Sv^*), d(Su, Su^*)\}$

$< \max\{d(Sv, Sv^*), d(Su, Su^*)\}$

$= \max\{d(v, v^*), d(u, u^*)\}$ (26)

Combining (25) and (26), we get $\max\{ d(u, u^*), d(v, v^*) \} < \max\{ d(u, u^*), d(v, v^*) \}$.
Therefore, $d(u, u^*)=0$ and $d(v, v^*) = 0$ implies that $u= u^*$ and $v = v^*$. Similarly, if $(u^*, v^*) \in B^2$, then we have $u= u^*$ and $v = v^*$.Then (u, v) $A^2 \cap B^2$ is unique common coupled fixed point of covariant mappings F and S.
Finally, we will show u=v.

$d(u, v)=d(F(u, v), F(v, u)) \leq \theta(d(Su, Sv), d(Sv, Su))\max\{ d(Su, Sv), d(Sv, Su)\}$

$< \max\{ d(Su, Sv), d(Sv, Su)\}$

$= \max\{ d(u, v), d(v, u)\}$ (27)

and

$d(v, u)=d(F(v, u), F(u, v)) \leq \theta(d(Sv, Su), d(Su, Sv))\max\{ d(Sv, Su), d(Su, Sv)\}$

$< \max\{ d(Sv, Su), d(Su, Sv)\}$

$= \max\{ d(v, u), d(u, v), \}$ (28)

Combining (27) and (28), we get $\max\{ d(u, v), d(v, u)\} < \max\{ d(u, v), d(v, u)\}$.
Therefore, d(u, v)=0 implies that u=v. Hence (u, u) is common fixed point of F and S.

Corollary 2.5: Let (A, B, d) be a complete bipolar metric space, F: $(A^2, B^2) \rightrightarrows (A, B)$ be a covariant mapping satisfying the condition

$$d(F(a, b),\ F(p, q)) \leq \theta(d(a, p),\ d(b, q)) \max\{d(a, p), d(b, q)\}$$

where $\theta \in \Theta$ and for all a, b$\in A$, p, q$\in B$. Then the mapping F: $A^2 \cup B^2 \rightarrow A \cup B$ has a unique fixed point.

Theorem 2.6: Let (A, B, d) be a complete bipolar metric space, F: $(A^2, B^2) \rightrightarrows (A, B)$ and S:(A, B) $\rightrightarrows (A, B)$ be two covariant mappings satisfying the following conditions

(ψ_0) $d(F(a, b),\ F(p, q)) \leq \theta(d(Sa, Sp),\ d(Sb, Sq))\max\{d(Sa, Sp), d(Sb, Sq)\}$
Where $\theta \in \Theta$ and a, b$\in A$, p, q$\in B$.

(ψ_1) $F(A^2 \cup B^2) \subseteq S(A \cup B)$.

(ψ_4) The pair (F, S) is weakly compatible.

(ψ_5) $S(A\cup B)$ is closed in $A\cup B$.

Then the mappings F: $A^2 \cup B^2 \rightarrow A \cup B$ and S: $A\cup B \rightarrow A\cup B$ have unique common fixed point.

Proof. Let $a_0, b_0 \in A$ and $p_0, q_0 \in B$ and from Theorem 2.4, we construct the bi-sequences, $(\{\omega_n\},\{\chi_n\})$ and $(\{\xi_n\},\{\kappa_n\})$ in (A, B) are Cauchy bi-sequences. Since (A, B, d) is complete, $(\{\omega_n\},\{\chi_n\})$ and $(\{\xi_n\},\{\kappa_n\})$ are converges sequences and its sub-sequences converges as follows

$\lim_{n\to\infty} \omega_{n+1}=\lim_{n\to\infty} F(a_{n+1}, b_{n+1})=\lim_{n\to\infty} Sa_{n+2}=w,$ $\lim_{n\to\infty} \xi_{n+1}=\lim_{n\to\infty} F(b_{n+1}, a_{n+1})=\lim_{n\to\infty} Sb_{n+2}=z,$

$\lim_{n\to\infty} \chi_{n+2}=\lim_{n\to\infty} F(p_{n+1}, q_{n+1})=\lim_{n\to\infty} Sp_{n+2}=u,$ $\lim_{n\to\infty} \kappa_{n+2}=\lim_{n\to\infty} F(q_{n+1}, p_{n+1})=\lim_{n\to\infty} Sq_{n+2}=v.$

Since $S(A\cup B)$ is closed in (A, B, d), so $\{\omega_{n+1}\},\{\chi_{m+1}\}$, $\{\xi_{n+1}\},\{\kappa_{m+1}\} \subseteq S(A\cup B)$ are converges in the complete bipolar metric spaces (S(A), S(B), d), therefore, there exist u, v$\in$ S(A), w, z$\in$ S(B) with

$$\lim_{n\to\infty} \omega_{n+1}=w,\ \lim_{n\to\infty} \xi_{n+1}=z \text{ and } \lim_{n\to\infty} \chi_{n+1}=u,\ \lim_{n\to\infty} \kappa_{n+1}=v.$$

Since S: $A\cup B \rightarrow A\cup B$ and u, v$\in$ S(A), w, z$\in$ S(B) , there exist l, m$\in A$ and r, s$\in$B such that Sl=u, Sm=v and Sr=w, Ss=z.

Putting (a, b)= (a_n, b_n), (p, q)=(r, s) in the inequality (ψ_0), we get

$d(F(a_n, b_n), F(r, s)) \leq \theta(d(Sa_n, Sr), d(Sb_n, Ss)) \max\{ d(Sa_n, Sr), d(Sb_n, Ss)\}$

$< \max\{ d(Sa_n, Sr), d(Sb_n, Ss)\}$.

Letting $n\rightarrow \infty$, it yields that $\lim_{n\to\infty} d(F(a_n, b_n), F(r, s)) \leq \lim_{n\to\infty} \max\{ d(Sa_n, Sr), d(Sb_n, Ss)\}=0.$

It follows that F(r, s)=w=Sr. Similarly, we can show F(s, r)=z=Ss, F(l, m)=u=Sl and F(m, l)=v=Sm.
Since (F, S) is ω- compatible mappings, we have F(u, v)=Su, F(v, u)=Sv and F(w, z)=Sw, F(z, w)=Sz.
Now we shall prove that Su=u, Sv=v and Sw=w, Sz.
Consider,

$d(Su, \chi_n)=d(F(u, v), F(p_n, q_n)) \leq \theta(d(Su, Sp_n),\ d(Sv, Sq_n))\max\{ d(Su, Sp_n),\ d(Sv, Sq_n)\}$

$< \max\{ d(Su, Sp_n),\ d(Sv, Sq_n)\}$.

Letting $n \rightarrow \infty$, we get $d(Su, u)<\max\{d(Su, u), d(Sv, v)\}$ (29)

and similarly, we shall show $d(Sv, v)<\max\{d(Sv, v), d(Su, u)\}$ (30)

Therefore, from (29) and (30), we get $\max\{d(Su, u), d(Sv, v)\}< \max\{d(Su, u), d(Sv, v)\}$, which implies that Su=u and Sv=v. Similarly, we can prove Sw=w and Sz=z.
Therefore,

$$F(r, s)=Sr=w=Sw=F(w, z), \qquad F(s, r)=Ss=z=Sz=F(z, w),$$

and

$$F(l, m)=Sl=u=Su=F(u, v), \qquad F(m, l)=Sm=v=Sv=F(v, u),$$

On the other hand

d(Sl, Sr)=d(u, w)=$d\left(\lim_{n\to\infty}\chi_n, \lim_{n\to\infty}\omega_n\right)=\lim_{n\to\infty} d(\omega_n,\chi_n)=0$

and

d(Sm, Ss)=d(v, z)= $d\left(\lim_{n\to\infty}\kappa_n, \lim_{n\to\infty}\xi_n\right)=\lim_{n\to\infty} d(\xi_n,\kappa_n)=0$.

So u=w and v=z. Therefore, (u, v) is coupled fixed point of F and S.

As in the proof of the Theorem 2.4, uniqueness of the coupled fixed point and unique common fixed point of F and S can be shown easily.

Example 2.7: Let A = {$U_m(R)/U_m(R)$ is upper triangular matrices over R} and B = {$L_m(R)/L_m(R)$ is lower triangular matrices over R}.

Define d: A× B→ $[0, \infty)$ as d (P, Q) = $\sum_{i,j=1}^{m} |p_{ij}-q_{ij}|$ for all P = $(p_{ij})_{m\times m} \in U_m(R)$ and Q = $(q_{ij})_{m\times m} \in L_m(R)$. Then obviously, (A, B, d) is a bipolar metric spaces.

Let the covariant maps F: $(A^2, B^2) \rightrightarrows$ (A, B) be defined as F (P, Q) = $\left(\frac{p_{ij}}{6}+\frac{q_{ij}}{2}\right)_{m\times m}$ (P = $(p_{ij})_{m\times m}$, Q= $(q_{ij})_{m\times m}$) $\in A^2 \cup B^2$ and S:(A, B) ⇉ (A, B) be defined as S(P)=$(3p_{ij})_{m\times m}$ where P = $(p_{ij})_{m\times m}$ ∈A∪B

$$d(F(P, Q), F(U, V)) = d\left(\left(\frac{p_{ij}}{6}+\frac{q_{ij}}{2}\right)_{m\times m}, \left(\frac{u_{ij}}{6}+\frac{v_{ij}}{2}\right)_{m\times m}\right)$$
$$= \sum_{i,j=1}^{m} \left|\left(\frac{p_{ij}}{6}+\frac{q_{ij}}{2}\right)-\left(\frac{u_{ij}}{6}+\frac{v_{ij}}{2}\right)\right|$$
$$\leq \frac{1}{6}\sum_{i,j=1}^{m} |p_{ij}-u_{ij}|+\frac{1}{2}\sum_{i,j=1}^{m} |q_{ij}-v_{ij}|$$
$$\leq \frac{1}{18}\sum_{i,j=1}^{m} |3p_{ij}-3u_{ij}|+\frac{1}{6}\sum_{i,j=1}^{m} |3q_{ij}-3v_{ij}|$$
$$\leq \frac{2}{9}\max\{d(SP, SU), d(SQ, SV)\}$$
$$\leq \theta(d(SP, SU), d(SQ, SV)) \max\{d(SP, SU), d(SQ, SV)\}.$$

Clearly F and S are satisfies all the conditions of Theorem 2.6 and$(O_{m\times m}, O_{m\times m})$ is the coupled fixed point.

Definition 2.8: Let (A, B, d) be a bipolar metric space, F: (A×B $B\times A$) ⇉(A, B) be a covariant mapping. If F(a, p)=a and F(p, a))=p for a∈ A and b∈B then (a, p) is called a coupled fixed point of F.

Theorem 2.9: Let (A, B, d) be a complete bipolar metric space, F: (A×B $B\times A$) ⇉(A, B) and S:(A, B) ⇉ (A, B)be two covariant mappings satisfying the following conditions

(Φ_0) $d(F(a,p), F(q,b)) \leq \theta(d(Sa,Sq), d(Sb,Sp))$ max{d(Sa, Sq), d(Sb, Sp)}
Where $\theta \in \Theta$ and a, b∈ A, p, q∈ B.

(Φ_1) F((A×B) $\cup (B\times A)$) ⊆S(A∪B).

(Φ_2) The pair (F, S) is compatible.

(Φ_3) S is continuous

Then the mappings F: (A×B) $\cup (B\times A)$→ A ∪ B and S: A∪B→ A∪B have unique common fixed point.

Theorem 2.10: Let (A, B, d) be a complete bipolar metric space, F: (A×B $B\times A$) ⇉(A, B) and S:(A, B) ⇉ (A, B)be two covariant mappings satisfying the following conditions

(Φ_0) $d(F(a,p), F(q,b)) \leq \theta(d(Sa,Sq), d(Sb,Sp))$ max{d(Sa, Sq), d(Sb, Sp)}
Where $\theta \in \Theta$ and a, b∈ A, p, q∈ B.

(Φ_1) F((A×B) $\cup (B\times A)$) ⊆S(A∪B).

(Φ_4) The pair (F, S) is ω-compatible.

(Φ_5) S(A∪B) is closed in A∪B

Then the mappings F: (A×B) $\cup (B\times A)$→ A ∪ B and S: A∪B→ A∪B have unique common fixed point.

III.APPLICATION TO HOMOTOPY

Theorem 3.1: Let (A, B, d) be a complete bipolar metric space, (U, V) be an open subset of (A, B) and $(\bar{U}, \bar{V})$ be closed subset of (A, B) such that (U, V)⊆ $(\bar{U}, \bar{V})$. Suppose H:$(\bar{U}^2 \cup \bar{V}^2)$×[0, 1]→A∪B be an operator with following conditions

(Ω_0) u≠H(u, v, κ) and v≠H(v, u, κ) for each u, v∈ ∂U∪ ∂V and κ∈[0, 1] (Here ∂U∪ ∂Vis boundrary of U∪V in A∪B)

(Ω_1) $d(H(u,v,\kappa), H(x,y,\kappa)) \leq \theta(d(u,x), d(v,y))$ max{ $d(u,x), d(v,y)$}
For all u, v∈ $\bar{U}$, x, y∈ $\bar{V}$ and θ ∈Θ, κ∈[0, 1]

(Ω_2) ∃ M>0 such that $d(H(u,v,\kappa), H(x,y,\zeta)) \leq M|\kappa-\zeta|$ for every u, v∈ $\bar{U}$, x, y∈ $\bar{V}$ and κ, ζ ∈[0, 1].

Then H(., 0) has a coupled fixed point $\Leftrightarrow$ H(., 1) has a coupled fixed point.
Proof. Let the set
X= $\{\kappa \in [0,1]: u = H(u,v,\kappa),\ v = H(v,u,\kappa)\ for\ some\ u,v \in U\}$ and
Y= $\{\zeta \in [0,1]: u = H(x,y,\zeta),\ y = H(y,x,\zeta)\ for\ some\ x,y \in V\}$.
Since H(., 0) has a coupled fixed point in $U^2 \cup V^2$, we have that (0, 0) $\in X^2 \cap Y^2$. So that $X^2 \cap Y^2$ is non-empty set.
Now we show that $X \cap Y$ is both closed and open in [0, 1] and hence by the connectedness X=Y=[0,1].
Let ($\{\kappa_n\}_{n=1}^{\infty}, \{\zeta_n\}_{n=1}^{\infty}$)$\subseteq$(X, Y) with $(\kappa_n, \zeta_n) \to (\kappa, \zeta) \in$ [0, 1] as n$\to \infty$.
We must show that κ=ζ∈ $X \cap Y$. Since $(\kappa_n, \zeta_n) \in$(X, Y) for n=0, 1, 2,…, there exist bi-sequences (u_n, x_n) and (v_n, y_n) with u_{n+1}=H(u_n, v_n, κ_n), v_{n+1}=H(v_n, u_n, κ_n) and x_{n+1}=H(x_n, y_n, ζ_n), y_{n+1}=H(y_n, x_n, ζ_n).
Consider,

$$d(u_n, x_{n+1}) = d(H(u_{n-1}, v_{n-1}, \kappa_{n-1}),\ H(x_n, y_n, \zeta_n))$$
$$\leq \theta(d(u_{n-1}, x_n), d(v_{n-1}, y_n)) \max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n)\}$$
$$< \max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n)\} \qquad (31)$$

and

$$d(v_n, y_{n+1}) = d(H(v_{n-1}, u_{n-1}, \kappa_{n-1}),\ H(y_n, x_n, \zeta_n))$$
$$\leq \theta(d(v_{n-1}, y_n), d(u_{n-1}, x_n)) \max\{ d(v_{n-1}, y_n), d(u_{n-1}, x_n) \}$$
$$< \max\{ d(v_{n-1}, y_n), d(u_{n-1}, x_n)\} \qquad (32)$$

Combining (31) and (32), we get

$$\frac{\max\{d(u_n, x_{n+1}), d(v_n, y_{n+1})\}}{\max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n)\}} \leq \theta(d(u_{n-1}, x_n), d(v_{n-1}, y_n)) < 1.$$

Letting n$\to \infty$, we get $\theta(d(u_{n-1}, x_n), d(v_{n-1}, y_n)) \to 1$. By the property of θ, we obtain that $d(u_{n-1}, x_n) \to 0$, $d(v_{n-1}, y_n) \to 0$ as n$\to \infty$.
Therefore, $\max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n)\} \to 0$ as n$\to \infty$. (33)
Similarly, , $\max\{ d(u_n, x_{n-1}), d(v_n, y_{n-1})\} \to 0$ as n$\to \infty$. (34)
and , $\max\{ d(u_n, x_n), d(v_n, y_n)\} \to 0$ as n$\to \infty$. (35)
For each n, m∈N, n<m. Using the property (B_3), we have

$$d(u_n, x_m) \leq d(u_n, x_{n+1}) + d(u_{n+1}, x_{n+1}) + \ldots\ldots + d(u_{m-1}, x_{m-1}) + d(u_{m-1}, x_m)$$
$$\leq \theta(d(u_{n-1}, x_n), d(v_{n-1}, y_n)) \max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n)\}$$
$$+M|\kappa_{n+1} - \zeta_{n+1}| + \ldots\ldots + M|\kappa_{m-1} - \zeta_{m-1}|$$
$$+ \theta(d(u_{m-2}, x_{m-1}), d(v_{m-2}, y_{m-1})) \max\{ d(u_{m-2}, x_{m-1}), d(v_{m-2}, y_{m-1})\}$$
$$< \max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n)\} + M|\kappa_{n+1} - \zeta_{n+1}| + \ldots\ldots + M|\kappa_{m-1} - \zeta_{m-1}|$$
$$+ \max\{ d(u_{m-2}, x_{m-1}), d(v_{m-2}, y_{m-1})\} \qquad (36)$$

and

$$d(v_n, y_m) \leq d(v_n, y_{n+1}) + d(v_{n+1}, y_{n+1}) + \ldots\ldots + d(v_{m-1}, y_{m-1}) + d(v_{m-1}, y_m)$$
$$\leq \theta(d(v_{n-1}, y_n), d(u_{n-1}, x_n)) \max\{ d(v_{n-1}, y_n), d(u_{n-1}, x_n)\}$$
$$+M|\kappa_{n+1} - \zeta_{n+1}| + \ldots\ldots + M|\kappa_{m-1} - \zeta_{m-1}|$$
$$+ \theta(d(v_{m-2}, y_{m-1}), d(u_{m-2}, x_{m-1})) \max\{ d(v_{m-2}, y_{m-1}), d(u_{m-2}, x_{m-1})\}$$
$$< \max\{ d(v_{n-1}, y_n), d(u_{n-1}, x_n)\} + M|\kappa_{n+1} - \zeta_{n+1}| + \ldots\ldots + M|\kappa_{m-1} - \zeta_{m-1}|$$
$$+ \max\{ d(v_{m-2}, y_{m-1}), d(u_{m-2}, x_{m-1})\} \qquad (37)$$

Combining (36) and (37), we get

$$\max\{ d(u_n, x_m), d(v_n, y_m)\} < \max\{ d(u_{n-1}, x_n), d(v_{n-1}, y_n)\} + M|\kappa_{n+1} - \zeta_{n+1}|$$
$$+ \ldots\ldots + M|\kappa_{m-1} - \zeta_{m-1}|$$
$$+ \max\{ d(u_{m-2}, x_{m-1}), d(v_{m-2}, y_{m-1})\} \to 0 \text{ as } n \to \infty.$$

It follows that $\lim_{n\to\infty} \max\{ d(u_n, x_m), d(v_n, y_m)\}$ =0. Similarly, we can prove that $\lim_{n\to\infty} \max\{ d(u_m, x_n), d(v_m, y_n)\}$ =0. Therefore, (u_n, x_n) and (v_n, y_n) are Cauchy bi-sequences in (U, V). By the completeness, there exist ξ, ν∈U and δ, η∈V with

$$\lim_{n\to\infty} u_n = \delta,\ \lim_{n\to\infty} v_n = \eta \text{ and } \lim_{n\to\infty} x_n = \xi,\ \lim_{n\to\infty} y_n = \nu. \qquad (38)$$

Now consider,

$$d(H(\xi, \nu, \kappa), \delta) \leq d(H(\xi, \nu, \kappa), x_{n+1}) + d(u_{n+1}, x_{n+1}) + d(u_{n+1}, \delta)$$
$$\leq d(H(\xi, \nu, \kappa), H(x_n, y_n, \kappa_n) + d(H(u_n, v_n, \kappa_n), H(x_n, y_n, \zeta_n) + d(u_{n+1}, \delta)$$
$$\leq \theta(d(\xi, x_n), d(\nu, y_n)) \max\{ d(\xi, x_n), d(\nu, y_n)\} + M|\kappa_n - \zeta_n|$$
$$+ d(u_{n+1}, \delta)$$
$$< \max\{ d(\xi, x_n), d(\nu, y_n)\} + M|\kappa_n - \zeta_n| + d(u_{n+1}, \delta) \to 0 \text{ as } n \to \infty.$$

It follows that d(H(ξ, ν, κ), δ)= 0 implies H(ξ, ν, κ)= δ. Similarly, we obtain H(ν , ξ, κ)= η and H(δ, η, ζ)=ξ, H(η, δ, ζ)=ν.

On the other hand

d(ξ, δ)=d$\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} u_n\right)$=$\lim_{n\to\infty} d(u_n, x_n)$=0

and

d(ν, η)= d$\left(\lim_{n\to\infty} y_n, \lim_{n\to\infty} v_n\right)$=$\lim_{n\to\infty} d(v_n, y_n)$=0.

Therefore, ξ=δ and ν=η and hence κ=ζ. Thus (κ, ζ) ∈ $X^2 \cap Y^2$. Clearly, $X^2 \cap Y^2$ is closed in [0, 1].

Let (κ_0, ζ_0)∈(X, Y), then there exist bi-sequences (u_0, x_0) and (v_0, y_0) with u_0=H(u_0, v_0, κ_0), v_0=H(v_0, u_0, κ_0) and x_0=H(x_0, y_0, ζ_0), y_0=H(y_0, x_0, ζ_0). Since $U^2 \cup V^2$ is open, then there exist r>0 such that $X_d(u_0$, r)⊆ $U^2 \cup V^2$, $X_d(v_0$, r)⊆ $U^2 \cup V^2$ and $X_d(x_0$, r)⊆ $U^2 \cup V^2$, $X_d(y_0$, r)⊆ $U^2 \cup V^2$.

Choose κ∈$(\zeta_0 - \epsilon, \zeta_0 + \epsilon)$, ζ∈$(\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ such that $|\kappa - \zeta_0| \le \frac{1}{M^n} < \frac{\epsilon}{2}$, $|\zeta - \kappa_0| \le \frac{1}{M^n} < \frac{\epsilon}{2}$ and $|\kappa_0 - \zeta_0| \le \frac{1}{M^n} < \frac{\epsilon}{2}$, then for x∈ $\overline{B_{X\cup Y}(u_0, r)}$={x, x_0 ∈V/d(u_0, x)≤r+d(u_0, x_0)},

y∈ $\overline{B_{X\cup Y}(v_0, r)}$={y, y_0 ∈V/d(v_0, y)≤r+d(v_0, y_0)}, u∈ $\overline{B_{X\cup Y}(r, x_0)}$={u, u_0 ∈V/d(u, x_0)≤r+d(u_0, x_0)} and v∈ $\overline{B_{X\cup Y}(r, y_0)}$={v, v_0 ∈V/d(v, y_0)≤r+d(v_0, y_0)}.

Also,

$$\begin{aligned} d(H(u, v, \kappa), x_0) &= d(H(u, v, \kappa), H(x_0, y_0, \zeta_0)) \\ &\le d(H(u, v, \kappa), H(x, y, , \zeta_0)) + d(H(u_0, v_0, \kappa), H(x, y, , \zeta_0)) \\ &\quad + d(H(u_0, v_0, \kappa), H(x_0, y_0, \zeta_0)) \\ &< \frac{2}{M^{n-1}} + \theta(d(u_0, x), d(v_0, y)) \max\{ d(u_0, x), d(v_0, y)\}. \end{aligned}$$

Letting n→ ∞, we get d(H(u, v, κ), x_0)< max{ d(u_0, x), d(v_0, y)}. (39)

Similarly, we show that d(H(v, u, κ), y_0)< max{ d(u_0, x), d(v_0, y)}. (40)

Combining (39) and (40), we get

$$\begin{aligned} \text{Max}\{ d(H(u, v, \kappa), x_0), d(H(v, u, \kappa), y_0) \} &< \max\{ d(u_0, x), d(v_0, y)\} \\ &< \max\{r+ d(u_0, x_0), r+d(v_0, y_0)\}. \end{aligned}$$

Similarly, we show that max{d(u_0, H(x, y, ζ)), d(v_0, H(y, x, ζ))}< max{r+ d(u_0, x_0), r+d(v_0, y_0)}.

On the other hand

d(u_0, x_0)=d(H(u_0, v_0, κ_0), H(x_0, y_0, ζ_0))≤M$|\kappa_0 - \zeta_0| \le M\frac{1}{M^n} < \frac{1}{M^{n-1}} \to 0$ as n→ ∞.

and

d(v_0, y_0)=d(H(v_0, u_0, κ_0), H(y_0, x_0, ζ_0))≤M$|\kappa_0 - \zeta_0| \le M\frac{1}{M^n} < \frac{1}{M^{n-1}} \to 0$ as n→ ∞.

So u_0= x_0 and v_0= y_0 and hence $\kappa = \zeta$. Thus for each fixed $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$, H(., κ): $\overline{B_{X\cup Y}(u_0, r)} \to \overline{B_{X\cup Y}(u_0, r)}$ and H(., κ): $\overline{B_{X\cup Y}(v_0, r)} \to \overline{B_{X\cup Y}(v_0, r)}$. Then all the conditions of Theorem 3.1 are satisfied. Thus we conclude that H(., κ) has a coupled fixed point in($\overline{U}^2 \cap \overline{V}^2$). But this must be in $U^2 \cap V^2$.

Therefore, (κ, κ)∈ $X^2 \cap Y^2$ for $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$. Hence $(\kappa_0 - \epsilon, \kappa_0 + \epsilon)$⊆ $X^2 \cap Y^2$. Clearly, $X^2 \cap Y^2$is open in [0, 1].

To prove the reverse, we can use the similar process.

IV. CONCLUSIONS

In this paper we conclude some applications to homotopy theory by using coupled fixed point theorems in Bipolar metric spaces.

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