A General Class of Multiple Eulerian Integrals Involving a Multivariable Gimel-Function with General Arguments

F.Y.Ayant

1 Teacher in High School, France

ABSTRACT
Recently, Raina and Srivastava [7] and Srivastava and Hussain [12] have provided closed-form expressions for a number of a general Eulerian integrals about the multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple Eulerian integrals involving a multivariable Gimel-function defined here with general arguments. These integrals will serve as a key formula from which one can deduce numerous useful integrals.

Keywords: Multivariable Gimel-function, multiple Eulerian integral , general polynomials, sequence of polynomials.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Intrduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a generalized transcendental function of several complex variables, see Ayant [1] for more details,

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1, \cdots, z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})]_{1,n_r},$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots; \quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})]_{n_r+1,p_r}:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)]_{1,n^{(1)}},[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})]_{n^{(1)}+1,p^{(1)}_i}$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})]_{1,q_r}:\ [(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)]_{1,m^{(1)}},[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})]_{m^{(1)}+1,q^{(1)}_i}$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)]_{1,m^{(r)}},[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})]_{m^{(r)}+1,p^{(r)}_i} \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)]_{1,n^{(r)}},[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})]_{n^{(r)}+1,q^{(r)}_i}\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [2].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [6].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [5].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [13,14].

The well-known Eulerian Beta integral

$$\int_a^b (z-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(Re(\alpha)>0, Re(\beta)>0, b>a) \tag{1.5}$$

is a basic result for evaluation of numerous other potentially useful integrals involving various special functions and polynomials. Raina and Srivastava [7], Saigo and Saxena [9], Srivastava and Hussain [12], Srivastava and Garg [11] etc have established a number of Eulerian integrals involving various general class of polynomials, Meijer's G-function and Fox's H-function of one and more variables with general arguments.

The explicit form of the generalized polynomial set [8, p.71, (2.3.4)] is

$$S_n^{\alpha,\beta,\tau}(x) = \sum_{e,p,u,v} C(e,p,u,v)x^R(1-\tau x^{\mathfrak{r}})^{\delta n-v} \tag{1.6}$$

where $C(e,p,u,v) = \frac{B^{qn}(-)^p(-p)_e(\alpha)_p(-v)_u(-\alpha-qn)_e\left(-\frac{\beta}{\tau}-sn\right)_v}{u!v!e!p!(1-\alpha-p)_e} l^n(-\tau)^v\left(\frac{e+k+\mathfrak{r}u}{l}\right)_n\left(\frac{A}{B}\right)^b$ (1.7)

where $\sum_{e,p,u,n} = \sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{p=0}^{n}\sum_{e=0}^{p}$ and $R = ln + \mathfrak{r}v + p$

We recall here the following definition of the general class of polynomials introduced and studied by Srivastava [10]

$$S_V^U(x) = \sum_{\eta=0}^{[V/U]} \frac{(-V)_{U\eta}A_{V,\eta}}{\eta!}x^\eta \tag{1.8}$$

where $V = 0,1,\cdots$ and U is an arbitrary positive integer. The coefficients $A_{V,\eta}(V,\eta \geqslant 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{V,\eta}$, $S_V^U(x)$ yields a number of known polynomials, these include the Jacobi polynomials, laguerre polynomials and others polynomials ([15], p. 158-161.)

The multivariable Gimel-function defined in the paper is a generalized transcendental function of several complex variables. It is defined in term of multiple Mellin-Barnes type integral :

2. Required result.

In this following section, we shall use the required integral and the notations (2.1) and (2.2).

$$X_j = (b_j-a_j)+\rho_j(t_j-a_j)+\sigma_j(b_j-t_j) \tag{2.1}$$

$$Y_j = \frac{(t_j-a_j)^{\gamma_j}(b_j-t_j)^{\delta_j}X_j^{1-\gamma_j-\delta_j}}{\beta_j(b_j-a_j)+(\beta_j\rho_j+\alpha_j-\beta_j)(t_j-a_j)+\beta_j\sigma_j(b_j-t_j)} \tag{2.2}$$

for $j = 1,\cdots,s$

Lemma. ([4] p.287)

$$\int_a^b \frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{\{b-a+\lambda(t-a)+\mu(b-t)\}^{\alpha+\beta}}\mathrm{d}t = \frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}\Gamma(\alpha)\Gamma(\beta)}{(b-a)\Gamma(\alpha+\beta)} \tag{2.3}$$

with $t \in [a;b]$ $a \neq b$, $Re(\alpha)>0, Re(\beta)>0,$ $\eta+\lambda(t-a)+\mu(b-t)\neq 0$

3. Main integral.

In this section, we shall establish the following Eulerian multiple integral of multivariable Gimel-function and we shall use the following notations (3.1) to (3.11). In your investigation, we shall use the following notations.

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{3.1}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)},0;A_{rj})]_{1,n_r},[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)},0;A_{rji_r})]_{\mathfrak{n}+1,p_{i_r}} \tag{3.2}$$

$$A_1 = (1-\tau_j-\zeta_j R;v_j',\cdots,v_j^{(r)},\zeta_j\mathfrak{r};1)_{1,s},(-\lambda_j-KS_j-\gamma_j\zeta_j R-\tau_j;\gamma_j v_j',\cdots,\gamma_j v_j^{(r)},\gamma_j\zeta_j\mathfrak{r};1)_{1,s}, \tag{3.3}$$

$$(-\mu_j-KT_j-\delta_j\zeta_j R-\tau_j;\delta_j v_j',\cdots,\delta_j v_j^{(r)},\delta_j\zeta_j\mathfrak{r};1)_{1,s} \tag{3.4}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}];(1-v+\delta\eta,1;1) \tag{3.5}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{3.6}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)},0;B_{rji_r})]_{1,q_{i_r}} \tag{3.7}$$

$$B_1=(-\lambda_j-\mu_j-K(S_j+T_j)-\zeta_j(\gamma_j+\delta_j)R-\tau_j-1;(\gamma_j+\delta_j)v_j',\cdots,(\gamma_j+\delta_j)v_j^{(r)},(\gamma_j+\delta_j)\zeta_j\mathfrak{r};1)_{1,s},$$

$$(1-\zeta_j R;v_j',\cdots,v_j^{(r)},\zeta_j\mathfrak{r};1)_{1,s} \tag{3.8}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,q_i^{(1)}};\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}};(0,1;1) \tag{3.9}$$

$$U=0,n_2;0,n_3;\cdots;0,n_r;V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{3.10}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{3.11}$$

Theorem.

We have the following result

$$\int_{a_1}^{b_1}\cdots\int_{a_s}^{b_s}\prod_{j=1}^{s}\frac{(t_j-a_j)^{\lambda_j}(b_j-t_j)^{\mu_j}}{X_j^{\lambda_j+\mu_j+2}}\,S_U^V\left[a\prod_{j=1}^{s}\frac{(t_j-a_j)^{S_j}(b_j-t_j)^{T_j}}{X_j^{S_j+T_j}}\right]$$

$$S_n^{\alpha,\beta,\tau}\left[b\prod_{j=1}^{s}Y^{\zeta_j};\mathfrak{r},t,q,A,B,k;l\right]\gimel\begin{pmatrix}\mathrm{z}_1\prod_{j=1}^{s}Y_j^{v'_j}\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{j=1}^{s}Y_j^{v_j^{(r)}}\end{pmatrix}\mathrm{d}t_1\cdots\mathrm{d}t_s$$

$$=\left\{\prod_{j=1}^{s}\{(b_j-a_j)^{-1}(1+\rho_j)^{-\lambda_j-1}(1+\sigma_j)^{-\mu_j-1}\right\}\sum_{K=0}^{[V/U]}\sum_{e,p,u,n}\sum_{\tau_1,\cdots,\tau_s=0}^{\infty}\frac{(-V)_{UK}A_{V,K}}{K!}$$

$$C(e,p,u,v)\left\{\prod_{j=1}^{s}\frac{(\beta_j-\alpha_j)^{\tau_j}(1+\rho_j)^{-K_jS_j-\gamma_j\zeta_jR-\tau_j}(1+\sigma_j)^{-KT_j-\delta_j\zeta_jR}}{\tau_j!\beta_j^{\tau_j+\zeta_jR}}\right\}a^Kb^R$$

$$\gimel^{U;0,n_r+3s:V;1,1}_{X;p_{i_r}+3s,q_{i_r}+2s,\tau_{i_r}:R_r:Y;1,1}\left(\begin{array}{c|c}\mathrm{z}_1\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-v'_j} & \mathbb{A};\mathrm{A}_1,\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-v_j^{(r)}} & \mathbb{B};\mathbf{B},\mathrm{B}_1:B\\ \mathrm{b}^{\mathfrak{r}}\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-\zeta_j\mathfrak{r}} & \end{array}\right) \quad (3.12)$$

We obtain a Gimel-function of $(r+1)-$variables

Provided that

(i) $\lambda_j,\mu_j,s_j,t_j,\zeta_j,v_j^{(i)}>0,\beta_j\neq 0,b_j-a_j\neq 0,\rho_j\neq -1,\sigma_j-1,$

$(b_j-a_j)+\rho_j(t_j-a_j)+\sigma_j(b_j-t_j)\neq 0,t_j\in[a_j,b_j]$ for $i=1,\cdots,r,j=1,\cdots,s$

(ii) $|(\beta_j-\alpha_j)(t_j-a_j)|<|\beta_j\{(b_j-a_j)+\rho_j(t_j-a_j)+\sigma_j(b_j-t_j)\}\,|;t_j\in[a_j,b_j]$ for $,j=1,\cdots,s$

(iii) When $\min(S_j,T_j)>0$

(a) $Re\left[\lambda_j+\gamma_j\zeta_j(ln+p)+\sum_{i=1}^{r}\gamma_jv_j^{(i)}\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+1\right]\right]>0$

(b) $Re\left[\mu_j+\delta_j\zeta_j(ln+p)+\sum_{i=1}^{r}\gamma_jv_j^{(i)}\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+1\right]\right]>0$

When $\max(S_j,T_j)<0$

(c) $Re\left[\lambda_j+S_j[V/U]+\gamma_j\zeta_j(ln+p)+\sum_{i=1}^{r}\gamma_jv_j^{(i)}\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+1\right]\right]>0$

(d) $Re\left[\mu_j+t_j[V/U]+\delta_j\zeta_j(ln+p)+\sum_{i=1}^{r}\gamma_jv_j^{(i)}\min_{1\leqslant j\leqslant m^{(i)}}\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+1\right]\right]>0$

When $S_j > 0, T_j < 0$ inequalities (a) and (d) are satisfied.

When $S_j < 0, T_j > 0$ inequalities (b) and (c) are satisfied

$\left|\arg\left(z_i\prod_{j=1}^{s}Y_j^{v_j^{(r)}}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

The multiple series of R.H.S. of (3.12) converges absolutely.

Proof

To establish the multiple integral formula (3.12), we first use the series representations for the polynomials sets $S_V^U(x)$ and $S_n^{\alpha,\beta,\tau}(x)$ respectively with the help of (1.8) and (1.7) respectively of in its left hand side. Further, using the Melin-Barnes multiple integrals contour representation for the multivariable Gimel-function and then interchanging the order of integrations and summations suitably, which is permissible under the conditions stated above, we find that

$$\text{L.H.S} = \sum_{K=0}^{[V/U]}\sum_{e,p,u,n}\frac{(-V)_{UK}A_{V,K}}{K!}\,a^Kb^RC(e,p,u,v)\,\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}$$

$$\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\int_{a_1}^{b_1}\cdots\int_{a_s}^{b_s}\prod_{j=1}^{s}\frac{(t_j-a_j)^{\lambda_j+KS_j}(b_j-t_j)^{\mu_j+KT_j}}{X_j^{\lambda_j+\mu_j+K(S_j+T_j)+2}}Y_j^{\zeta_jR+\sum_{i=1}^{r}s_iv_j^{(i)}}$$

$$\left(1-\tau x^{\mathfrak{r}}\prod_{j=1}^{s}Y_j^{\zeta_jq}\right)^{\delta n-v}\mathrm{d}t_1\cdots,\mathrm{d}t_s\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.13}$$

Now by writing $\left(1-\tau x^{\mathfrak{r}}\prod_{j=1}^{s}Y_j^{\zeta_jq}\right)^{\delta n-v}$ in terms of contour integral and changing the order of integration therein, we obtain

$$\text{L.H.S} = \sum_{K=0}^{[V/U]}\sum_{e,p,u,n}\frac{(-V)_{UK}A_{V,K}}{K!}\,a^Kb^R\,C(e,p,u,v)\,\frac{1}{(2\pi\omega)^{r+1}}\int_{L_1}\cdots\int_{L_r}\int_{L_{r+1}}$$

$$\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\,(-\tau b^{\mathfrak{r}})^{s_{r+1}}\,\Gamma(-s_{r+1})\Gamma(v-\delta n+s_{r+1})\left[\int_{a_1}^{b_1}\cdots\int_{a_s}^{b_s}\right.$$

$$\left.\left\{\prod_{j=1}^{s}\frac{(t_j-a_j)^{\lambda_j+KS_j}(b_j-t_j)^{\mu_j+KT_j}}{X_j^{\lambda_j+\mu_j+K(S_j+T_j)+2}}Y_j^{\zeta_jR+\sum_{i=1}^{r}s_iv_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1}}\right\}\mathrm{d}t_1\cdots,\mathrm{d}t_s\right]\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}s_{r+1} \tag{3.14}$$

Substituting the value of Y_j from (2.2) and after simplifications, we get

$$\text{L.H.S} = \sum_{K=0}^{[V/U]}\sum_{e,p,u,n}\frac{(-V)_{UK}A_{V,K}}{K!}\,a^Kb^R\,C(e,p,u,v)\,\frac{1}{(2\pi\omega)^{r+1}}\int_{L_1}\cdots\int_{L_r}\int_{L_{r+1}}$$

$$\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\left(-\tau b^{\mathfrak{r}}\right)^{s_{r+1}}\Gamma(-s_{r+1})\Gamma(v-\delta n+s_{r+1})$$

$$\left[\int_{a_1}^{b_1}\cdots\int_{a_s}^{b_s}\left\{\prod_{j=1}^{s}\frac{(t_j-a_j)^{\lambda_j+KS_j+\gamma_j\sum_{i=1}^{r}\xi_i v_j^{(i)}+\gamma_j\zeta_j(R+\mathfrak{r}s_{r+1})}}{X_j^{\lambda_j+\mu_j+K(S_j+T_j)+2+(\gamma_j+\delta_j)(R\zeta_j+\sum_{i=1}^{r}s_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}}\right.\right.$$

$$\left.\frac{(b_j-t_j)^{\mu_j+KT_j+\delta_j\sum_{i=1}^{r}s_i v_j^{(i)}+\gamma_j\zeta_j(R+\mathfrak{r}s_{r+1})}}{\beta_j^{(R\zeta_j+\sum_{i=1}^{r}s_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}}\left(1-\frac{(\beta_j-\alpha_j)(t_j-a_j)}{\beta_j X_j}\right)^{-(\zeta_j R+\sum_{i=1}^{r}s_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}\right\}$$

$$\left.\mathrm{d}t_1\cdots,\mathrm{d}t_s\right]\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}s_{r+1} \tag{3.15}$$

If $\dfrac{(\beta_j-\alpha_j)(t_j-a_j)}{\beta_j X_j}<1, t_j\in[a_j;b_j]$ for $j=1,\cdots,s$

then use the binomial expansion is valid and we thus find that

$$\text{L.H.S} = \sum_{K=0}^{[V/U]}\sum_{e,p,u,n}\frac{(-V)_{UK}A_{V,K}}{K!}a^K b^R\,C(e,p,u,v)\prod_{j=1}^{s}\left\{\frac{(\beta_j-\alpha_j)^{\tau_j}}{\beta_j^{\tau_j}\tau_j!}\right\}$$

$$\frac{1}{(2\pi\omega)^{r+1}}\int_{L_1}\cdots\int_{L_r}\int_{L_{r+1}}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\left(-\tau b^{\mathfrak{r}}\right)^{s_{r+1}}\Gamma(-s_{r+1})\Gamma(v-\delta n+s_{r+1})$$

$$\left[\int_{a_1}^{b_1}\cdots\int_{a_s}^{b_s}\left\{\prod_{j=1}^{s}\frac{(t_j-a_j)^{\lambda_j+KS_j+\gamma_j\sum_{i=1}^{r}s_i v_j^{(i)}+\gamma_j\zeta_j(R+\mathfrak{r}s_{r+1})+\tau_j}}{X_j^{\lambda_j+\mu_j+K(S_j+T_j)+2+(\gamma_j+\delta_j)(R\zeta_j+\sum_{i=1}^{r}s_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})+\tau_j}}\right.\right.$$

$$\prod_{i=1}^{s}\left\{\frac{\Gamma(\tau_j+R\zeta_j+\sum_{i=1}^{r}s_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}{\Gamma(R\zeta_j+\sum_{i=1}^{r}s_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}\beta_j^{-(R\zeta_j+\sum_{i=1}^{r}s_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}\right\}$$

$$\left.(b_j-x_j)^{\mu_j+KT_j+\delta_j\sum_{i=1}^{r}s_i v_j^{(i)}+\delta_j\zeta_j(R+\mathfrak{r}s_{r+1})}\mathrm{d}t_1\cdots,\mathrm{d}t_s\right]\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}s_{r+1} \tag{3.16}$$

Now using (2.1) and then evaluating the inner-most integral by using the lemma (2.3), we get

$$\text{L.H.S} = \left\{\prod_{j=1}^{s}\left\{(b_j-a_j)^{-1}(1+\rho_j)^{-\lambda_j-1}(1+\sigma_j)^{-\mu_j-1}\right\}\right\}\sum_{K=0}^{[V/U]}\sum_{e,p,u,n}\frac{(-V)_{UK}A_{V,K}}{K!}a^K b^R$$

$$C(e,p,u,v)\left\{\prod_{j=1}^{s}\frac{(\beta_j-\alpha_j)^{\tau_j}(1+\rho_j)^{-K_jS_j-\gamma_j\zeta_jR-\tau_j}(1+\sigma_j)^{-KT_j-\delta_j\zeta_jR}}{\tau_j!\beta_j^{\tau_j+\zeta_jR}}\right\}$$

$$\frac{1}{(2\pi\omega)^{r+1}}\int_{L_1}\cdots\int_{L_r}\int_{L_{r+1}}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\,(-\tau b^{\mathfrak{r}})^{s_{r+1}}\,\Gamma(-s_{r+1})\Gamma(v-\delta n+s_{r+1})$$

$$\prod_{i=1}^{s}\left\{\frac{\Gamma(\tau_j+R\zeta_j+\sum_{i=1}^{r}\xi_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}{\Gamma(R\zeta_j+\sum_{i=1}^{r}s_i v_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}\beta_j^{-(R\zeta_j+\sum_{i=1}^{r}s_iv_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})}\right\}$$

$$\prod_{j=1}^{s}\left\{\frac{\Gamma(\tau_j+\lambda_j+KS_j+\gamma_j\zeta_jR+\gamma_j\sum_{i=1}^{r}s_iv_j^{(i)}+\gamma_j\zeta_j\mathfrak{r}s_{r+1}+1)}{\Gamma(\lambda_j+\mu_j+K(S_j+T_j)+(\gamma_j+\delta_j)(\zeta_jR+\sum_{i=1}^{r}s_iv_j^{(i)}+\zeta_j\mathfrak{r}s_{r+1})+\tau_j+2)}\right.$$

$$\left.\Gamma(-s_{r+1})\Gamma(v-\delta n+s_{r+1})\Gamma(\mu_j+K_{tj}+\delta_j\zeta_jR+\delta_j\sum_{i=1}^{r}s_iv_j^{(i)}+\delta_js_j\mathfrak{r}\zeta_{r+1}+1)\right\}$$

$$\prod_{j=1}^{s}\left\{\frac{(1+\rho_j)^{-\gamma_j}(1+\sigma_j)^{-\delta_j}}{\beta_j}\right\}^{\sum_{i=1}^{r}s_iv_j^{(i)}}\prod_{j=1}^{s}\left\{\frac{(1+\rho_j)^{-\gamma_j\zeta_jq}(1+\sigma_j)^{-\delta_j\zeta_jq}(-\tau b^{\mathfrak{r}})}{\beta_j^{\zeta_j\mathfrak{r}}}\right\}^{s_{r+1}}\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}s_{r+1}\qquad(3.17)$$

Finally, reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable Gimel-function, we obtain the result (2.12).

4. Particular cases

The multivariable Gimel-function occurring in the main integral can be suitably specialized to a remarkably wide variety of special functions which are expressible in terms of E, G, H and I-function of one and several variables. Again by suitably specializing various parameters and coefficients, the general class of polynomials and the general sequence of functions can be reduced to a large number of orthogonal polynomials and hypergeometric polynomials. Thus using various special cases of these special functions, we can obtain a large number of others integrals involving simpler special functions and polynomials of one and several variables.

On taking $V=0, U=1$ and $A_{0,0}$ in (3.12), the general class of polynomials $S_V^U(x)$ reduces to unity an we get

Corollary 1.

$$\int_{a_1}^{b_1}\cdots\int_{a_s}^{b_s}\prod_{j=1}^{s}\frac{(t_j-a_j)^{\lambda_j}(b_j-t_j)^{\mu_j}}{X_j^{\lambda_j+\mu_j+2}}\;S_n^{\alpha,\beta,\tau}\left[b\prod_{j=1}^{s}Y^{\zeta_j};\mathfrak{r},t,q,A,B,k;l\right]\gimel\begin{pmatrix}\mathrm{z}_1\prod_{j=1}^{s}Y_j^{v_j'}\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{j=1}^{s}Y_j^{v_j^{(r)}}\end{pmatrix}\mathrm{d}t_1\cdots\mathrm{d}t_s$$

$$=\left\{\prod_{j=1}^{s}\{(b_j-a_j)^{-1}(1+\rho_j)^{-\lambda_j-1}(1+\sigma_j)^{-\mu_j-1}\right\}\sum_{e,p,u,n}\sum_{\tau_1,\cdots,\tau_s=0}^{\infty}$$

$$C(e,p,u,v)\left\{\prod_{j=1}^{s}\frac{(\beta_j-\alpha_j)^{\tau_j}(1+\rho_j)^{-\gamma_j\zeta_jR-\tau_j}(1+\sigma_j)^{-KT_j-\delta_j\zeta_jR}}{\tau_j!\beta_j^{\tau_j+\zeta_jR}}\right\}b^R$$

$$\beth_{X;p_{i_r}+3s,q_{i_r}+2s,\tau_{i_r}:R_r:Y;1,1}^{U;0,n_r+3s:V;1,1}\left(\begin{array}{c|c} \mathrm{z}_1\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-v'_j} & \mathbb{A};\mathrm{A}_2,\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-v_j^{(r)}} & \mathbb{B};\mathbf{B},\mathrm{B}_2:B \\ \mathrm{b}^{\mathfrak{r}}\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-\zeta_j\mathfrak{r}} & \end{array}\right) \tag{4.1}$$

where

$$A_2=(1-\tau_j-\zeta_jR;v'_j,\cdots,v_j^{(r)},\zeta_j\mathfrak{r};1)_{1,s},(-\lambda_j-\gamma_j\zeta_jR-\tau_j;\gamma_jv'_j,\cdots,\gamma_jv_j^{(r)},\gamma_j\zeta_j\mathfrak{r};1)_{1,s},$$

$$(-\mu_j-\delta_j\zeta_jR-\tau_j;\delta_jv'_j,\cdots,\delta_jv_j^{(r)},\delta_j\zeta_j\mathfrak{r};1)_{1,s} \tag{4.2}$$

$$B_2=(-\lambda_j-\mu_j-\zeta_j(\gamma_j+\delta_j)R-\tau_j-1;(\gamma_j+\delta_j)v'_j,\cdots,(\gamma_j+\delta_j)v_j^{(r)},(\gamma_j+\delta_j)\zeta_j\mathfrak{r};1)_{1,s},$$

$$(1-\zeta_jR;v'_j,\cdots,v_j^{(r)},\zeta_j\mathfrak{r};1)_{1,s} \tag{4.3}$$

with the same notations and corresponding validity conditions that (3.12).

Putting $s=1$ in (3.12), we arrive at the following integral form

Corollary 2.

$$\int_{a_1}^{b_1}\frac{(t-a_1)^{\lambda}(b_1-t)^{\mu}}{X_j^{\lambda+\mu+2}}S_U^V\left[a\frac{(t-a_1)^{S_j}(b_1-t)^T}{X^{S+T}}\right]S_n^{\alpha,\beta,\tau}\left[bY^{\zeta};\mathfrak{r},t,q,A,B,k;l\right]$$

$$\beth\left(\begin{array}{c}\mathrm{z}_1Y^{v'}\\ \cdot\\ \cdot\\ \mathrm{z}_rY^{v^{(r)}}\end{array}\right)\mathrm{d}t_1\cdots\mathrm{d}t_s=\left\{(b_1-a_1)^{-1}(1+\rho)^{-\lambda-1}(1+\sigma)^{-\mu-1}\right\}$$

$$\sum_{K=0}^{[V/U]}\sum_{e,p,u,n}\sum_{\tau_1=0}^{\infty}\frac{(-V)_{UK}A_{V,K}}{K!}C(e,p,u,v)\left\{\frac{(\beta-\alpha)^{\tau}(1+\rho)^{-KS-\gamma-\tau}(1+\sigma)^{-KT-\delta\zeta R}}{\tau!\beta^{\tau+\zeta R}}\right\}a^Kb^R$$

$$\beth_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y;1,1}^{U;0,n_r+3:V;1,1}\left(\begin{array}{c|c} \mathrm{z}_1\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-v'_j} & \mathbb{A};\mathrm{A}_3;\mathbf{A},:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-v_j^{(r)}} & \mathbb{B};\mathbf{B},\mathrm{B}_3:B \\ \mathrm{b}^{\mathfrak{r}}\prod_{j=1}^{s}\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-\zeta_j\mathfrak{r}} & \end{array}\right) \tag{4.4}$$

where

$$\mathbb{A}_3=(1-\tau_1-\zeta R;v',\cdots,v^{(r)},\zeta\mathfrak{r};1),(-\lambda-KS-\gamma\zeta R-\tau_1;\gamma v',\cdots,\gamma v^{(r)},\gamma\zeta\mathfrak{r};1),$$

$$(-\mu-KT-\delta\zeta R-\tau;\delta v',\cdots,\delta v_j^{(r)},\delta\zeta\mathfrak{r};1). \tag{4.5}$$

$$\mathbb{B}_3=(-\lambda-\mu-K(S+T)-\zeta(\gamma+\delta)R-\tau_1-1;(\gamma+\delta)v',\cdots,(\gamma+\delta)v^{(r)},(\gamma+\delta)\zeta\mathfrak{r};1),$$

$$(1-\zeta R;v',\cdots,v^{(r)},\zeta\mathfrak{r};1). \tag{4.6}$$

with the same notations and corresponding validity conditions that (3.12).

Putting $t_j=b_j(b_j-a_j)v_j; j=1,\cdots,s$ in (2.12), we obtain the following result.

Corollary 3.

$$\int_0^1\cdots\int_0^1\prod_{j=1}^s\frac{(1-v_j)^{\lambda_j}v_j^{\mu_j}}{X_j'^{\lambda_j+\mu_j+2}}S_U^V\left[a\prod_{j=1}^s\frac{(1-v_j)^{S_j}v_j^{T_j}}{X_j'^{S_j+T_j}}\right]S_n^{\alpha,\beta,\tau}\left[b\prod_{j=1}^sY^{\zeta_j};\mathfrak{r},t,q,A,B,k;l\right]$$

$$\gimel\begin{pmatrix}\mathrm{z}_1\prod_{j=1}^sY_j^{v_j'}\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{j=1}^sY_j^{v_j^{(r)}}\end{pmatrix}\mathrm{d}t_1\cdots\mathrm{d}t_s=\left\{\prod_{j=1}^s\{(1+\rho_j)^{-\lambda_j-1}(1+\sigma_j)^{-\mu_j-1}\right\}\sum_{K=0}^{[V/U]}\sum_{e,p,u,n}$$

$$\sum_{\tau_1,\cdots,\tau_s=0}^{\infty}\frac{(-V)_{UK}A_{V,K}}{K!}C(e,p,u,v)\left\{\prod_{j=1}^s\frac{(\beta_j-\alpha_j)^{\tau_j}(1+\rho_j)^{-K_jS_j-\gamma_j\zeta_jR-\tau_j}(1+\sigma_j)^{-KT_j-\delta_j\zeta_jR}}{\tau_j!\beta_j^{\tau_j+\zeta_jR}}\right\}a^Kb^R$$

$$\gimel^{U;0,n_r+3s:V;1,1}_{X;p_{i_r}+3s,q_{i_r}+2s,\tau_{i_r}:R_r:Y;1,1}\left(\begin{array}{c|c}\mathrm{z}_1\prod_{j=1}^s\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-v_j'} & \mathbb{A};\mathrm{A}_1,\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\prod_{j=1}^s\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-v_j^{(r)}} & \mathbb{B};\mathbf{B},\mathrm{B}_1:B\\ \mathrm{b}^{\mathfrak{r}}\prod_{j=1}^s\left\{\beta_j(1+\rho_j)^{\gamma_j}(1+\sigma_j)^{\delta_j}\right\}^{-\zeta_j\mathfrak{r}} & \end{array}\right) \tag{4.7}$$

where

$$X_j'=v_j(\rho_j-\sigma_j)+\rho_j+1 \tag{4.8}$$

and

$$Y_j=\frac{((1-v_j)^{\lambda_j}v_j^{\delta_j}(X_j')^{1-\gamma_j-\delta_j}}{(\alpha_j+\beta_j\rho_j)(1-v_j)+(1+\sigma_j)\beta_jv_j} \tag{4.9}$$

for $j=1,\cdots,s$

with the same notations and corresponding validity conditions that (3.12).

Remark : We obtain the same multiple Eulerian integrals about the functions cited in the section I.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of the multiple Eulerian integrals with general class of polynomials and general arguments utilized in this study, we can obtain a large variety of single, double and multiple Eulerian integrals. Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

REFERENCES.

[1] F. Ayant, An expansion formula for multivariable Gimel-function involving generalized Legendre Associated function, International Journal of Mathematics Trends and Technology (IJMTT), 56(4) (2018), 223-228.

[2] F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

[3] B.L.J. Braaksma, Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[4] I.S. Gradsteyn and I.M. Ryxhik, Table of integrals, series and products: Academic press, New York , (1980).

[5] Y.N. Prasad, Multivariable I-function , Vijnana Parisha Anusandhan Patrika 29 (1986), 231-237.

[6] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 1-12.

[7] R.K. Raina and H.M. Srivastava, Evaluation of certain class of Eulerian integrals. J. phys. A: Math.Gen. 26 (1993), 691-696.

[8] S.K. Raizada, A study of unified representation of special functions of mathematical physics and their use in statistical and boundary value problems, Ph.D. Thesis, Bundelkhand University, Jhansi, India, 1991.

[9] M. Saigo, and R.K. Saxena, Unified fractional integral formulas forthe multivariable H-function. J.Fractional Calculus 15 (l999), 91-107.

[10] H.M. Srivastava, A contour integral involving Fox's H-function, Indian. J. Math. 14(1972), 1-6.

[11] H.M. Srivastava and M. Garg, Some integrals involving general class of polynomials and the multivariable H-function. Rev. Roumaine. Phys. 32 (1987) 685-692.

[12] H.M. Srivastava and M.A. Hussain, Fractional integration of the H-function of several variables. Comput. Math. Appl. 30 (9) (1995),73-85.

[13] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[14] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.

[15] H.M. Srivastava and N.P. Singh, The integration of certains products of the multivariable H-function with a general class of polynomials, Rend. Circ. Mat. Palermo. 32(2)(1983), 157-187.