On k-Hyperperfect and Super Hyperperfect Numbers

V Puneeth[1]
[1]*Department of Mathematics, Christ (Deemed to be University)*
India

Abstract:
A positive integer n is said to be a superperfect number, if $\sigma(\sigma(n)) = 2n$. k-hyperperfect number, if $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$ where the function $\sigma(n)$ is the sum of all positive divisors of n. In this paper we investigate some general results on k-hyperperfect numbers and super hyperperfect numbers.

Keywords – *Divisor function, Mersenne prime, Perfect number, Super perfect number, Hyperperfect number and super hyperperfect number .*

I. INTRODUCTION

For a natural number n we denote the sum of positive divisors by $\sigma(n) = \sum_{d|n} d$. A positive integer n is said to be a perfect number if the sum of all its positive divisors is equal to two times the number. For any perfect number n, $\sigma(n) = 2n$.

All perfect numbers known are even. The question of existence of an odd perfect number still remains open. Euler proved that an even perfect number is of the form $2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are prime[6]. Prime numbers of the form $2^p - 1$where p is a prime are called Mersenne primes.

D.Suryanarayana[2] introduced the notion of superperfect numbers in 1969. A positive integer n is called superperfect if $\sigma^2(n) = \sigma(\sigma(n)) = 2n$. Even super perfect numbers are of the form 2^{p-1}, where $2^p - 1$ is a Mersenne prime. If any odd superperfect number exists then they are square numbers and either n or $\sigma(n)$ is divisible by atleast three distinct primes.

Manoli and Bear[3] introduced the concept of k-hyperperfect number and they conjectured that there are k-hyperperfect numbers for every k. A positive integer n is said to be k-hyperperfect number if $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$. A number is perfect if and only if it is 1-hyperperfect number.

D Suryanarayan[2] introduced the concept of super perfect numbers in the year 1968 and at first he considered only 2-superperfect numbers. He proved that the numbers of the form 2^{p-1} are 2-super perfect numbers only if $2^p - 1$ is a prime. Later in 2009, Antal Bege and Kinga Fogarasi have investigated some new ways of generalizing the perfect numbers. They have proved that if a number is of the form $3^{k-1}(3^k - 2)$ where $3^k - 2$ is a prime then the number is a 2-hyperperfect number. They have proved that if a number is of the form $3^p - 1$ where p and $\frac{3^p-1}{2}$ are primes then the number is a super hyperperfect number.

Main Results

Theorem 1. If k+1=p is a prime then the number of the form $\boldsymbol{n} = p^{q-1}(p^q - k)$ where $(p^q - k)$ is a prime is a k-hyperperfect number.

Proof:

$$\begin{aligned} n &= p^{q-1}(p^q - k) \\ \sigma(n) &= \sigma\left(p^{q-1}(p^q - k)\right) \\ &= \sigma(p^{q-1})\sigma(p^q - k) \\ &= \frac{p^q - 1}{p - 1}(p^q - k + 1) \\ &= \frac{p.p^{q-1}(p^q - k)}{p - 1} + \frac{k - 1}{p - 1} \\ &= \frac{p}{p - 1}n + \frac{k - 1}{p - 1} \\ &= \frac{k + 1}{k}n + \frac{k - 1}{k} \end{aligned}$$

Conjecture 1. Every k-hyperperfect numbers are of the form $\boldsymbol{n = p^{q-1}(p^q - k)}$, where $\boldsymbol{p^q - k}$ is a prime.

Theorem 2. Every 2-hyperperfect number is an odd number.

Proof: Let n be a 2-hyperperfect number. Then n is of the form $\mathbf{3^{q-1}(3^q-2)}$ such that $\mathbf{3^q-2}$ is a prime.
We shall show that $\boldsymbol{n \equiv 1(mod\ 2)}$.

We know that
$$\mathbf{3 \equiv 1(mod\ 2)}$$
$$\mathbf{3^q \equiv 1(mod\ 2)}$$
and also
$$\mathbf{3^{-1} \equiv 1(mod\ 2)}$$
$$\mathbf{3^{q-1} \equiv 1(mod\ 2)}$$
Since $\mathbf{3^q}$ is odd for every $\boldsymbol{q \in N}$ we have,
$$\mathbf{3^q-2 \equiv 1(mod\ 2)}$$
$$\mathbf{3^{q-1}(3^q-2) \equiv 1(mod\ 2)}$$
$$\boldsymbol{n \equiv 1(mod\ 2)}$$
Hence the proof.

Theorem 3. The distance between 2-hyperperfect numbers is a multiple of 4.

Proof: Let $\boldsymbol{\zeta(q_1) = 3^{q1-1}(3^{q1}-2)}$ and $\boldsymbol{\zeta(q_2) = 3^{q2-1}(3^{q2}-2)}$ be any two 2-hyperperfect numbers and $\boldsymbol{\zeta(q) = |\zeta(q_1)-\zeta(q_2)|}$ be the distance between them

WLOG let $\boldsymbol{\zeta(q_1) > \zeta(q_2)}$
$$\begin{aligned}\zeta(q) &= \zeta(q_1)-\zeta(q_2)\\ &= 3^{q1-1}(3^{q1}-2)-3^{q2-1}(3^{q2}-2)\\ &= 3^{2q1-1}-3^{2q2-1}+2.3^{-1}(3^{2q1}-3^{2q2})\end{aligned}$$
We know that $\mathbf{3^m} = \begin{cases} \mathbf{1(mod\ 4)} & \boldsymbol{if\ m \equiv 0\ (mod\ 2)} \\ \mathbf{-1(mod\ 4)} & \boldsymbol{if\ m \equiv 1\ (mod\ 2)} \end{cases}$

Case 1: $\boldsymbol{q_1 \equiv 0(mod\ 2)}$ and $\boldsymbol{q_2 \equiv 0(mod\ 2)}$
$$\begin{aligned}\zeta(q) &= \zeta(q_1)-\zeta(q_2)\\ &= 3^{2q1-1}-3^{2q2-1}+2.3^{-1}(3^{2q1}-3^{2q2})\\ &\equiv 0\ (mod\ 4)\end{aligned}$$

Case 2: $\boldsymbol{q_1 \equiv 1(mod\ 2)}$ and $\boldsymbol{q_2 \equiv 1(mod\ 2)}$
$$\begin{aligned}\zeta(q) &= \zeta(q_1)-\zeta(q_2)\\ &= 3^{2q1-1}-3^{2q2-1}+2.3^{-1}(3^{2q1}-3^{2q2})\\ &\equiv 0\ (mod\ 4)\end{aligned}$$

Case 3: $\boldsymbol{q_1 \equiv 1(mod\ 2)}$ and $\boldsymbol{q_2 \equiv 0(mod\ 2)}$
$$\begin{aligned}\zeta(q) &= \zeta(q_1)-\zeta(q_2)\\ &= 3^{2q1-1}-3^{2q2-1}+2.3^{-1}(3^{2q1}-3^{2q2})\\ &\equiv 4.3^{-1}\ (mod\ 4)\end{aligned}$$
But $\mathbf{3^{-1} \equiv 3\ (mod\ 4)}$
$$\equiv \mathbf{0}\ (\boldsymbol{mod}\ \mathbf{4})$$

Case 4: $\boldsymbol{q_1 \equiv 0(mod\ 2)}$ and $\boldsymbol{q_2 \equiv 1(mod\ 2)}$
$$\begin{aligned}\zeta(q) &= \zeta(q_1)-\zeta(q_2)\\ &= 3^{2q1-1}-3^{2q2-1}+2.3^{-1}(3^{2q1}-3^{2q2})\\ &\equiv 4.3^{-1}\ (mod\ 4)\end{aligned}$$
But $\mathbf{3^{-1} \equiv 3\ (mod\ 4)}$
$$\equiv \mathbf{0}\ (\boldsymbol{mod}\ \mathbf{4})$$
Hence the result

Theorem 4. If $\frac{p^{q-1}}{p-1}$ and $k+1=p$ are primes then $m=p^{q-1}$ is a super k-hyperperfect number.

Proof:

$$m = p^q - 1$$
$$\sigma(m) = \frac{p^q - 1}{p - 1}$$
$$\sigma(\sigma(m)) = \frac{p^q - 1}{p - 1} + 1$$
$$= \frac{p.p^{q-1}}{p - 1} + \frac{p - 2}{p - 1}$$
$$= \frac{p}{p - 1} m + \frac{p - 2}{p - 1}$$
$$= \frac{k + 1}{k} m + \frac{k - 1}{k}$$

Hence the proof.

Conjecture 2. Every super k-hyperperfect number is of the form p^{q-1} where p and $\frac{p^{q-1}}{p-1}$ are primes.

II. CONCLUSIONS

In this paper we have seen few generalized results for hyperperfect and super hyperperfect numbers. We can see some rigorous works done in perfect numbers and these topics are evolving and are interesting to work. We can see more conjectures rising in this field. We can also see couple of conjecture in this article. Solving these would lead to more interesting results.

ACKNOWLEDGMENT

I would like to thank the department of Mathematics Christ (Deemed to be University) for their continuous support. I would like to extend my gratitude to my beloved parents who have been a constant source of energy throughout the course. Last but not the least the motivation behind my work without whom this wouldn't have been possible, a sincere thanks to my wife, Ms.Poorvi for her constant guidance and support.

REFERENCES

[1] B.Das and H.K.Saikia, "Identities for near and deficient hyperperfect numbers," Indian, J. Math. Soft Comput., vol. 3, pp. 214-134, 2016.
[2] D.Suryanarayana, "Superperfect numbers," elem. Math., vol. 24, pp. 213-225, 1969.
[3] D.Manoli and R.Bear, "Hyperperfect numbers," Pi Mu Epsilon J., vol. 6, pp. 153-157, 1975.
[4] V.Puneeth and T.V.Joseph, "On k-Near Perfect Numbers," Int. J. Math. And. Appl., vol. 1, pp. 61-65, 2018.
[5] A.Bege and K.Fogarasi, "Generalised Perfect Numbers," Acta Univ. Sapientiae. Mathematica., vol. 1, pp. 73-82, 2009.
[6] L.E.Dickenson, History of theory of numbers, vol 1, Newyork: Chelsea Publishing Co., 1966.