Integrals and Fourier Series Involving the Product of Biorthogonal Polynomials and Special Functions

F.Y. AYANT[1]

1 Teacher in High School , France

ABSTRACT
In this paper, first we present two integrals pertaining to some special functions like biorthogonal polynomials, Aleph-function, multivariable aleph-function and the general class of multivariable polynomials. Later, we give two Fourier series for the product of some special functions like biorthogonal polynomials, Aleph-function, multivariable aleph-function and the general class of multivariable polynomials which have been obtained by the application of the integrals derived in first time.

Keywords : Aleph-function, general polynomials, multivariable Aleph-function, integrals, biorthogonal polynomials, Fourier series, M-series.

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1. Introduction

We use the following results in our investigations.
$U_n(x,k)$ and $V_n(x,k)$ pair of polynomials sets have been given by Prabhakar and Tomar [3]. Where

$$U_n(x,k)=\sum_{j=0}^{n}(-)^j\binom{n}{j}\frac{(\frac{j+1}{k})_n}{(1/k)n}\left(\frac{1-x}{2}\right)^j \tag{1.1}$$

and

$$V_n(x,k)=\sum_{j=0}^{n}(-)^j\binom{n}{j}\frac{(1+n)_{kj}}{(1)_{kj}}\left(\frac{1-x}{2}\right)^{kj} \tag{1.2}$$

The generalized polynomials defined by Srivastava ([6],p. 251, Eq. (C.1)), is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.3}$$

we shall note

$$a_1=\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \tag{1.4}$$

where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex. I we take $s=1$ in the (1.3) and denote $A[N,K]$ thus obtained by $A_{N,K}$, we arrive at general class of polynomials $S_N^M(x)$ study by Srivastava ([5],p. 1, Eq. 1).

The Aleph- function , introduced by Südland et al. [9,10], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z\ \middle|\ \begin{matrix}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r'}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r'}\end{matrix}\right)=\frac{1}{2\pi\omega}\int_L\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)z^{-s}\mathrm{d}s \tag{1.5}$$

for all z different to 0 and

$$\Omega^{M,N}_{P_i,Q_i,c_i;r'}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r'}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \quad (1.6)$$

With $|argz| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M}B_j + \sum_{j=1}^{N}A_j - c_i\left(\sum_{j=M+1}^{Q_i}B_{ji} + \sum_{j=N+1}^{P_i}A_{ji}\right) > 0, i = 1,\cdots,r'$

For convergence conditions and other details of Aleph-function , see Südland et al [9,10]. The serie representation of Aleph-function is given by Chaurasia and Singh [2].

$$\aleph^{M,N}_{P_i,Q_i,c_i;r'}(z) = \sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)}{B_G g!}z^{-s} \quad (1.7)$$

With $s = \eta_{G,g} = \frac{b_G+g}{B_G}, P_i < Q_i, |z| < 1$ and $\Omega^{M,N}_{P_i,Q_i,c_i;r'}(s)$ is given in (1.2) (1.8)

We will use the contracted form about the multivariable aleph-fonction $\aleph(z_1,\cdots,z_u)$ by :

$$\aleph(z_1,\cdots,z_r) = \aleph^{0,\mathfrak{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} z_1 & A\ ;\ C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ z_r & B;\ D \end{array}\right) = \frac{1}{(2\pi\omega)^u}\int_{L_1}\cdots\int_{L_u}\psi'(s_1,\cdots,s_u)\prod_{k=1}^{u}\theta'_k(s_k)z_k^{s_k}\mathrm{d}s_1\cdots\mathrm{d}s_u \quad (1.9)$$

with $\omega = \sqrt{-1}$

See Ayant [1], concerning the definition of the following quantities $V, W, \psi'(s_1,\cdots,s_r), A, B, C, D$ and $\theta'_k(s_k)$.

The complex numbers z_i are not zero. Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

A M-series was introduced and developed by Sharma and Jain [4] as

$$_p\overset{a}{M}_q(x) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{\Gamma(ak+1)} \quad (1.10)$$

Here, α ∈ C, $Re(\alpha) > 0$ and $(a_j)_k, (b_j)_k$ are the Pochammer symbols. The series (1.10) is defined when none of the parameters b_j s, j = 1, 2, , q, is a negative integer or zero. If any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. From the ratio test it is evident that the series in (1.10) is convergent for all z if p ≤ q, it is convergent for if p = q + 1 and divergent, if p > q + 1. When p = q + 1 and |z| = 1, the series can converge in some cases. Let $\beta = \sum_{j=1}^{p}a_j - \sum_{j=1}^{q}b_j$. It can be shown that when p = q + 1 the series is absolutely convergent for |z| = 1 if $Re(\beta) < 0$, conditionally convergent for z = −1 if 0 ≤ $Re(\beta) < 1$, and divergent for |z| = 1 if 1 ≤ $Re(\beta)$.

We shall note $c_k = \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{1}{\Gamma(ak+1)}$

2. Main integrals.

In this section, we shall establish two integrals pertaining to some special functions like biorthogonal polynomials, Aleph-function, multivariable aleph-function and the general class of multivariable polynomials.

Theorem 1.

$$\int_0^{\frac{\pi}{2}} \cos 2u\theta (sin\theta)^v U_n(1-2\sin^{2h}\theta;k) \aleph^{M,N}_{P_i,Q_i,c_i;r'}(z(\sin\theta)^{2\rho'_1})_p \overset{a}{M}_q(y(\sin\theta)^{2\rho'_2})$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}[y_1(\sin\theta)^{2\rho_1},\cdots,y_s(sin\theta)^{2\rho_s}]\aleph[z_1(\sin\theta)^{2\sigma_1},\cdots,z_r(sin\theta)^{2\sigma_r}]\mathrm{d}\theta =$$

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}(-)^j\binom{n}{j}\frac{\left(\frac{j+1}{k}\right)y^j}{(1/k)_n}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)}{B_G g!}z^{-s}$$

$$\frac{(-)^s\Gamma\left(\frac{1}{2}\pm u\right)}{2^{v+2hj+2\rho'_1\eta_{G;g}+2\rho'_2k+\sum_{i=1}^s 2\rho_iK_i}}a_1c_ky^ky_1^{K_1}\cdots y_s^{K_s}\aleph^{0,\mathfrak{n}+1:V}_{p_i+1,q_i+2,\tau_i;R:W}$$

$$\left(\begin{array}{c|c} 2^{-\sigma_1}x_1 & (\text{-v-2hj-}2\rho'_1\eta_{G,g}-2\rho'_2k-\sum_{l=1}^s 2\rho_lK_l:2\sigma_1,\cdots,2\sigma_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{-\sigma_r}x_r & \mathrm{B},\left(-\frac{v}{2}\pm\frac{u}{2}-hj-\rho'_1\eta_{G,g}-\rho'_2k-\sum_{l=1}^s 2\rho_lK_l:\sigma_1,\cdots,\sigma_r\right),D \end{array}\right) \tag{2.1}$$

provided that

$\sigma_i > 0, i=1,\cdots,r$ and $Re(v+2\rho'_1\eta_{G,g}+2\rho'_2k)+2\sum_{i=1}^r\sigma_i\min_{1\leqslant j\leqslant m_i} Re\left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$

$|argz(\sin\theta)^{2\rho'_1}| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^M B_j + \sum_{j=1}^N A_j - c_i\left(\sum_{j=M+1}^{Q_i} B_{ji} + \sum_{j=N+1}^{P_i} A_{ji}\right) > 0, i=1,\cdots,r'$

$\left|\arg(z_i(\sin\theta)^{2\sigma_i})\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by Ayant [1]

Proof

Express $U_n(1-2\sin^{2h}\theta;k)$, the Aleph-function, the M-serie and the multivariable polynomial in series with the help (1.1), (1,7), (1,10) and (1.3) respectively and expressing the multivariable Aleph-function in multiple Mellin-Barnes integrals. Interchange the series and Mellin-Barnes integrals and interchanging the multiple Mellin-Barnes integrals and θ-integral due to absolute convergence of the series and integrals involved in the process. Now evaluate the inner θ -integral with the help of the following integral

$$\int_0^{\pi/2}\cos 2u\theta(\sin\theta)^v\mathrm{d}\theta = \frac{\Gamma(v+1)\Gamma\left(\frac{1}{2}\pm u\right)}{2^{v+1}\Gamma\left(\frac{v}{2}\pm\frac{u}{2}+1\right)}, Re(v)>0$$

and interpret the expression in multivariable Aleph-function, after algebraic manipulations, we obtain the result (2.1) with the help of the above integral.

Theorem 2

$$\int_0^{\frac{\pi}{2}} \cos 2u\theta (sin\theta)^v V_n(1-2\sin^{2h}\theta;k) \aleph^{M,N}_{P_i,Q_i,c_i;r'}(z(\sin\theta)^{2\rho'_1})_p \overset{a}{M}_q(y(\sin\theta)^{2\rho'_2})$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}[y_1(\sin\theta)^{2\rho_1},\cdots,y_s(sin\theta)^{2\rho_s}]\aleph[z_1(\sin\theta)^{2\sigma_1},\cdots,z_r(sin\theta)^{2\sigma_r}]\mathrm{d}\theta =$$

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}(-)^j\binom{n}{j}\frac{(1+n)_{kj}}{(1)_{kj}}y^{kj}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)}{B_G g!}z^{-s}$$

$$\frac{(-)^s\Gamma\left(\frac{1}{2}\pm u\right)}{2^{v+2hj+2\rho'_1\eta_{G;g}+2\rho'_2k''+\sum_{i=1}^s 2\rho_iK_i+1}}a_1c_ky^ky_1^{K_1}\cdots y_s^{K_s}\aleph^{0,\mathfrak{n}+1:V}_{p_i+1,q_i+2,\tau_i;R:W}$$

$$\left(\begin{array}{c|c} 4^{-\sigma_1}x_1 & (\text{-v-2hkj-}2\rho_1'\eta_{G,g}-2\rho_2'k-\sum_{l=1}^{s}\rho_lK_l:2\sigma_1,\cdots,2\sigma_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{-\sigma_r}x_r & \text{B,}\left(-\frac{v}{2}\pm\frac{u}{2}-hkj-\rho_1'\eta_{G,g}-\rho_2'k-\sum_{l=1}^{s}\rho_lK_l:\sigma_1,\cdots,\sigma_r\right),D \end{array}\right) \tag{2.2}$$

provided that

$\sigma_i > 0, i = 1, \cdots, r$ and $Re(v + 2\rho_1'\eta_{G,g} + 2\rho_2'k) + 2\sum_{i=1}^{r}\sigma_i \min_{1\leqslant j\leqslant m_i} Re\left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$

$|argz(\sin\theta)^{2\rho_1'}| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M}B_j + \sum_{j=1}^{N}A_j - c_i\left(\sum_{j=M+1}^{Q_i}B_{ji} + \sum_{j=N+1}^{P_i}A_{ji}\right) > 0, i = 1, \cdots, r'$

$\left|\arg(z_i(\sin\theta)^{2\sigma_i})\right| < \frac{1}{2}A_i^{(k)}\pi$

To prove the theorem 2, we use the similar method that theorem 1.

Remarks:
We can calculate the integrals of following functions:

$g(\theta) = \cos 2u\theta(cos\theta)^v U_n(1 - 2\cos^{2h}\theta; k)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(\cos\theta)^{2\rho_1'})S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1(\cos\theta)^{2\rho_1}, \cdots, y_s(cos\theta)^{2\rho_s}]$
${}_p\overset{a}{M}_q(y(\cos\theta)^{2\rho_2'})\aleph[z_1(\cos\theta)^{2\sigma_1}, \cdots, z_r(cos\theta)^{2\sigma_r}]$

and

$h(\theta) = \cos 2u\theta(cos\theta)^v V_n(1 - 2\cos^{2h}\theta; k)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(\cos\theta)^{2\rho_1'})\aleph[z_1(\cos\theta)^{2\sigma_1}, \cdots, z_r(cos\theta)^{2\sigma_r}]$
$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1(\cos\theta)^{2\rho_1}, \cdots, y_s(cos\theta)^{2\rho_s}]_p\overset{a}{M}_q(y(\cos\theta)^{2\rho_2'})$

Concerning the proof of these integrals, the method is similar, we evaluate the inner θ-integral with the help of the following integral

$$\int_0^{\pi/2}\cos 2u\theta(\cos\theta)^v \mathrm{d}\theta = \frac{\pi\Gamma(u+1)}{2^{v+1}\Gamma\left(\frac{v}{2}\pm\frac{u}{2}+1\right)}, Re(v) > 0$$

3. Fourier series.

The following results related to Fourier series for several special functions and product of biorthogonal polynomials have been established in this section by making use of the results derivated in the section 2.

Theorem 3.

$$(sin\theta)^v U_n(1 - 2\sin^{2h}\theta; k)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(\sin\theta)^{2\rho_1'})_p\overset{a}{M}_q(y(\sin\theta)^{2\rho_2'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1(\sin\theta)^{2\rho_1}, \cdots, y_s(sin\theta)^{2\rho_s}]\aleph[z_1(\sin\theta)^{2\sigma_1}, \cdots, z_r(sin\theta)^{2\sigma_r}] =$$

$$\frac{1}{\pi 2^{v-1}}\sum_{m'=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}(-)^{\tau_2}\binom{n}{j}\frac{\left(\frac{j+1}{k}\right)y^j}{(1/k)_n}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!}z^{-s}$$

$$\frac{(-)^s\Gamma\left(\frac{1}{2}\pm m'\right)\cos 2m'\theta}{2^{v+2hj+2\rho_1'\eta_{G;g}+2\rho_2'k''+\sum_{i=1}^{s}2\rho_iK_i}}a_1c_k y^k y_1^{K_1}\cdots y_s^{K_s}\aleph_{p_i+1,q_i+2,\tau_i;R:W}^{0,\mathfrak{n}+1:V}$$

$$\left(\begin{array}{c|c} 4^{-\sigma_1}x_1 & (\text{-v-2hj-}2\rho_1'\eta_{G,g}-2\rho_2'k-\sum_{l=1}^s 2\rho_l K_l:2\sigma_1,\cdots,2\sigma_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{-\sigma_r}x_r & \text{B},\left(-\frac{v}{2}\pm\frac{u}{2}-hj-\rho_1'\eta_{G,g}-\rho_2'k-\sum_{l=1}^s 2\rho_l K_l:\sigma_1,\cdots,\sigma_r\right),D \end{array}\right) \tag{3.1}$$

under the same conditions that theorem 1.

Proof

Let

$$f(\theta)=(sin\theta)^v U_n(1-2\sin^{2h}\theta;k)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(\sin\theta)^{2\rho_1'})_p\overset{a}{M}_q(y(\sin\theta)^{2\rho_2'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1(\sin\theta)^{2\rho_1},\cdots,y_s(sin\theta)^{2\rho_s}]\aleph[z_1(\sin\theta)^{2\sigma_1},\cdots,z_r(sin\theta)^{2\sigma_r}]=\sum_{m'=0}^{\infty}A_{m'}\cos 2m'\theta, 0<\theta\pi \tag{3.2}$$

The above equation is valid since $f(\theta)$ is continuous and bounded variation in the open interval $\left(0,\frac{\pi}{2}\right)$. Now multiplying both sides of (3.2) by $\cos 2u\theta$ and integrating with respect to θ from 0 to $\pi/2$, we obtain

$$\int_0^{\frac{\pi}{2}}\cos 2u\theta(sin\theta)^v U_n(1-2\sin^{2h}\theta;k)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(\sin\theta)^{2\rho_1'})_p\overset{a}{M}_q(y(\sin\theta)^{2\rho_2'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1(\sin\theta)^{2\rho_1},\cdots,y_s(sin\theta)^{2\rho_s}]\aleph[z_1(\sin\theta)^{2\sigma_1},\cdots,z_r(sin\theta)^{2\sigma_r}]\mathrm{d}\theta=$$

$$\int_0^{\pi/2}\sum_{m'=0}^{\infty}A_{m'}\cos 2m'\theta\cos 2u\theta\mathrm{d}\theta=\sum_{m'=0}^{\infty}A_{m'}\int_0^{\pi/2}(\cos 2m'\theta)^2\mathrm{d}\theta,\text{ if } m'=u$$

$$=\sum_{m'=0}^{\infty}A_{m'}\frac{\pi}{4} \tag{3.3}$$

Now using the theorem 1 and the above equation, we obtain

$$A_{m'}=\frac{1}{\pi 2^{v-1}}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}(-)^{\tau_2}\binom{n}{j}\frac{\left(\frac{j+1}{k}\right)y^j}{(1/k)_n}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!}z^{-s}$$

$$\frac{(-)^s\Gamma\left(\frac{1}{2}\pm m'\right)}{\tau_1!\eta_{G,g}!g!2^{v+2hj+2\rho_1'\eta_{G;g}+2\rho_2'k''+\sum_{i=1}^s 2\rho_i K_i}}a_1 c_k y^k y_1^{K_1}\cdots y_s^{K_s}\aleph_{p_i+1,q_i+2,\tau_i;R:W}^{0,\mathfrak{n}+1:V}$$

$$\left(\begin{array}{c|c} 4^{-\sigma_1}x_1 & (\text{-v-2hj-}2\rho_1'\eta_{G,g}-2\rho_2'k-\sum_{l=1}^s 2\rho_l K_l:2\sigma_1,\cdots,2\sigma_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{-\sigma_r}x_r & \text{B},\left(-\frac{v}{2}\pm\frac{u}{2}-hj-\rho_1'\eta_{G,g}-\rho_2'k-\sum_{l=1}^s 2\rho_l K_l:\sigma_1,\cdots,\sigma_r\right),D \end{array}\right) \tag{3.4}$$

With the aid of equation (3.3) and (3.4) ,we obtain the result.

Theorem 4.

$$(sin\theta)^v V_n(1-2\sin^{2h}\theta;k)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(\sin\theta)^{2\rho_1'})_p\overset{a}{M}_q(y(\sin\theta)^{2\rho_2'})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1(\sin\theta)^{2\rho_1},\cdots,y_s(sin\theta)^{2\rho_s}]\aleph[z_1(\sin\theta)^{2\sigma_1},\cdots,z_r(sin\theta)^{2\sigma_r}]=$$

$$\sum_{m'=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}(-)^j\binom{n}{j}\frac{(1+n)_{kj}}{(1)_{kj}}y^{kj}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!}z^{-s}$$

$$\frac{(-)^s\Gamma\left(\frac{1}{2}\pm m'\right)\cos 2m'\theta}{\tau_1!\eta_{G,g}!g!2^{v+2hj+2\rho_1'\eta_{G;g}+2\rho_2'k''+\sum_{i=1}^{s}2\rho_iK_i+1}}a_1c_k y^k y_1^{K_1}\cdots y_s^{K_s}\aleph_{p_i+1,q_i+2,\tau_i;R:W}^{0,\mathfrak{n}+1:V}$$

$$\left(\begin{array}{c|c} 4^{-\sigma_1}x_1 & (\text{-v-2hkj-}2\rho_1'\eta_{G,g}-2\rho_2'k-\sum_{l=1}^{s}2\rho_lK_l:2\sigma_1,\cdots,2\sigma_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 4^{-\sigma_r}x_r & \mathrm{B},(-\frac{v}{2}\pm\frac{u}{2}-hkj-\rho_1'\eta_{G,g}-\rho_2'k-\sum_{l=1}^{s}2\rho_lK_l:\sigma_1,\cdots,\sigma_r),D \end{array}\right) \quad (3.5)$$

under the same conditions that theorem 1. The proof is similar that theorem 3.

Remarks :
We can calculate the Fourier series by the same method of following functions:

$$g(\theta)=\cos 2u\theta(cos\theta)^v U_n(1-2\cos^{2h}\theta;k)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(\cos\theta)^{2\rho_1'})\,{}_p\overset{a}{M}_q(y(\cos\theta)^{2\rho_2'})$$
$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1(\cos\theta)^{2\rho_1},\cdots,y_s(cos\theta)^{2\rho_s}]\aleph[z_1(\cos\theta)^{2\sigma_1},\cdots,z_r(cos\theta)^{2\sigma_r}]$$

and

$$h(\theta)=\cos 2u\theta(cos\theta)^v V_n(1-2\cos^{2h}\theta;k)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(\cos\theta)^{2\rho_1'})\,{}_p\overset{a}{M}_q(y(\cos\theta)^{2\rho_2'})$$
$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1(\cos\theta)^{2\rho_1},\cdots,y_s(cos\theta)^{2\rho_s}]\aleph[z_1(\cos\theta)^{2\sigma_1},\cdots,z_r(cos\theta)^{2\sigma_r}]$$

We obtain the same relation with the multivariable H-function defined by Srivastava and Panda [7,8].

4.Conclusion.

By specializing the various parameters as well as variables in the multivariable Aleph-function, the aleph-function, the M-serie, the biorthogonal polynomials and the multivariable polynomial, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics. We can obtain a large number of integrals and Fourier series.

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