Strongly *- labeling

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Abstract

*A graph G =(V,E) is said to be strongly *-graph if there exists a bijection $f: V \to \{1,2,\ldots,n\}$ in such a way that when an edge, whose end vertices are labeled i and j, is labeled with the value i+j+ij , all edge labels are distinct. In this paper we prove that $< K_{1,n}^{(1)}: K_{1,n}^{(2)} >$, $< K_{1,n}^{(1)}: K_{1,n}^{(2)}: \ldots: K_{1,n}^{(k)} >$, $< K_{1,n}^{(1)}: K_{1,n}^{(2)} >$ with cycle and $< K_{1,n}^{(1)}: K_{1,n}^{(2)}: \ldots: K_{1,n}^{(k)} >$ with cycle are strongly *- graph.*

Keywords - *Labeling, strongly *- labeling, strongly *- graph, star graph.*

I. INTRODUCTION

Graph labeling was first introduced in the 1960’s. Graph labeling is one of the fascinating areas of graph theory with wide range of applications. An enormous body of literature has grown around graph labeling in the last four decades. For more results on graph labeling one may refer to Gallian Survey [5].
Agida and Somashekara [1] have shown that all trees, cycles and grids are strongly *- graphs. Baskar Babujee and Vishnupriya [2] have proved that $C_n \times P_2, (P_2 \cup \overline{K_m}) + \overline{K_2}$, windmills $K_3^{(n)}$ and jelly fish graphs $J(m,n)$ are strongly *- graphs. Baskar Babujee and Beaula [3] proved that cycles and complete bipartite graphs are vertex strongly *- graphs. Baskar Babujee, Kannan and Vishnupriya [4] proved that wheels, paths, fans, crowns, $(P_2 \cup mK1+K2$ and umbrellas are vertex strongly *- graphs. Seoud and Mahran [6] give some technical necessary conditions for a graph to be strongly *- graph.

II. MAIN RESULTS

We begin this section with definition of strongly *- labeling.

***Definition* 2.1.** A graph $G = (V,E)$ is said to be strongly *-graph if there exists a bijection $f: V \to \{1,2,\ldots,n\}$ in such a way that when an edge, whose end vertices are labeled i and j, is labeled with the value $i+j+ij$, all edge labels are distinct.

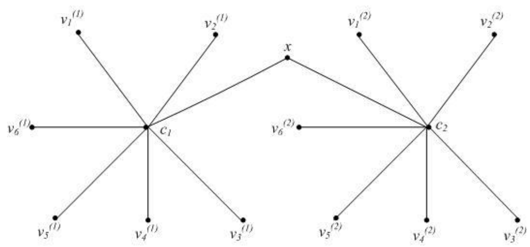
***Definition* 2.2.** Consider two stars $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ then $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)} >$ is the graph obtained by joining apex vertices of stars to a new vertex x. G has $2n+3$ vertices and $2n+2$ edges.

***Definition* 2.3.** Consider k copies of stars namely $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(k)}$ then $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)}: \ldots: K_{1,n}^{(k)} >$ is a graph obtained by joining apex vertices of each $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ to a new vertex x_{p-1}, where $2 \leq p \leq k$. G has $k(n+2)$ -1 vertices and $k(n+1)+1$ edges.

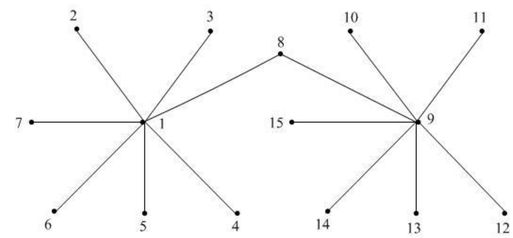
***Definition* 2.4.** Consider two stars $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ then $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)} >$ with cycle is the graph obtained by joining apex vertices of stars to two new vertices x and y. G has $2n+4$ vertices and $2n+4$ edges.

***Definition* 2.5.** Consider k copies of stars namely $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(k)}$ then $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)}: \ldots: K_{1,n}^{(k)} >$ with cycle is a graph obtained by joining apex vertices of each $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ to two new vertices x_{p-1} and x'_{p-1} where $2 \leq p \leq k$. G has $n(k+1)+1$ vertices and $n(k+1)+2$ edges.

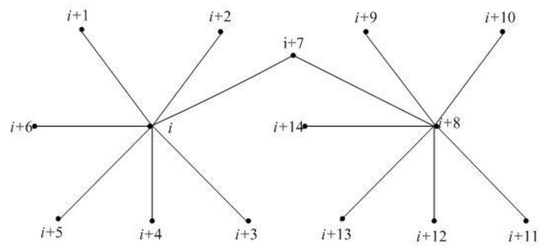
***Theorem* 2.1.** $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)} >$ is a strongly *-graph for $n \geq 2$.



(*a*)



(*b*)



(*c*)

Fig. 1 (a) $< \mathbf{K_{1,6}^{(1)}}: \mathbf{K_{1,6}^{(2)}} >$ **with vertices defined (b) strongly *- labeling of** $< \mathbf{K_{1,6}^{(1)}}: \mathbf{K_{1,6}^{(2)}} >$ **(c) labeling pattern of** $< \mathbf{K_{1,6}^{(1)}}: \mathbf{K_{1,6}^{(2)}} >$**.**

***Proof*:**

Let $v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)}$ be the pendant vertices of $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)}$ be the pendant vertices of $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and let they be adjacent to a new common vertex x. See Fig. 1(a).

Define $f: V(G) \to \{1,2, \dots, 2n+3\}$ as follows:

$$f(c_1) = 1$$

$$f\left(v_i^{(1)}\right) = i+1, 1 \leq i \leq n$$
$$f(x) = f\left(v_n^{(1)}\right) + 1$$

$$f(c_2) = f(x) + 1$$

$$f\left(v_i^{(2)}\right) = i + f(c_2), 1 \leq i \leq n$$

See Fig. 1(b). Then the edge labels take the value

$$f\left(c_1, v_i^{(1)}\right) = f(c_1) + f\left(v_i^{(1)}\right) + f(c_1) f\left(v_i^{(1)}\right), \ 1 \leq i \leq n$$

$$f\left(c_2, v_i^{(2)}\right) = f(c_2) + f\left(v_i^{(2)}\right) + f(c_2) f\left(v_i^{(2)}\right), \ 1 \leq i \leq n$$

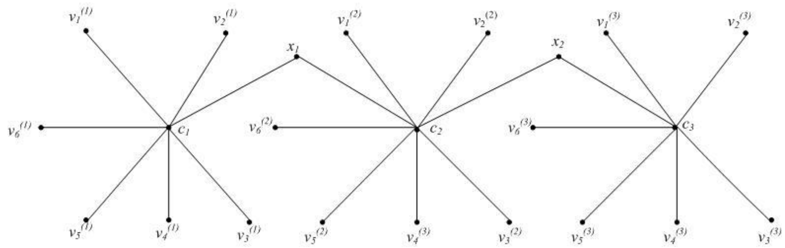
$$f(c_1, x) = f(c_1) + f(x) + f(c_1) f(x),$$

$$f(c_2, x) = f(c_2) + f(x) + f(c_2) f(x).$$

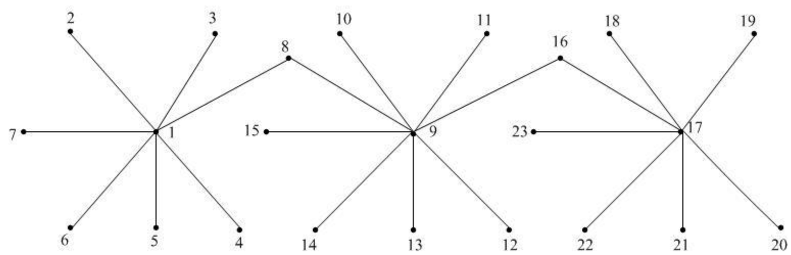
From Fig.1(b) it is clear that the edge labels produced are all distinct as the vertices are labeled with consecutive integers. So the labeling pattern defined above satisfies the conditions of strongly *-graph. Hence $G = < K_{1,n}^{(1)}: K_{1,n}^{(2)} >$ admits strongly *- labeling.

***Proof of correctness*:** Consider $G = < K_{1,n}^{(1)} : K_{1,n}^{(2)} >$. Let (c_1, x) and $(c_1, v_2^{(1)})$ be distinct edges of G. We have c_1 adjacent to x and c_1 adjacent to $v_2^{(1)}$ then, the label of the edge incident to the vertices c_1 and x may be equal to that of the edge incident to the vertices c_1 and $v_2^{(1)}$. Using the labeling function, we have $f(v_2^{(1)}) = i + 2$, $f(c_1) = i$, $f(x) = i + 7$ as shown in Fig. 1(c). Then the label of the two described edges are $(i+2)+(i)+(i+2)(i)$ and $(i)+(i+7)+(i)(i+7)$ which gives $4i^2+2+i^2$ and $9i+7+i^2$ which are not equal. Hence it is clear that the edge labels are distinct. In a similar manner we can check the labels of other edges.

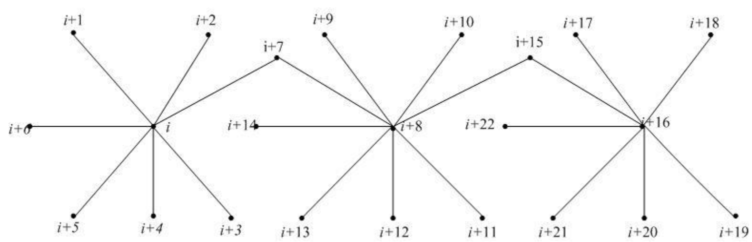
Theorem 2.2. $G = < K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$ is a strongly *- graph for $n \geq 2$ and $k \geq 3$.



(*a*)



(*b*)



(*c*)

***Fig.* 2 (*a*) $G = < K_{1,6}^{(1)} : K_{1,6}^{(2)} : \ldots : K_{1,6}^{(k)} >$ with vertices labeled (*b*) strongly *- labeling of $G = < K_{1,6}^{(1)} : K_{1,6}^{(2)} : \ldots : K_{1,6}^{(k)} >$ (*c*) labeling pattern of $G = < K_{1,6}^{(1)} : K_{1,6}^{(2)} : \ldots : K_{1,6}^{(k)} >$.**

***Proof*:** Let $K_{1,n}^{(k)}$ be k copies of star $K_{1,n}$, $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(j)}$ and c_j be the apex vertex of $K_{1,n}^{(j)}$ (Here $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$). Let $x_1, x_2, \ldots, x_{k-1}$ be the vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \leq p \leq k$. See Fig. 2(*a*). Define $f: V(G) \rightarrow \{1,2, \ldots, 3n+5\}$ as follows:

$$f(c_1) = 1$$

$$f(c_2) = f(x_1) + 1$$

$$f(c_3) = f(x_2) + 1$$

$$f\left(v_i^{(1)}\right) = i + 1, 1 \le i \le n$$

$$f\left(v_i^{(2)}\right) = i + f(c_2), 1 \le i \le n$$

$$f\left(v_i^{(3)}\right) = i + f(c_3), 1 \le i \le n$$

$$f(x_1) = f(c_1) - 1$$

$$f(x_2) = f(c_2) - 1$$

The vertices are labeled in the clockwise direction. See Fig. 2(*b*). Then the edge labels take the value

$$f\left(c_1, v_i^{(1)}\right) = f(c_1) + f\left(v_i^{(1)}\right) + f(c_1)f\left(v_i^{(1)}\right), \quad 1 \le i \le n$$

$$f\left(c_2, v_i^{(2)}\right) = f(c_2) + f\left(v_i^{(2)}\right) + f(c_2)f\left(v_i^{(2)}\right), \ 1 \le i \le n$$

$$f\left(c_3, v_i^{(3)}\right) = f(c_3) + f\left(v_i^{(3)}\right) + f(c_3)f\left(v_i^{(3)}\right), \ 1 \le i \le n$$

$$f(c_1x_1) = f(c_1) + f(x_1) + f(c_1)f(x_1),$$

$$f(c_2x_1) = f(c_3) + f(x_1) + f(c_3)f(x_1),$$

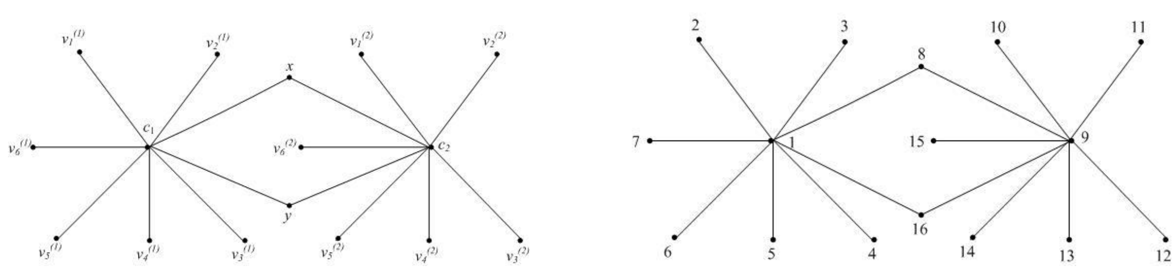
$$f(c_2x_2) = f(c_2) + f(x_2) + f(c_2)f(x_2),$$

$$f(c_3x_2) = f(c_3) + f(x_2) + f(c_3)f(x_2).$$

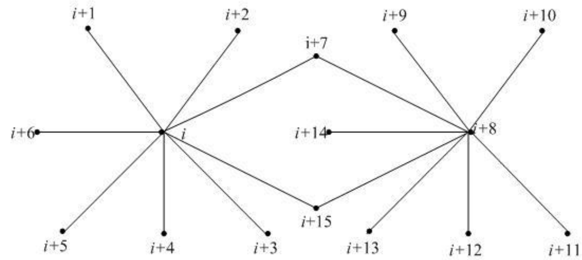
From Fig. 2(*b*) it is clear that the edge labels produced are all distinct as the vertices are labeled with consecutive integers. So the labeling pattern defined above satisfies the conditions of strongly *-graph. Hence $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$ admits strongly *- labeling.

***Proof of correctness*:** Consider $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$. Let ($c_1$, $v_1^{(1)}$) and (c_1, $v_6^{(1)}$) be distinct edges of *G*. We have c_1 adjacent to $v_1^{(1)}$ and c_1 adjacent to $v_6^{(1)}$ then, the label of the edge incident to the vertices c_1 and $v_1^{(1)}$ may be equal to that of the edge incident to the vertices c_1 and $v_6^{(1)}$. Using the labeling function, we have $f\left(v_6^{(1)}\right) = i + 6$, $f(c_1) = i$, $f\left(v_1^{(1)}\right) = i + 1$ as shown in Fig. 2(*c*). Then the label of the two described edges are $(i+6) + (i) + (i+6)(i)$ and $(i) + (i+1) + (i)(i+1)$ which gives $8i+i^2+6$ and $3i+1+i^2$ which are not equal. Hence it is clear that the edge labels are distinct. In a similar manner we can check the labels of other edges.

***Theorem* 2. 3.** $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ with cycle is a strongly *- graph for $n \ge 2$.



(*a*) (*b*)



(c)

Fig. 3 (a) $\mathbf{G} =< \mathbf{K}_{1,6}^{(1)}: \mathbf{K}_{1,6}^{(2)} >$ with vertices defined (b) strongly *- labeling of $\mathbf{G} =< \mathbf{K}_{1,6}^{(1)}: \mathbf{K}_{1,6}^{(2)} >$ with cycle (c) labeling pattern of $\mathbf{G} =< \mathbf{K}_{1,6}^{(1)}: \mathbf{K}_{1,6}^{(2)} >$.

***Proof*:** Let $v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)}$ be the pendant vertices of $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)}$ be the pendant vertices of $K_{1,n}^{(2)}$. Let c_1 and c_2 be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively and let they be adjacent to two new vertices x and y. See Fig. 3(a).

Define $f: V(G) \to \{1,2, \dots, 2n+4\}$ as follows:

$$f(c_1) = 1$$

$$f\left(v_i^{(1)}\right) = i + 1, 1 \le i \le n$$

$$f(x) = f\left(v_n^{(1)}\right) + 1$$

$$f(y) = f\left(v_n^{(2)}\right) + 1$$

$$f(c_2) = f(x) + 1$$

$$f\left(v_i^{(2)}\right) = i + f(c_2), 1 \le i \le n$$

See Fig. 3(b). Then the edge labels take the value

$$f\left(c_1, v_i^{(1)}\right) = f(c_1) + f\left(v_i^{(1)}\right) + f(c_1) f\left(v_i^{(1)}\right), \ 1 \le i \le n$$

$$f\left(c_2, v_i^{(2)}\right) = f(c_2) + f\left(v_i^{(2)}\right) + f(c_2) f\left(v_i^{(2)}\right), \ 1 \le i \le n$$

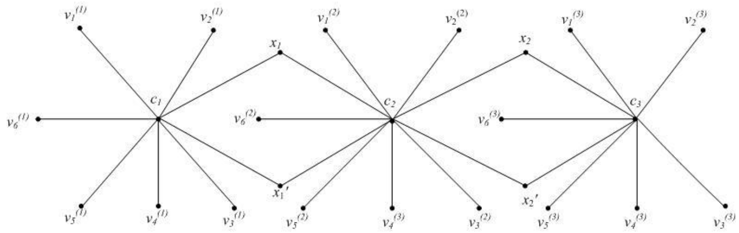
$$f(c_1 x) = f(c_1) + f(x) + f(c_1) f(x),$$

$$f(c_2 x) = f(c_2) + f(x) + f(c_2) f(x).$$

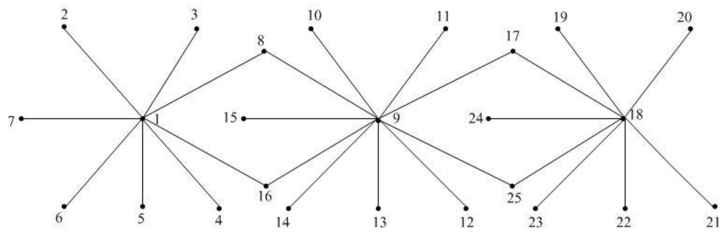
From Fig. 3(b) it is clear that the edge labels produced are all distinct as the vertices are labeled with consecutive integers. So the labeling pattern defined above satisfies the conditions of strongly *-graph. Hence $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)} >$ with cycle admits strongly *- labeling.

***Proof of correctness*:** Consider $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)} >$ with cycle. Let (c_1, $v_3^{(1)}$) and (c_1, $v_4^{(1)}$) be distinct edges of G. We have c_1 adjacent to $v_3^{(1)}$ and c_1 adjacent to $v_4^{(1)}$ then, the label of the edge incident to the vertices c_1 and $v_3^{(1)}$ may be equal to that of the edge incident to the vertices c_1 and $v_4^{(1)}$. Using the labeling function, we have $f\left(v_4^{(1)}\right) = i + 4$, $f(c_1) = i$, $f\left(v_3^{(1)}\right) = i + 3$ as shown in Fig. 3(c). Then the label of the two described edges are (i+4)+ (i) + (i+4)(i) and (i) + (i+3) + (i)(i+3) which gives 6i+4+i^2 and 5i+3+i^2 which are not equal. Hence it is clear that the edge labels are distinct. In a similar manner we can check the labels of the other edges.

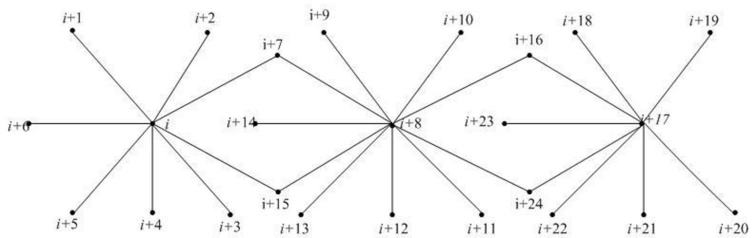
***Theorem* 2.4.** $G =< K_{1,n}^{(1)}: K_{1,n}^{(2)}: \ldots: K_{1,n}^{(k)} >$ with cycle is a strongly *-graph for $n \geq 2$ and $k \geq 3$.



(a)



(b)



(c)

***Fig.* 4 (a) $G =< K_{1,6}^{(1)}: K_{1,6}^{(2)}: \ldots: K_{1,6}^{(k)} >$ with cycle vertices labeled (b) strongly *- labeling of $G =< K_{1,6}^{(1)}: K_{1,6}^{(2)}: \ldots: K_{1,6}^{(k)} >$ with cycle (c) labeling pattern of $G =< K_{1,6}^{(1)}: K_{1,6}^{(2)}: \ldots: K_{1,6}^{(k)} >$.**

***Proof*:** Let $K_{1,n}^{(k)}$ be k copies of star $K_{1,n}$, $v_i^{(j)}$ be the pendant vertices of $K_{1,n}^{(i)}$ and c_j be the apex vertex of $K_{1,n}^{(j)}$ (Here i = 1, 2, …, n and j = 1, 2, …, k). Let x_1, x_2, …, x_{k-1} be the vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \leq p \leq k$. Let x_1', x_2', …, x_{k-1}' be vertices such that c_{p-1} and c_p are adjacent to x_{p-1} where $2 \leq p \leq k$. See Fig. 4(a). Define $f: V(G) \rightarrow \{1,2, \ldots ,4n + 1\}$ as follows:

$$f(c_1) = 1$$

$$f(c_2) = f(x_1) + 1$$

$$f(c_3) = f(x_2) + 1$$

$$f(v_i^{(1)}) = i + 1, 1 \leq i \leq n$$

$$f(v_i^{(2)}) = i + f(c_2), 1 \leq i \leq n$$

$$f(v_i^{(3)}) = i + f(c_3), 1 \leq i \leq n$$

$$f(x_1) = f(c_1) - 1$$

$$f(x_2) = f(c_2) - 1$$

$$f(x_1') = f\left(v_n^{(1)}\right) + 1$$

$$f(x_2') = f\left(v_n^{(2)}\right) + 1$$

The vertices are labeled in the clockwise direction. See Fig. 4(b).Then the edge labels take the value

$$f\left(c_1, v_i^{(1)}\right) = f(c_1) + f\left(v_i^{(1)}\right) + f(c_1)f\left(v_i^{(1)}\right),\ 1 \le i \le n$$

$$f\left(c_2, v_i^{(2)}\right) = f(c_2) + f\left(v_i^{(2)}\right) + f(c_2)f\left(v_i^{(2)}\right),\ 1 \le i \le n$$

$$f\left(c_3, v_i^{(3)}\right) = f(c_3) + f\left(v_i^{(3)}\right) + f(c_3)f\left(v_i^{(3)}\right),\ 1 \le i \le n$$

$$f(c_1x_1) = f(c_1) + f(x_1) + f(c_1)f(x_1),$$

$$f(c_2x_1) = f(c_3) + f(x_1) + f(c_3)f(x_1),$$

$$f(c_2x_2) = f(c_2) + f(x_2) + f(c_2)f(x_2),$$

$$f(c_3x_2) = f(c_3) + f(x_2) + f(c_3)f(x_2).$$

From Fig. 4(b) it is clear that the edge labels produced are all distinct as the vertices are labeled with consecutive integers. So the labeling pattern defined above satisfies the conditions of strongly *-graph. Hence $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$ with cycle admits strongly *- labeling.

***Proof of correctness*:** Consider $G =< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$ with cycle. Let (c_2, x_1) and (c_2, $v_1^{(2)}$) be distinct edges of G. We have c_2 adjacent to x_1 and c_2 adjacent to $v_1^{(2)}$ then, the label of the edge incident to the vertices c_1 and x_1 may be equal to that of the edge incident to the vertices c_1 and $v_1^{(2)}$. Using the labeling function, we have $f\left(v_1^{(2)}\right) = i + 9$, $f(c_2) = i + 8$, $f(x_1) = i + 7$ as shown in Fig. 4(c). Then the label of the two described edges are (i+9)+ (i+8) + (i+9)(i+8) and (i+8) + (i+7) + (i+8)(i+7) which gives $19i+89+i^2$ and $17i+71+i^2$ which are not equal. Hence it is clear that the edge labels are distinct. In a similar way we can check for other distinct edges.

III. CONCLUSION

In this paper we prove that $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$, $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$, $< K_{1,n}^{(1)} : K_{1,n}^{(2)} >$ with cycle and $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$ with cycle admits strongly *- labeling. Further our study of strongly *- graph of Interconnection networks is under investigation.

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