About 2- Isolate Inclusive Sets In Graphs

D.K.Thakkar[1] and N.J.Savaliya[2]
Department of Mathematics,Saurashtra University Campus,
University Road, Rajkot – 360005, Gujarat, India

Abstract:

In this paper we introduce in new concept call 2-isolate inclusive sets in graphs. Every 2-isolate inclusive set is an isolate inclusive set of G . We characterize maximal 2-isolate inclusive set of a graph. We deduce that every maximal 2-isolate inclusive sets of G is a distance-2 dominating set of G. We also define 2-isolate inclusive number of a graph and we observe that it is less then or equal to isolate inclusive number of the graph. We also prove that if the $< S >$ has the maximum number of 2-isolated vertices among all the 2-isolate inclusive sets then S is a maximum 2-packing of G. We also prove several other related results.

Keywords: *2-isolated vertex, 2-packing, 2-isolate inclusive set, maximum 2-isolate inclusive set, maximal 2-isolate inclusive set, distance-k dominating set, distance-2 open neighbourhood, 2-degree of vertex,2-isolate distance-2 dominating set, distance-2 private neighbourhood.*

AMS Subject Classification: 05C69

I. INTRODUCTION

The concept of isolate inclusive set was introduced in [3]. Several interesting results have been proved about isolate inclusive sets. We now introduce a new concept called 2-isolate inclusive sets in graphs. If $S \subset V(G)$ and $v \in S$ then v is said to be 2-isolated vertex in S. If $d(v,u) > 2$ distance between v and u stractly grather then 2, for all $u \in S$ if $u \neq v$. A set S of vertices is said to be 2-isolate inclusive set if it contains a 2-isolated vertex. We consider maximum 2-isolate inclusive sets and maximal 2-isolate inclusive sets in graphs. We prove that every maximal 2-isolate inclusive set is a distance-2 dominating set of G.

We observe that isolate inclusive number [3] of any graph is at least as be as 2-isolate inclusive number of the graph. We further observe that if a graph has an isolated vertex then it has only one 2-isolate inclusive set namely vertex set of the graph.

Here, we also introduce 2-isolate distance-2 dominating set in graphs. We further study the effect of removing a vertex from the graph on 2-isolate distance-2 domination number of a graph. We also consider the operation of edge removal an observe its effect on 2-isolate distance-2 domination number of the graph.

II. PRELIMINARIES AND NOTATIONS

If G is a graph then $V(G)$ denotes the vertex set of the graph G and $E(G)$ denotes the edge set of the graph G. If v is vertex of the graph G then $G - v$ is the subgraph of G induced by all the vertices different from v.
We will consider only simple undirected graphs with finite vertex set.

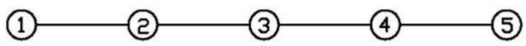
III. DEFINITIONS AND EXAMPLES

Definition 3.1 (2-isolated vertex) :
Let G be a graph and $S \subset V(G)$ a vertex $v \in S$ is said to be 2-isolated vertex of S if $d(v,u) > 2$, for all $u \in S$ with $u \neq v$.

Definition 3.2 (2-isolate inclusive set) :
Let G be a graph and $S \subset V(G)$ then S is said to be 2-isolate inclusive set if S contains a 2-isolated vertex.

It is obvious that every 2-isolated vertex of S is an isolated vertex of S and every 2-isolate inclusive is an isolate inclusive set.

Example 3.3: Consider the path graph P_5 with 5 vertices $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$



Let $S = \{1, 2, 5\}$.

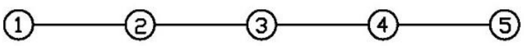
In S, 5 is a 2-isolated vertex of S and therefore S is a 2-isolate inclusive set.

Consider the path graph P_5 as above.
And let $T = \{1, 2, 4\}$. Then 4 is an isolate in T but it is not 2-isolate of T.

Remark 3.4:
Let G be a graph and $v \in V(G)$. Then v is 2-isolated vertex of $V(G)$ if and only if v is an isolated vertex of G.
Definition 3.5 (2-packing) :[7]
Let G be a graph and $S \subset V(G)$ then S is said to be a 2-packing if $d(u, v) > 2$, for all $u, v \in S$.
Let G be a graph. A 2-packing of G with maximum cardinality is called maximum 2-packing of G. The cardinality of a maximum 2-packing is called the packing number of G and it is denoted as $\delta(G)$.
Remark 3.6:
Let S be a 2-packing of G then every vertex of S is a 2-isolated vertex of S.

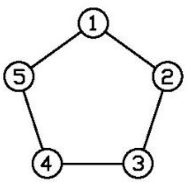
Definition 3.7 (maximum 2-isoinc set) :
Let G be a graph. A 2-isolate inclusive set with maximum cardinality is called a maximum 2-isoinc set and its cardinality is denoted as $\beta_{2is}(G)$.

Example 3.8: Consider the path graph P_5 with 5 vertices {1 , 2 , 3 , 4, 5}



Let $S = \{1, 2, 5\}$ then S is a maximum 2-isoinc set and $\beta_{2is}(5) = 3$.
$\beta_{2is}(G) = |S| \leq \beta_{is}(G)$
$\beta_{2is}(G) \leq \beta_{is}(G)$

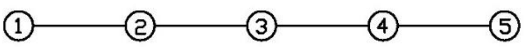
Example 3.9: Consider the cycle graph C_5 with 5 vertices {1 , 2 , 3 , 4, 5}



Let $S = \{1, 2, 3, 5\}$ then $\beta_{is}(G) = 4$ and
Let $S = \{1, 2, 5\}$ then $\beta_{2is}(G) = 3$.
Therefore $\beta_{2is}(G) < \beta_{is}(G)$.

Definition 3.10 (maximal 2-isoinc set) :
Let G be a graph and $S \subset V(G)$ be a 2-isoinc set then S is said to be a maximal 2-isoinc set if it is not properly contain in any isoinc set. Obviously every maximum 2-isoinc set is a maximal isoinc set.

Example 3.11: Consider the path graph P_5 with 5 vertices {1 , 2 , 3 , 4, 5}



Let $S = \{2, 5\}$ then S is a maximal 2-isoinc set but it is not a maximum 2-isoinc set.

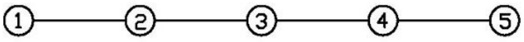
Definition 3.12 (distance-k dominating set) :[7]
Let G be a graph and $S \subset V(G)$. Then S is said to be a distance-k dominating set, if for every $v \in V(G) - S$, there is a vertex u in S such that $d(v, u) \leq k, (k \geq 1)$.

Definition 3.13 (distance-2 open neighbourhood) :[7]
Let G be a graph and $v \in V(G)$. Then the distance-2 open neighbourhood of v,

$N_2(v) = \{u \in V(G) \ni u \neq v \, \& \, d(u,v) \leq 2\}$ also the distance- 2 close neighbourhood of v , $N_2[v] = N_2(v) \cup \{v\}$.

Definition 3.14 (2-degree of vertex) :
Let G be a graph and $v \in \mathrm{V(G)}$. Then the cardinality of $|N_2(v)|$ will be called the 2-degree of vertex. The minimum 2-degree of a graph G will be denoted as $\delta_2(G)$.

Example 3.15: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$



Note that if v is an isolated vertex then 2-degree of $v = 0$. Conversely, also if 2-degree of $v = 0$ then v is an isolated vertex.
If $d(v) = 1$ then it is not necessary that $d_2(v) = 1$.

Example 3.16: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

Here, $d(1) = 1$ but $d_2(1) = 1$.

Definition 3.17 (distance-2 dominating set) :[7]
Let G be a graph and $S \subset V(G)$. Then S is said to be a distance-2 dominating set if for every $v \in V(G) - S$, there is a vertex u in S such that $d(v,u) \leq 2$.

A distance-2 dominating set with minimum cardinality is called a minimum distance-2 dominating set.

The cardinality of a minimum distance-2 dominating set is called the distance-2 domination number of the graph and it is denoted as $\gamma_{\leq 2}(G)$.

Definition 3.18 (minimal distance-2 dominating set) :[7]
A distance-2 dominating set S is said to be a minimal distance-2 dominating set if $S - \{v\}$ is not a distance-2 dominating set, for each $v \in S$.

Note that every minimum distance-2 dominating set is minimal distance-2 dominating set.

Definition 3.19 (2-isolate distance-2 dominating set) :
Let G be a graph and $S \subset V(G)$. Then S is said to be a 2-isolate distance-2 dominating set if
(1) S is a distance-2 dominating set
(2) $< S >$ contains a 2-isolated vertex.

Let G be a graph. A 2-isolate distance-2 dominating set with minimum cardinality is called a minimum 2-isolate distance-2 dominating set. It is denoted as $\gamma_{0\leq 2}$-set.

The cardinality of a $\gamma_{0\leq 2}$-set is called the 2-isolate distance-2 domination number of the graph and it is denoted as $\gamma_{0\leq 2}(G)$.

Example 3.20: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

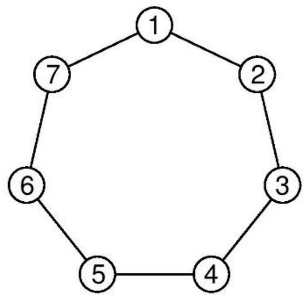
Let $S = \{2, 5\}$ then S is a minimum 2-isolate distance-2 dominating set of a graph.

Let $T = \{1, 4, 7\}$ then T is a minimum 2-isolate distance-2 dominating set of a graph.
Note that $|T| < |S|$.

Definition 3.21 (distance-2 private neighbourhood) :
Let G be a graph and S ⊂ V(G) and $v \in S$. Then distance-2 private neighbourhood of v with respect to the set S is equal to $P_{rnd\,2}[v, S] = \{w \in V(G) \ni N_2[w] \cap S = \{v\}\}$.

Remark 3.22:
Let G be a graph, $v \in V(G)$ and $v \in S$.
(1) If $d(v, u) > 2$ for every $u \in S$ with $u \neq v$ then $v \in P_{rnd\,2}[v, S]$.
(2) If $x \in S$ and $x \neq v$ then $x \notin P_{rnd\,2}[v, S]$.
(3) If $w \in V(G) - S$ then $w \in P_{rnd\,2}[v, S]$ if and only if v is the only vertex in S whose distance from w is ≤ 2.

Example 3.23: Consider the cycle graph C_7 with 7 vertices {1 , 2 , 3 , 4, 5, 6, 7}



Let $S = \{2, 5\}$. Let $v = \{2\}$.
(1) $v \notin P_{rnd\,2}[v, S]$ because $v \in S, 1 \in S$ and $d(v, 1) \leq 2$.
(2) $1 \notin P_{rnd\,2}[v, S]$ because $1 \in S$ and $1 \neq S$.
(3) $3 \notin S$ but $3 \in P_{rnd\,2}[v, S]$ because $d(3, 2) \leq 2$ and also $d(3, 1) \leq 2$.
(4) $4 \notin S$ but $4 \in P_{rnd\,2}[v, S]$ because $d(4, v) = 2$ and also $d(4, 1) = 3$. Which is > 2.
Similarly $5 \in P_{rnd\,2}[v, S]$.
(5) $6 \notin S$ but $6 \notin P_{rnd\,2}[v, S]$ because $d(6, v) = 3 > 2$.

IV. MAIN RESULT

Proposition 4.1: Let G be a graph, S ⊂ V(G) and $v \in$ S then v is a 2-isolated vertex of S if and only if $N(v) \cap N(u) = \emptyset$, for every $u \in S$ with $u \neq v$.

Proof: Suppose v is 2 -isolated in S and suppose u is in S, $N(v) \cap N(u) \neq \emptyset$.
Let $2 \in N(v) \cap N(u)$. Then v is adjacent to 2 and 2 is adjacent to u.
Therefore $d(v, u) \leq d(v, z) + d(z, u) = 1 + 1 = 2$.
This is a contradiction.
Therefore for every $u \in S$ with $u \neq v$.
$N(v) \cap N(u) = \emptyset$.

Conversely, suppose condition is holds.
Then $d(v, u) > 2$, for every $u \in S$ with $u \neq v$.
Therefore v is 2 isolated in S ▌

Proposition 4.2: Let S be a 2-isoinc set and $v \in V(G) - S$. Then $S \cup \{v\}$ is not a 2-isoinc set if and only if $d(v, u) \leq 2$, for all 2-isolated verties u of S.

Proof: Suppose $S \cup \{v\}$ is not a 2-isoinc set.
Then $d(u, x) \leq 2$, for every isolate u of S and for some $x \in S \cup \{v\}$ but $d(u, w) > 2$.
For each $w \in S$ therefore $d(u, v) \leq 2$, for each 2-isolated vertex u of S.

Conversely, suppose $d(u, v) \leq 2$, for each 2-isolated vertex u of S.
Then obviously $S \cup \{v\}$ does not have any 2-isolated vertex. ▌

Theorem 4.3: Let G be a graph and S be a 2-isoinc set of G. Then S is a maximal 2-isoinc set if and only if for every $v \in V(G) - S, S \cup \{v\}$ is not a 2-isoinc set of G.

Proof: Suppose S is maximal 2-isoinc set.
Let $v \in V(G) - S$. Since $S \cup \{v\}$ properly contain S, $S \cup \{v\}$ cannot be a 2-isoinc set of G.

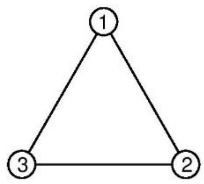
Conversely, suppose the condition holds.
Suppose $T \subset V(G)$ is such that S is a proper subset of T if $T = S \cup \{v\}$ for some $u \in V(G) - S$ then by the given condition T cannot be a 2-isoinc set of G.
Therefore we may assume that $|T| - |S| \geq 2$.
Let $v \in T - S$ by the given condition $S \cup \{v\}$ does not have any 2-isolated vertex.
Let $u \in T - S$ be such that $S \cup \{u\}$ does not have any 2-isolated vertex.
Continuing this way, we see that $T = S \cup \{x_1, x_2, \dots \dots \dots \dots, x_k\}$ is not a 2 -isoinc set $T - S = \{x_1, x_2, \dots \dots \dots \dots, x_k\}$.
Thus, the theorem is prove. ▌

Theorem 4.4: Let G be a graph with $\beta_{2is}(G) \geq 2$ then $\beta_{2is}(G) < \beta_{is}(G)$.

Proof: Let S be a maximum 2-isoinc set of G.
Let $v, v' \in S$ and assume that v is 2-isolated vertex of S. Then $d(v, v') > 2$.
Suppose $d(v, v') = 3$.
Let vu_1u_2v' with a shortest path joining v & v' in G. Then $u_1 \notin S$ and $u_2 \notin S$.
Also note that u_2 is not adjacent to v.
Let $S_1 = S \cup \{u_2\}$. Since v is not adjacent to u_2 and v is also not adjacent to any vertex of S. It follows that v is not adjacent to any vertex of S_1.
Therefore S_1 is an isoinc set of G.
Therefore $\beta_{is}(G) \geq |S_1| > |S| = \beta_{2is}(G)$.
Thus $\beta_{2is}(G) < \beta_{is}(G)$. ▌

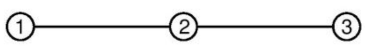
Remark : If $\beta_{2is}(G) = 1$ then the above theorem is not true.

Example 4.5: Consider the triangle with vertices {1 , 2 , 3}



Then $\beta_{is}(G) = 1$ and $\beta_{2is}(G) = 1$.
However, it is also not true that $\beta_{is}(G) = \beta_{2is}(G)$ if $\beta_{2is}(G) = 1$.

Example 4.6: Consider the path graph $G = P_3$ with 3 vertices {1 , 2 , 3}



Here, $\beta_{2is}(G) = 1$ and $\beta_{is}(G) = 2$.

Theorem 4.7: Let G be a graph and $S \subset V(G)$ be a 2-isoinc set of G. Then S is a maximal 2-isoinc set if and only if for each $v \in V(G) - S, d(v,u) \leq 2$ for every 2-isolated vertex u of S.

Proof: Suppose S is a maximal and $v \in V(G) - S$.
Let $S_1 = S \cup \{v\}$. Then S_1 does not have any 2-isolated vertex. This means that $d(v,u) \leq 2$ for every 2-isolated vertex u of S, $S \cup \{v\}$ cannot have any 2-isolated vertex.
Thus, S is a maximal 2-isoinc set of G.

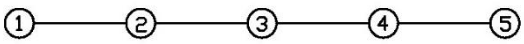
Corollary 4.8: Let G be a graph and v be an isolated vertex of G. If S is any maximal 2-isoinc set then $v \in S$.

Proof: Suppose for some maximal 2-isoinc set S, $v \notin S$. Then $d(v,u) \leq 2$, for every 2-isolated vertex u of S.
This implies that v is not a isolated vertex in G.
Which is a contradiction.
Thus the result is proved.

Corollary 4.9: Let G be a graph and $S \subset V(G)$ be a maximal 2-isoinc set of G. Then S is a distance-2 dominating set of G.

Proof: Let $v \in V(G) - S$.
By theorem-4.5, $d(v,u) \leq 2$ for every 2-isolated vertex u of S.
There is a vertex u in S which is a 2-isolated vertex of S.
Therefore $d(v,u) \leq 2$.
Thus S is a distance-2 dominating set of G. ▌

Example 4.10: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$



Note that if v is an isolated vertex then $\delta_2(v) = 0$. Conversely also if $\delta_2(v) = 0$ then v is an isolated vertex.
If $d(v) = 1$ then it is not necessary that $d_2(v) = 1$.

Theorem 4.11: Let G be a graph and $v \in V(G) \ni d_2(v) = \delta_2(v)$. Let $T = V(G) - N_2(v)$ then T is a maximum 2-isoinc set of G.

Proof: Obviously, T is a 2-isoinc set of G.
Suppose T is not a maximum 2-isoinc set of G.
Then there is a 2-isoinc set of G such that $|S| > |T|$.
Let x be any 2-isolated vertex of S. Then
(1) $d_2(v) = \delta_2(v)$.
(2) $N_2(x) \subset V(G) - S$.
Therefore $S \subset V(G) - N_2(x) \subset V(G) - N_2(x) = T$ and
Therefore $|S| \leq |T|$.
Which is a contradiction.
Therefore T is a maximum 2-isoinc set of G. ▌

Theorem 4.12: Let G be a graph and T be a maximum 2-isoinc set of G. Then there is $v \in T \ni d_2(v) = \delta_2(G)$ and $T = V(G) - N_2(v)$.

Proof: Let v be any isolated vertex of $< T >$ then $N_2(v) \subset V(G) - T$ or $T \subset V(G) - N_2(v)$.
Now $V(G) - N_2(v)$ is an isoinc set of G and also note that $|T| \leq |V(G) - N_2(v)|$.

Therefore T be a maximum 2-isoinc set of G.
$|T| = |V(G) - N_2(v)|$. Since $T \subset V(G) - N_2(v)$, $T = V(G) - N_2(v)$.
Suppose $d_2(v) > \delta_2(G)$.
Let x be a vertex of G such that $d_2(x) = \delta_2(G)$, by above theorem-4.8 $V(G) - N_2(v)$is a maximum 2-isoinc set of G. Since $d_2(v) > d_2(x)$.
$|V(G) - N_2(v)| < |V(G) - N_2(x)|$.
Then this implies that $V(G) - N_2(v)$ is not a maximum 2-isoinc set of G.
Which is a contradiction.
Thus $d_2(v) = \delta_2(G)$. ▌

Corollary 4.13: Let G be a graph and v be an isolated vertex then $V(G)$ is the only maximum 2-isoinc set of G.

Proof: Since v is an isolated vertex, $N_2(v) = \emptyset$ and by the above theorem, $V(G) - N_2(v) = V(G)$ is a maximum 2-isoinc set of G.
If S is a proper subset of $V(G)$ then obviously S cannot be a maximum 2-isoinc set of G.
Thus $V(G)$ is the only maximum 2-isoinc set of G. ▌

Theorem 4.14: Let G be a graph and $S \subset V(G)$ be such that $< S >$ has the maximum number of 2-isolated vertex among all the 2-isoinc set of G. Then S is a maximum 2-paking of G.

Proof: Let S_1 be the set of all 2-isolated vertices of $< S >$.
Let M be a maximum 2-paking of G. Then $|S_1| \geq |M| = \delta(G)$.
Note that S_1 itself 2-paking of G with $|S_1| \geq |M|$.
Therefore $|S_1| = |M|$.
Thus S_1 is a maximum 2-paking of G.
Suppose $|S| > |S_1|$. Let $x \in S \ni x \notin S$. Since $S_1$is a maximum 2-paking, $d(x, y) \leq 2$, for some y in S_1 but then this means that y is not a 2-isolated vertices in the $< S >$.
Which is a contradiction.
Thus $|S_1| = |S|$ and hence $S_1 = S$ and
Therefore S is a maximum 2-paking of G. ▌

Proposition 4.15: If S is a maximum 2-isoinc set then S is a 2-isolate distance-2 dominating set of G.

Proof: Let $v \in V(G) - S$.
Then by theorem-4.5, there is a vertex u in S such that $d(v, u) \leq 2$.
Thus S is a 2-isolate distance-2 dominating set of G. ▌

Theorem 4.16: Let S is a maximum 2-isoinc set of G
(1) For each 2-isolated vertex v of S, $N_2(v) = V(G) - S$.
(2) If u and v are 2-isolates of S then $d_2(u) = d_2(v)$.

Proof: (1) Let v be an 2-isolated vertex of S then $N_2(v) \subset V(G) - S$.
Let $x \in V(G) - S$.
Since S is maximal, $d(x, w) \leq 2$, every 2-isolated vertex w of S.
Therefore $x \in N_2(v)$.
Thus $N_2(v) = V(G) - S$.

(2) If u and v be two 2-isolates of S then
$d_2(u) = |N_2(u)| = |V(G) - S| = |N_2(v)| = d_2(v)$.
Thus $d_2(u) = d_2(v)$. ▌

Remark 4.17: Let G be a graph and v be an isolated vertex in G. It is obvious that $d(x,y)$ in G is equal to $d(x,y)$ in $G-v$ as v is an isolated vertex in G.

Theorem 4.18: Let G be a graph and v be an isolated vertex in G. Then $\gamma_{\trianglelefteq 2}(G) \leq \gamma_{\trianglelefteq 2}(G-v)$ if and only if for every minimum 2-isolate distance-2 dominating set S of G. The following condition is satisfied
C: v is the only 2-isolated vertex in the $< S >$.

Proof: Suppose $\gamma_{\trianglelefteq 2}(G) \leq \gamma_{\trianglelefteq 2}(G-v)$.
Let S be any minimum 2-isolate distance-2 dominating set of G. Since v is an isolated vertex of $G, v \in S$.
Suppose there is vertex $v' \in S \ni v' = v \;\&\; v'$ is 2-isolated vertex of S.
Now let $S_1 = S - \{v\}$. Consider the subgraph $G - v$.
Let $x \notin S_1$. Then there is a vertex y in S such that $d(x,y) \leq 2$ in G. Obviously $y \neq v$.
Therefore $d(x,y) \leq 2$ in $G - v$ also.
Thus S_1 is a 2-isolate distance-2 dominating set of $G - v$.
Therefore $\gamma_{\trianglelefteq 2}(G-v) \leq |S_1| < |S| = \gamma_{\trianglelefteq 2}(G)$.
This contradict the hypothesis that $\gamma_{\trianglelefteq 2}(G) \leq \gamma_{\trianglelefteq 2}(G-v)$.
Therefore v is the only 2-isolated vertex of S.

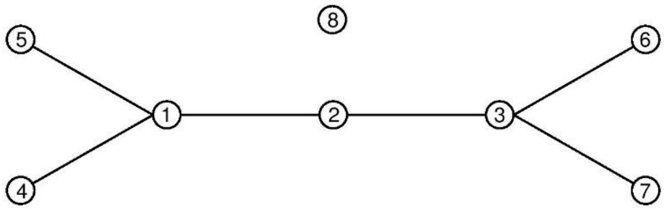
Conversely, suppose the condition is satisfied.
For any minimum 2-isolate distance-2 dominating set of G.
Let T be any set of vertices of $G - v$ such that $|T| < \gamma_{\trianglelefteq 2}(G)$.
Suppose T is a 2-isolate distance-2 dominating set in $G - v$.
Let x be any vertex of G such that $x \neq v$ and $x \notin T$. There is a vertex y in T such that $d(x,y) \leq 2$ in $G - v$.
Since v is an isolated vertex, by the above remark $d(x,y) \leq 2$ in G also.
Note let $S = T \cup \{v\}$ then S is a minimum 2-isolate distance-2 dominating set of G.
Then $|S| = \gamma_{\trianglelefteq 2}(G)$ and S contains two 2-isolated vertices including v.
Which is a contradicyion.
Therefore if $|T| < \gamma_{\trianglelefteq 2}(G)$ then T cannot be 2-isolate distance-2 dominating set of G.
Therefore $|T| \geq \gamma_{\trianglelefteq 2}(G)$.
Therefore $\gamma_{\trianglelefteq 2}(G-v) \geq \gamma_{\trianglelefteq 2}(G)$. ▌

Theorem 4.19: Let G be a graph and v be an isolated vertex of G. Then $\gamma_{\trianglelefteq 2}(G-v) < \gamma_{\trianglelefteq 2}(G)$ if and only if there is a minimum 2-isolate distance-2 dominating set S of G such that S contains an isolate different from v.

Proof: Suppose $\gamma_{\trianglelefteq 2}(G-v) < \gamma_{\trianglelefteq 2}(G)$.
Let S_1 be a minimum 2-isolate distance-2 dominating set of $G - v$.
Then S_1 cannot be 2-isolate distance-2 dominating set of G because $|S_1| = \gamma_{\trianglelefteq 2}(G-v) < \gamma_{\trianglelefteq 2}(G)$.
Let $S = S_1 \cup \{v\}$.
Let x be a vertex of S_1 such that $x \notin S$ and $x \notin S_1$ also. Since S_1 is a 2-isolate distance-2 dominating set of $G - v, d(x,y) \leq 2$ in $G - v$, for some y in S. Then $d(x,y) \leq 2$ in G also.
Thus S is a 2-isolate distance-2 dominating set of G such that $v \in S$. Since $|S| = |S_1| + 1$, S is a minimum 2-isolate distance-2 dominating set of G.
Let v' be if 2-isolated vertex of S_1 then v' is also 2-isolated vertex of S as v is an isolated vertex of G.
Thus S is a minimum 2-isolate distance-2 dominating set of G which contains an isolate different from v.

Conversely, suppose there is a minimum 2-isolate distance-2 dominating set S of G such that S contains a 2-isolate different from v. Since v is an isolated vertex in $G, v \in S$.
Let $S_1 = S - \{v\}$. Then S_1 contains a 2-isolate (which is different from v).
Therefore S_1 is a 2-isolate distance-2 dominating set of $G - v$.
Thus $\gamma_{\trianglelefteq 2}(G-v) \leq |S_1| < |S| = \gamma_{\trianglelefteq 2}(G)$.
Thus $\gamma_{\trianglelefteq 2}(G-v) < \gamma_{\trianglelefteq 2}(G)$. ▌

Example 4.20: Consider the graph G with vertices $\{1, 2, 3, 4, 5, 6, 7, 8\}$ mansion blow.



We may note that the set $S = \{1, 2, 8\}$ is a minimum 2-isolate distance-2 dominating set of G. Also note that for any minimum set T of G, 8 is the only 2-isolated vertex of T.

Now consider the subgraph $G - 8$. Consider the set $S_1 = \{1, 6, 7\}$. Then S_1 is a minimum 2-isolate distance-2 dominating set of $G - 8$.
Thus $\gamma_{\trianglelefteq 2}(G - 8) = \gamma_{\trianglelefteq 2}(G)$.

Example 4.21: Consider the graph G with vertices $\{1, 2, 3, 4, 5\}$ mansion blow.

Let $S = \{1, 4, 5\}$. Then S is a minimum 2-isolate distance-2 dominating set of G. Note that S contains 2-isolate different from 5.(infact 1 and 4 are both 2-isolates of S)

Now consider the subgraph $G - 5$. Let $T = \{1, 4\}$. Then T is a minimum 2-isolate distance-2 dominating set of $G - 5$. Therefore thus $\gamma_{\trianglelefteq 2}(G - 5) = 2 < 3 = \gamma_{\trianglelefteq 2}(G)$.

Corollary 4.22: Let G be a graph and $v_1, v_2, \dots\dots\dots, v_k$ be all the isolated vertices of $G (k \geq 2)$. Then $\gamma_{\trianglelefteq 2}(G - v_i) < \gamma_{\trianglelefteq 2}(G)$, for all $i = 1, 2, \dots\dots\dots, k$.

Proof: Let S be a minimum 2-isolate distance-2 dominating set of G.
Then $v_i \in S$, for every $i = 1, 2, \dots\dots\dots, k$.
Then by above corollary,
$\gamma_{\trianglelefteq 2}(G - v_i) < \gamma_{\trianglelefteq 2}(G)$, for all $i = 1, 2, \dots\dots\dots, k$. ▌

Corollary 4.23: If there is a 2-isolated vertex v such that $\gamma_{\trianglelefteq 2}(G - v) \geq \gamma_{\trianglelefteq 2}(G)$ then the graph has only one 2-isolated vertex namely v.

Proof: Obvious . ▌

Theorem 4.24: Let G be a graph and v be a non isolated vertex in G. Then $\gamma_{\trianglelefteq 2}(G - v) > \gamma_{\trianglelefteq 2}(G)$ if and only if the following two conditions are satisfied.
(1) For every a minimum 2-isolate distance-2 dominating set S of G, $d(v, S) \leq 1$
(2) There is no subset S of $V(G) - N_2(v)$ such that $|S| \leq \gamma_{\trianglelefteq 2}(G)$ and S is a minimum 2-isolate distance-2 dominating set of $G - v$.

Proof: Suppose $\gamma_{\trianglelefteq 2}(G - v) > \gamma_{\trianglelefteq 2}(G)$.
(1) Let S be a minimum 2-isolate distance-2 dominating set of G.
If $v \in S$ then $d(v, S) = 0 < 1$.
Suppose $v \notin S$.
Consider the subgraph $G - v$. Since $|S| \leq \gamma_{\trianglelefteq 2}(G - v)$, S cannot be 2-isolate distance-2 dominating set of $G - v$.
Note that any 2-isolate of S in G is also 2-isolate of S in $G - v$.

Therefore the $< S >$ in $G - v$ contains 2-isolated vertices. It follows that S is not a distance-2 dominating set of $G - v$.
Therefore there is a vertex x in $G - v$ such that $x \notin S$ and $d(x, S) \geq 3$ in $G - v$. Since S is a distance-2 dominating set of G, $d(x, S) \geq 2$ in G.
Let P be a path joining x to some vertex z of S such that length of $P \leq 2$. Obviously this path contain v.Since $x \neq v, x$ is adjacent to v and v is adjacent to z in G.
Thus $d(v, S) = 1$.
Therefore from both the above cases it follows that $d(v, S) \leq 1$.

(2) Suppose there is a set $S \subset V(G) - N_2[v]$ such that $|S| \leq \gamma_{\leq 2}(G)$ and S is a 2-isolate distance-2 dominating set of $G - v$. Then $\gamma_{\leq 2}(G - v) \leq |S| \leq \gamma_{\leq 2}(G)$. Which implies that $\gamma_{\leq 2}(G - v) \leq \gamma_{\leq 2}(G)$.
which is a contradiction.
Therefore (2) is also proved.

Thus the theorem is proved.

Now we state and prove a necessary and sufficient condition under which 2-isolate distance-2 domination number decreases when vertex is remove from the graph.

Theorem 4.25: Let G be a graph and $v \in V(G)$. Then $\gamma_{0 \leq 2}(G - v) > \gamma_{0 \leq 2}(G)$ if and only if there is a minimum 2-isolate distance-2 dominating set of S such that
(1) S contains a 2-isolate different from v.
(2) $v \in S$ and $P_{rnd\,2}[v, S] = \{v\}$.

Proof: Suppose $\gamma_{0 \leq 2}(G - v) < \gamma_{0 \leq 2}(G)$.
Let S_1 be a minimum 2-isolate distance-2 dominating set of $G - v$. Let z be a 2-isolate of S_1 in $G - v$. Since $|S_1| = \gamma_{0 \leq 2}(G - v) < \gamma_{0 \leq 2}(G)$, S_1 cannot be a 2-isolate distance-2 dominating set of G.
Therefore there is a vertex x in G such that $d(x, S_1) > 2$ in G.
If $x \neq v$ then $d(x, S_1) \leq 2$ in $G - v$ because S_1 is a 2-isolate distance-2 dominating set of $G - v$. Then $d(x, S_1)$ in G is ≤ 2.
Which is a contradiction.
Therefore $x \neq v$ is not possible.
Therefore $x = v$ and $d(x, S_1) > 2$ in G.
Let $S = S_1 \cup \{v\}$. Then $v \in S$. Since $d(v, z) > 2$ in G, z is also a 2-isolate of S in G.
If $y \in V(G)$ and $y \notin S$ then as prove (1) above $d(y, S_1) \leq 2$ in G.
Therefore $d(y, S) \leq 2$ in G.Therefore S is a 2-isolate distance-2 dominating set of G containing v. Since $d(v, S_1) > 2, d(v, u) > 2$, for all $u \in S$ with $u \neq v$.
Therefore $v \in P_{rnd\,2}[v, S]$.
Let $T \in V(G) - S$ such that $d(T, v) \leq 2$. Now $T \notin S_1$ and S_1 is a distance-2 dominating set of $G - v$. Therefore there is a vertex T' in S_1 such that $d(T, T') \leq 2$ in G also.
Thus we have proved that $d(T, v) \leq 2$ in G.
Therefore $T \notin P_{rnd\,2}[v, S]$.
Thus $P_{rnd\,2}[v, S] = \{v\}$. Also note that S contain a 2-isolate different from v.

Conversely, suppose there is a minimum 2-isolate distance-2 dominating set S of G such that
(1) S contain a 2-isolate different from v.
(2) $P_{rnd\,2}[v, S] = \{v\}$.
Let $S_1 = S \cup \{v\}$. Let z be 2-isolate of S different from v. Then $z \in S_1$ and z is a 2-isolate of S_1 in $G - v$.
Let x be any vertex of $G - v$ such that $x \notin S$ also. Since S is a distance-2 dominating set of G. There is some y in S such that $d(x, y) \leq 2$ in G.

Case(1): suppose $y = v$.

Since $x \notin P_{rnd\,2}[v,S]$. There is a vertex y' in S such that $y' \neq v$ and $d(x,y') \leq 2$ in G. Any path in G joining x to y' whose length is ≤ 2 cannot contain v as an internal vertex because $d(x,y') > 2$. Therefore Any path joining x to y' in G having length ≤ 2 is also a path in $G - v$.
Therefore $d(x,y') \leq 2$ in $G - v$.

Case(2): suppose $y \neq v$.
Then $y \in S_1$ and $d(x,\ y) \leq 2$ in G. By the same argument given above $d(x,\ y) \leq 2$ in $G - v$ also.
Thus we have proved that any vertex of $G - v$ which is not in S_1 satisfies $d(x,\ y) \leq 2$ in $G - v$, for some y in S_1.
Therefore S_1 is a 2-isolate distance-2 dominating set of $G - v$. Therefore $\gamma_{0\leq 2}(G - v) \leq |S_1| < |S| = \gamma_{0\leq 2}(G)$.
Hence, $\gamma_{0\leq 2}(G - v) < \gamma_{0\leq 2}(G)$.
Thus the theorem is prove.

Example 4.26: Consider the path graph P_6 with 6 vertices $\{1, 2, 3, 4, 5, 6\}$

Let $S = \{3, 6\}$. It is obvious that S is a minimum 2-isolate distance-2 dominating set of G. Now consider the graph $G - 6$ which is the path graph P_5 with vertices $\{1, 2, 3, 4, 5\}$ 2-isolate distance-2 domination number =1
Thus $\gamma_{0\leq 2}(G - v) < \gamma_{0\leq 2}(G)$.
Observe that $6 \in S$ and $P_{rnd\,2}[6,S] = \{6\}$.
Also S contains a 2-isolate different from 6.

Corollary 4.27: Let G be a graph without isolated vertices. Suppose $\gamma_{0\leq 2}(G - v) < \gamma_{0\leq 2}(G)$. Then there is a minimum 2-isolate distance-2 dominating set S of G such that $v \notin S$.

Proof: Since $\gamma_{0\leq 2}(G - v) < \gamma_{0\leq 2}(G)$.
There is a minimum 2-isolate distance-2 dominating set of $S_1$of G such that (which contains isolate different from v) $P_{rnd\,2}[v,\ S] = \{v\}$.
Since v is not an isolated vertex of G. There is a vertex v' such that $d(v,\ v') \leq 2$. Obviously $v' \notin S_1$.
Let $S = (S_1 - \{v\}) \cup \{v'\}$. Then $|S| = |S_1|$.
Let x be any vertex of G then $x \notin S$.
If $x = v$ then $d(v,\ v') \leq 2$ and $v' \in S$.
If $x \neq v$ then $x \notin S_1$. Since S_1 is a 2-isolate distance-2 dominating set of G. There is a vertex z in S_1 such that $d(x,z) \leq 2$.
If $z = v$ then there is another vertex w in S_1 such that $d(x,w) \leq 2$ in G because $x \notin P_{rnd\,2}[v,\ S_1]$ then $w \in S$ and $d(x,\ w) \leq 2$.
If $d(x,\ v') \leq 2$ in G then $v' \in S$ and $d(x,\ v') \leq 2$.
Thus for any x not in S. There is some vertex y in S such that $d(x,\ y) \leq 2$ in G. This proves that S is a minimum 2-isolate distance-2 dominating set of G. Note that $v \notin S$.

Proposition 4.28: Let G be a graph and v be an isolated vertex of G. If S is a 2-isolate distance-2 dominating set of G then $v \in S$.

Proof: Suppose $v \notin S$. Then $d(v,\ S) \leq 2$. Then there is a vertex $u \in S$ such that $d(v,\ u) \leq 2$. This implies that v is not isolated vertex.
Which is a contradiction.
Therefore $v \in S$.

Theorem 4.29: Let G be a graph and $v_1, v_2, \dots\dots\dots, v_k$ be all the isolated vertices of $G\ (k \geq 2)$. Then $\gamma_{0\leq 2}(G - v_i) < \gamma_{0\leq 2}(G)$ for $i = 1, 2, \dots\dots\dots, k$.

Proof: Let S be a minimum 2-isolate distance-2 dominating set of G . By the above proposition 28, $v_i \in S$, for all $i = 1, 2, \dots \dots \dots, k$.
Consider v_i. Now $v_1 \in S$ and S also contains a 2-isolate vertex of S different from v_1. Also $P_{rnd\,2}[v_1, S] = \{v_1\}$.
Therefore $\gamma_{0\lesssim 2}(G - v_1) < \gamma_{0\lesssim 2}(G)$.
Similarly, it can be proved that $\gamma_{0\lesssim 2}(G - v_i) < \gamma_{0\lesssim 2}(G)$ for $i = 1, 2, \dots \dots \dots, k$.
Thus the theorem is proved.

Corollary 4.30: Let G be a graph and v be an isolated vertex of G. If $\gamma_{0\lesssim 2}(G - v) \geq \gamma_{0\lesssim 2}(G)$. Then v is the only isolated vertex of G .

Proof: Suppose there is vertex v' of G such that $v' \neq v$ and v' is also isolated vertex.
Let S be a minimum 2-isolate distance-2 dominating set of G. Then $v, v' \in S$.
Also $P_{rnd\,2}[v, S] = \{v\}$. And therefore $\gamma_{0\lesssim 2}(G - v) < \gamma_{0\lesssim 2}(G)$.
Which is a contradiction.
Then v is the only isolated vertex of G.

Now we consider the operation of edge removal in graph.
Proposition 4.31: Let G be a graph and e be an edge of G. If $u, v \in V(G)$. Then $d(u, v)$in $G - e \geq d(u, v)$ in$G$.

Proof: If there is no path joining u and v in G . Then there is no path joining u and v in $G - e, d(u, v) = \infty$ in G. In this case there is no path joining u and v in $G - e$ also. And therefore $d(u, v) = \infty$ in $G - e$ also.
Thus the result is prove in this case.
Suppose $d(u, v) = k$ in $G - e$, for some positive integer k. Then there is a path of length k joining u and v in $G - e$. This is also a path joining u and v in $G - e$.
Therefore $d(u, v)$ in G is $\leq$ the length of the path P which is = k which is = $d(u, v)$ in $G - e$.
Therefore $d(u, v)$ in $G \leq d(u, v)$ in $G - e$.
Thus the result is proved.

Now we prove the following theorem.

Theorem 4.32: Let G be a graph and $e = \{uv\}$ be an edge of G. Then $\beta_{2is}(G - e) \geq \beta_{2is}(G)$.

Proof: Let S be a maximum 2-isoinc set of G. Let $u \in S$ be a 2-isolated vertex of S. Then $d(u, x) > 2$ in G, for every $x \in S$ with $x \neq u$. Let w be 2-isolated vertex of S. Then $d(w, x) > 2$ in G, for every $x \in S$ with $x \neq w$.
Then $d(w, x) > 2$ in $G - e$ also, for every $x \in S$ with $x \neq w$.
Thus w is a 2-isolated vertex of S in $G - e$.
Moreover, if $a, b \in S$ then $d(a, b) > 2$ in G, for all $a, b \in S$. Therefore $d(a, b) > 2$, for all $a, b \in S$ in $G - e$ also.
Thus S is a 2-isoinc set in $G - e$ also.
Therefore $\beta_{2is}(G - e) \geq |S| = \beta_{2is}(G)$.
Therefore $\beta_{2is}(G - e) \geq \beta_{2is}(G)$.

Now we state and prove a necessary and sufficient condition under which 2-isoinc number of a graph increases when an edge is remove on the graph.

Theorem 4.33: Let G be a graph and and $e = \{uv\}$ be an edge of G. Then $\beta_{2is}(G - e) > \beta_{2is}(G)$ if and only if for every maximum 2-isoinc S of $G - e$. The following conditions are satisfied.
(1) If $u, v \in S$ then for every 2-isolate z of S. There is a vertex w in $V(G) - S$, which is adjacent to z and u or there is a vertex w' in $V(G) - S$, which is adjacent to z and v.
(2) If $u \notin S$ and $v \in S$ then for ervery 2-isolate z of S, z is adjacent to u.
(3) If $v \notin S$ and $u \in S$ then for ervery 2-isolate z of S, z is adjacent to v.

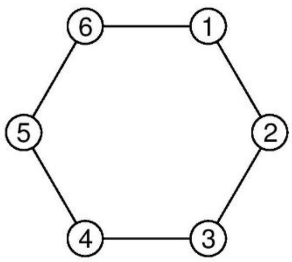
Proof: First suppose that $\beta_{2is}(G-e) \geq \beta_{2is}(G)$.
Let S be a maximum 2-isoinc of $G-e$. Since $|S| > \beta_{2is}(G)$, S cannot be a 2-isoinc of G.
Therefore if z is any 2-isolate of S in $G-e$ then z cannot be a 2-isolated vertex S in G.
Therefore $d(z,x) \leq 2$ in G some x in S.
Case(1): $u \in S$ & $v \in S$
It follows that $u = x$ or $v = x$.
If $x = u$ then there is a vertex w in $V(G) - S$ such that w is adjacent to both u & z.
If $x = v$ then there is a vertex w in $V(G) - S$ such that w' is adjacent to both v & z.

Case(2): $u \notin S$ & $v \in S$
In this case it follows that $x = v$. Since $d(z,v) > 2$ in $G-e$ and $d(z,v) \leq 2, z$ must be adjacent to u.

Case(3): $v \notin S$ & $u \in S$
In this case it follows that $x = u$. Since $d(z,u) > 2$ in $G-e$ and $d(z,u) \leq 2, z$ must be adjacent to v.

Conversely, suppose conditions (1), (2) and (3) are any one is satisfied, for any maximum 2-isoinc set S of $G-e$.
Let S be subset of $V(G)$ such that $|S| \geq \beta_{2is}(G-e)$. Suppose S is a 2-isoinc of G, S must be a 2-isoinc set S of $G-e$ also.
Thus here $|S| \geq \beta_{2is}(G-e)$ and S is a 2-isoinc set S of $G-e$.
This is a contradiction.
Thus S cannot be a 2-isoinc set S of G if $|S| \geq \beta_{2is}(G-e)$.
Suppose $|S| = \beta_{2is}(G-e)$. Suppose S is a 2-isoinc of G. Now S is also a maximum 2-isoinc of $G-e$. By the assumption conditions (1), (2) or (3) are satisfied by S and therefore S cannot be a 2-isoinc set of G.
Which is again contradiction.
Thus we have prove that $|S| \geq \beta_{2is}(G-e)$ then S cannot be a 2-isoinc set of G.
Therefore if T is any maximum 2-isoinc set of G then $|T| < \beta_{2is}(G)$.
Therefore $\beta_{2is}(G-e) > \beta_{2is}(G)$.

Example 4.34: Let G be the cycle graph C_6 with 6 vertices $\{1,2,3,4,5,6\}$



In this graph $S = \{1,4\}, \{2,5\}, \{3,6\}$ are the only maximum 2-isoinc sets of G.
Now consider the graph $G-e$. Which is the path graph P_6 with 6 vertices $\{1,2,3,4,5,6\}$.

In this graph $S_1 = \{1,4,5,6\}$ and $S_1 = \{1,2,3,6\}$ are the only maximum 2-isoinc sets of $G-e$.
Note that $\{1,6\} \in S_1$ and $\{1,6\} \in S_2$ also. Let $w = 2$ then w is the adjacent to 1 for the set S_1. Let $w = 5$ then w is the adjacent to 6 for the set S_2.

Example 4.35: Consider the complete graph K_n with $n \geq 3$.

Then $\beta_{2is}(K_n) = 1$. Remove any edge from the graph K_n then $\beta_{2is}(K_n - e) = 1$.
Here, 2-isoinc number does not increases when any edge remove from the graph.

V. CONCLUDING REMARKS

In this paper we have consider 2-isolate inclusive set. It may be possible to study those sets which do not contain 2-isolated vertices. These sets can be studied and can be compared with totally dominating sets.

ACKNOWLEDGMENT

The authors of this paper would like to thanks the editors and the refries for their comments and valuable suggestions.

REFERENCES

[1] D. K. Thakkar and J. C. Bosamiya Graph critical with respect to independent domination, Journal of Discrete Mathematical Sciences and Cryptography Vol.16(2013),179-186

[2] D. K. Thakkar and N. J. Savaliya , About Isolate Domination in Graphs , International Journal of Mathematical Archive Vol.8 (11) (2017), 171-178

[3] D. K. Thakkar and N. J. Savaliya , On Isolate Inclusive Sets in Graphs , International Journal of Innovation in Science and Mathematics Vol.5(3) (2017), 74-76

[4] I. Sahul Hamid and S. Balamurugan, Isolate Domination Number and Maximum Degree, bulletin of the international mathematical virtual institute Vol.3(2013),127-133

[5] I. Sahul Hamid and S. Balamurugan, Isolate Domination in Graphs, Arab Journal of Mathematical Sciences Math Sci 22(2016),232-241

[6] Michael A. Henning., Anders Yeo Total Domination in Graphs, Springer, New York, (2013)

[7] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamental of Domination In graphs, Marcel Dekker, New York, (1998)

[8] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Domination In Graphs Advanced Topics , Marcel Dekker, New York, (1998)