Application of Q-Leibniz Rule to Transformations Involving Basic Analogue of Multivariable H-Function

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ABSTRACT
In this paper, we give applications of the q-Leibniz rule to evaluate fractional order derivatives and to derive certain transformations involving basic analogue of multivariable H-function. At the end, we shall see several particular cases.

Keywords : Fractional q-derivative operator, q-Leibniz rule, basic analogue of multivariable H-function.

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1. Introduction and preliminaries.

Recently, Yadav and Purohit [9,10] have investigated the application of q-Leibniz rule for fractional order derivatives and discussed several interesting transformations involving various basic hypergeometric functions of one variable including the basic analogue of Fox's H-function. In this study, we give the application of q-Leibniz rule for fractional order derivatives concerning the basic analogue of multivariable H-function.

In the theory of q-series, for real or complex a and $|q|<1$, the q-shifted factorial is defined as :

$$(a;q)_n=\prod_{i=1}^{n-1}(1-aq^i)=\frac{(a;q)_\infty}{(aq^n;q)_\infty}\qquad(n\in\mathbb{N})\tag{1.1}$$

so that $(a;q)_0=1$,

or equivalently

$$(a,q)_n=\frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}\qquad(a\neq 0,-1,-2,\cdots).\tag{1.2}$$

The q-gamma function [4] is given by

$$\Gamma_q(\alpha)=\frac{(q;q)_\infty(1-q)^{\alpha-1}}{(q^\alpha q)_\infty}=\frac{[1-q]_{\alpha-1}}{(1-q)^{\alpha-1}}=\frac{(q;q)_{\alpha-1}}{(1-q)^{\alpha-1}}(\alpha\neq 0,-1,-2,\cdots).\tag{1.3}$$

The fractional q-differential operator of arbitrary order α, see Al-Salam [2] and Agarwal [1] is defined as :

$$D_{t,q}^{\alpha}f(t)=\frac{1}{\Gamma_q(-\alpha)}\int_0^t[t-yq]_{-\alpha-1}f(y)d(y;q)\tag{1.4}$$

where $Re(\alpha)<0,|q|<1$ and

$$[x-y]_v=x^v\prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{n+v}}\right]\tag{1.5}$$

Also the basic integral, see Gasper and Rahman [3] are given by

$$\int_0^x f(t)d_qt=x(1-q)\sum_{k=0}^{\infty}q^kf(xq^k)\tag{1.6}$$

The equation (1.4) in conjonction with (1.6) yield the following series representation of the Riemann-Liouville fractional integral operator

$$I_q^{\mu} f(x) = \frac{x^{\mu}(1-q)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^k [1-q^{k+1}]_{\mu-1} f(xq^k) \tag{1.7}$$

In particular, for $f(t) = t^{\mu-1}$, the above

$$D_{t,q}^{\alpha}(t^{\mu-1}) = \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} t^{\mu-\alpha-1} \tag{1.8}$$

In view of Agarwal [1], we have the q-extension of the Leibniz rule for the fractional order q-derivatives for a product of two functions in terms of a series involving fractional q derivatives of the functions in the following manner :

$$D_{t,q}^{\lambda}[U(t)V(t)] = \sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}(q^{-\lambda};q)_n}{(q;q)_n} D_{t,q}^{\lambda-n} U[tq^n] D_{t,q}^n V[t] \tag{1.9}$$

where $U(t)$ and $V(t)$ are two basic functions.

2.Basic analogue multivariable H-function.

In this section, we introduce the basic analogue of multivariable H-function defined by Srivastava and Panda [7,8].

We note

$$G(q^a) = \left[\prod_{n=0}^{\infty}(1-q^{a+n})\right]^{-1} = \frac{1}{(q^a;q)_{\infty}} \tag{2.1}$$

$$H(z_1,\cdots,z_r;q) = H_{p,q':p_1,q_1;\cdots;p_r,q_r}^{0,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}; \mathrm{q} \,\middle|\, \begin{array}{c} (\mathrm{a}_j;\alpha_j',\cdots,\alpha_j^{(r)})_{1,p} : (c_j',\gamma')_{1,p_1},\cdots,(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ \cdot \\ \cdot \\ (\mathrm{b}_j;\beta_j',\cdots,\beta_j^{(r)})_{1,q'} : (d_j',\delta')_{1,q_1},\cdots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{array}\right)$$

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \pi^r \phi(s_1,\cdots,s_r;q)\prod_{i=1}^{r}\theta_i(s_i;q) x_1^{s_1}\cdots x_r^{s_r} d_q s_1\cdots d_q s_r \tag{2.2}$$

where

$$\phi(s_1,\cdots,s_r;q) = \frac{\prod_{j=1}^{n} G(q^{1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i})}{\prod_{j=n+1}^{p} G(q^{a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i})\prod_{j=1}^{q'} G(q^{1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i})} \tag{2.3}$$

$$\theta_i(s_i;q) = \frac{\prod_{j=1}^{m_i} G(q^{d_j^{(i)}-\delta_j^{(i)}s_i})\prod_{j=1}^{n_i} G(q^{1-c_j^{(i)}+\gamma_j^{(i)}s_i})}{\prod_{j=m_i+1}^{q_i} G(q^{1-d_j^{(i)}+\delta_j^{(i)}s_i})\prod_{j=n_i+1}^{p_i} G(q^{c_j^{(i)}-\gamma_j^{(i)}s_i}) G(q^{1-s_i})\sin\pi s_i} \tag{2.4}$$

$i = 1,\cdots,r$

where the integers n,p,q,m_i,n_i,p_i,q_i are constrained by the inequalities $0 \leq n \leq p, 0 \leq q', 1 \leq m_i \leq q_i$ and $0 \leq n_i \leq p_i, i = 1,\cdots,r$. The poles of the integrand are assumed to be simple.
The quantities, $a_j, j=1,\cdots,p; c_j^{(i)}, j=1,\cdots,p_i; b_j, j=1,\cdots,q'; d_j^{(i)}, j=1,\cdots,q_i, i=1,\cdots,r$ are complex numbers and the following quantities $\alpha_j^{(i)}, j=1,\cdots,p; \gamma_j^{(i)}, j=1,\cdots,p_i; \beta_j^{(i)}, j=1,\cdots,q'; \delta_j^{(i)}, j=1,\cdots,q_i$, $i=1,\cdots,r$ are positive real numbers.

The contour L_i in the complex s_i-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $\omega\infty$ with indentations, if necessary to ensure that all the poles of $G(q^{d_j^{(i)}+\delta_j^{(i)}s_i}), j=1,\cdots,m_i$ are separated from those of$G(q^{1-c_j^{(i)}+\gamma_j^{(i)}s_i})$, $i=1,\cdots,n_i$, $G(q^{1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i})$, $j=1,\cdots,n, i=1,\cdots,r$. For large values of $|s_i|$ the integrals converge if $Re(slog(z_i)-log\sin\pi s_i)<0, i=1,\cdots,r$.

3. Main results.

Let

$$U=m_1,n_1;\cdots;m_r,n_r;V=p_1,q_1;\cdots;p_r,q_r \tag{3.1}$$

$$\mathrm{A}=(\mathrm{a}_j;\alpha'_j,\cdots,\alpha_j^{(r)})_{1,p}:B=(c'_j,\gamma')_{1,p_1},\cdots,(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \tag{3.2}$$

$$\mathrm{C}=(\mathrm{b}_j;\beta'_j,\cdots,\beta_j^{(r)})_{1,q'}:D=(d'_j,\delta')_{1,q_1},\cdots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \tag{3.3}$$

In this section, we shall establish certain transformations associated with the basic analogue of multivariable H-function ans an application of the q-Leibnitz rule for the fractional derivatives of a product of two basic functions.

Theorem 1.

$$D_{t,q}^{s}\left\{t^{\delta+\lambda-1}H_{p,q':V}^{0,n:U}\left(\begin{array}{c}\mathrm{x}_1t^{k_1}\\ \cdot\\ \cdot\\ \mathrm{x}_rt^{k_r}\end{array};\mathrm{q}\left|\begin{array}{c}\mathrm{A: B}\\ \cdot\\ \cdot\\ \mathrm{C: D}\end{array}\right.\right)\right\}=\frac{t^{\delta+\lambda-s-1}\Gamma_q(\delta+\lambda)}{\Gamma_q(\lambda+\delta-s)}$$

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c}\mathrm{x}_1t^{k_1}\\ \cdot\\ \cdot\\ \mathrm{x}_rt^{k_r}\end{array};\mathrm{q}\left|\begin{array}{c}(\text{-}\delta-\lambda;k_1,\cdots,k_r),A:B\\ \cdot\\ \cdot\\ \mathrm{C},(\mathrm{s}\text{-}\delta-\lambda;k_1,\cdots,k_r):D\end{array}\right.\right) \tag{3.4}$$

where $\mathrm{Re}(\lambda+\delta)>s>0, Re(slog(z_i)-log\sin\pi s_i)<0, i=1,\cdots,r$

Proof

We consider the left hand side and apply the definition of the basic analogue of multivariable H-function, we have

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\phi(s_1,\cdots,s_r;q)\prod_{i=1}^{r}\theta_i(s_i;q)x_1^{s_1}\cdots x_r^{s_r}D_{t,q}^{s}[t^{k_1s_1+\cdots+k_rs_r+\delta+\lambda-1}]d_qs_1\cdots d_qs_r \tag{3.5}$$

Now, we use the result (1.8), interpreting the mellin-Barnes integrals contour in basic analogue of multivariable H-function, we obtain the result after algebraic manipulations.

Theorem 2.

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c}\mathrm{x}_1t^{k_1}\\ \cdot\\ \cdot\\ \mathrm{x}_rt^{k_r}\end{array};\mathrm{q}\left|\begin{array}{c}(\text{-}\delta-\lambda;k_1,\cdots,k_r),A:B\\ \cdot\\ \cdot\\ \mathrm{C},(\text{-}\delta-\mu;k_1,\cdots,k_r):D\end{array}\right.\right)=\frac{\Gamma_q(\delta+\mu)\Gamma_q(\delta+1)}{\Gamma_q(\delta+\lambda)\Gamma_q(\delta+\mu+1-\lambda)}$$

$$\sum_{s=0}^{\infty}\frac{q^{s(\delta+\lambda)}(q^{\mu-\lambda};q)_s(q^{1-\lambda};q)_s}{(q;q)_s(q^{\delta+\mu+1-\lambda};q)_s}H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c}\mathrm{x}_1t^{k_1}\\ \cdot\\ \cdot\\ \mathrm{x}_rt^{k_r}\end{array};\mathrm{q}\left|\begin{array}{c}(\text{-}\lambda;k_1,\cdots,k_r),A:B\\ \cdot\\ \cdot\\ \mathrm{C},(\mathrm{s}\text{-}\lambda;k_1,\cdots,k_r):D\end{array}\right.\right) \tag{3.6}$$

where $(\lambda)>s>0, Re(\delta+\mu)>0, Re(\delta+1)>0$ and $Re(slog(z_i)-log\sin\pi s_i)<0, i=1,\cdots,r$

Proof

To prove the theorem 2, we consider the equation (1.9) with $U(t) = t^\delta$

$$V(t) = t^{\lambda-1}H^{0,n:U}_{p,q':V}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ; q \;\middle|\; \begin{array}{c} ,A: B \\ \cdot \\ \cdot \\ C: D \end{array}\right)$$

Replacing λ by $\lambda - \mu$, we obtain

$$D^{\lambda-\mu}_{t,q}\left\{t^{\delta+\lambda-1}H^{0,n:U}_{p,q':V}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ; q \;\middle|\; \begin{array}{c} A: B \\ \cdot \\ \cdot \\ C: D \end{array}\right)\right\} = \sum_{s=0}^{\infty}\frac{(-)^s q^{s(s+1)/2}(q^{-\lambda};q)_s}{(q;q)_s} D^{\lambda-\mu-s}_{t,q}((tq^s)^\delta)$$

$$D^{s}_{t,q}\left\{ t^{\lambda-1}H^{0,n:U}_{p,q':V}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ; q \;\middle|\; \begin{array}{c} ,A: B \\ \cdot \\ \cdot \\ C: D \end{array}\right)\right\} \tag{3.7}$$

In order to find the values of various order fractional q-derivatives terms involved in the above equation, we shall use the theorem 1.

Let $s = \lambda - \mu$ in the theorem 1, we have the following result

Corollary 1.

$$D^{\lambda-\mu}_{t,q}\left\{t^{\delta+\lambda-1}H^{0,n:U}_{p,q':V}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ; q \;\middle|\; \begin{array}{c} A: B \\ \cdot \\ \cdot \\ C: D \end{array}\right)\right\} = \frac{t^{\delta+\mu-1}\Gamma_q(\delta+\lambda)}{\Gamma_q(\mu+\delta)}$$

$$H^{0,n+1:U}_{p+1,q'+1:V}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ; q \;\middle|\; \begin{array}{c} (-\delta-\lambda;k_1,\cdots,k_r),A:B \\ \cdot \\ \cdot \\ C,(-\delta-\mu;k_1,\cdots,k_r):D \end{array}\right) \tag{3.8}$$

where $Re(slog(z_i) - log\sin\pi s_i) < 0, i = 1,\cdots,r$

Taking $\delta = 0$ in theorem 1, we have

Corollary 2.

$$D^{s}_{t,q}\left\{t^{\lambda-1}H^{0,n:U}_{p,q':V}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ; q \;\middle|\; \begin{array}{c} A: B \\ \cdot \\ \cdot \\ C: D \end{array}\right)\right\} = \frac{t^{\lambda-s-1}\Gamma_q(\lambda)}{\Gamma_q(\lambda-s)}$$

$$H^{0,n+1:U}_{p+1,q'+1:V}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ; q \;\middle|\; \begin{array}{c} (-\lambda;k_1,\cdots,k_r),A:B \\ \cdot \\ \cdot \\ C,(s-\lambda;k_1,\cdots,k_r):D \end{array}\right) \tag{3.9}$$

where $Re(slog(z_i) - log\sin\pi s_i) < 0, i = 1,\cdots,r$

Further on making use the equation (1.8) one can evaluate

$$D_{t,q}^{\lambda-\mu-s}\{(tq^s)^\delta\} = \frac{q^{s\delta}(1-q)^{\mu-\lambda+s}}{(q^{\delta+1;q})_{\mu-\lambda}(q^{\delta+\mu+1-\lambda;q})_s} t^{\delta+\mu-\lambda+s} \tag{3.10}$$

On substituting the values of the various q-derivatives expression involved in the equation (3.6), from equations (3.7), (3.8), (3.9) and (3.10), we obtain

$$\frac{\Gamma_q(\delta+\lambda)}{\Gamma_q(\delta+\mu)} t^{\delta+\mu-1}\ H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ;\ q \left| \begin{array}{c} (-\delta-\lambda;k_1,\cdots,k_r),A:B \\ \cdot \\ \cdot \\ C,(s-\delta-\mu;k_1,\cdots,k_r):D \end{array}\right.\right) = \sum_{s=0}^{\infty} \frac{(-)^s q^{s(s+1)/2}(q^{\mu-\lambda};q)_s}{(q;q)_s}$$

$$\frac{q^{s\delta}(1-q)^{\mu-\lambda+s}}{(q^{\delta+1};q)_{\mu-\lambda}(q^{\delta+\mu+1-\lambda};q)_s}\ t^{\delta+\mu-\lambda-s}\ \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda-s)} t^{\lambda-s-1}$$

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ;\ q \left| \begin{array}{c} (-\lambda;k_1,\cdots,k_r),A:B \\ \cdot \\ \cdot \\ C,(s-\lambda;k_1,\cdots,k_r):D \end{array}\right.\right) \tag{3.11}$$

which on further simplification, leads to the theorem 2.

Let $s=\lambda-\mu, \delta=0$ in theorem 1, we obtain

Corollary 3

$$D_{t,q}^{\lambda-\mu}\left\{t^{\delta+\lambda-1}H_{p,q':V}^{0,n:U}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ;\ q \left| \begin{array}{c} A:B \\ \cdot \\ \cdot \\ C:D \end{array}\right.\right)\right\} = \frac{t^{\mu-1}\Gamma_q(\lambda)}{\Gamma_q(\mu)}$$

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c} x_1 t^{k_1} \\ \cdot \\ \cdot \\ x_r t^{k_r} \end{array} ;\ q \left| \begin{array}{c} (-\lambda;k_1,\cdots,k_r),A:B \\ \cdot \\ \cdot \\ C,(-\mu;k_1,\cdots,k_r):D \end{array}\right.\right) \tag{3.12}$$

where $Re(slog(z_i) - log\sin\pi s_i) < 0, i = 1,\cdots,r$

Remarks :
If the basic analogue of multivariable H-function reduces to basic of Srivastava-Daoust function, see the work of Purohit et al. [4].
We obtain the same relations with basic analogue of H-function of two variables defined by Saxena et al. [6], the basic analogue of Fox's H-function defined by Saxena et al. [5].

4. Conclusion.

The importance of our all the results lies in their manifold generality. By specialising the various parameters as well as variables in the basic analogue of multivariable H-function, we obtain a large number of results involving remarkably wide variety of useful basic functions (or product of such basic functions) which are expressible in terms of basic H-function, Basic Meijer's G-function, Basic E-function, basic hypergeometric function of one and two variables and simpler special basic functions of one and two variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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