The (n+1)-Dimensional Equal Width Wave Equation by Differential Transform Method (DTM)

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Abstract:

Differential transform method is used to obtain the closed form solution of the (n+1) dimensional Equal Width Wave equation.In this method the solution is calculated in the form of the convergent Power series with easily computable Components.The results demonstrate the reliability and efficiency of this method.

Keywords:

$(n+1)$-dimensional Equal Width wave equation, differential transform method.

1 Introduction

In the past few decades, the traditional integral transform methods such as Fourier and Laplace transform have commonly been used to solve the engineering problems.These methods transform the differential equations into algebraic equations which are easier to deal with.

The well -known Korteweg and de varies (KdV) equation is $u_t + uu_x + u_{xxx} = 0$ is a non linear Partial differential equation (PDE) that models the time-dependent motion of shallow water waves in one space dimension. Morrison et al[1] proposed the one dimensional PDE $u_t + uu_x - \mu u_{xxt} = 0$ as an equally valid and accurate model for the same wave phenomena simulated by the KdV equation.This PDE is called the Equal width Wave equation because the solutions for solitary waves with a permanent form and speed, for a given value of the parameter μ are waves with an equal width or wavelength for all wave amplitudes.

In this paper, we have employed the differential transform method to solve the (n+1)-dimensional equal width wave equation. Few numerical examples are carried out to validate and illustrate the method.

Let us consider the (n+1)-dimensional Equal Width wave equation as,

$$\frac{\partial u}{\partial t} = \mu_1 \frac{\partial^3 u}{\partial x_1^2 \partial t} + \mu_2 \frac{\partial^3 u}{\partial x_2^2 \partial t} + ... + \mu_n \frac{\partial^3 u}{\partial x_n^2 \partial t} + \gamma u \frac{\partial u}{\partial x_1} \tag{1}$$

under the initial condition

$$u(x_1, x_2, \ldots, x_n, 0) = u_0(x_1, x_2, \ldots, x_n) \tag{2}$$

where μ_i's , $i = 1, 2, ..., n$ and γ is a constant

Zhou [2] was the first one to use the differential transform method (DTM) in engineering applications. He employed DTM in solution of initial boundary value problems in electric circuit analysis. In recent years concept of DTM has broadend to problems involving partial differential equations and system of differential equations [3-5]. Some researchers have lately applied DTM for analysis of uniform and non-uniform beams [6-10].

2 Differential Transform Method (DTM)

In this section, We give some basic definitions of the differential transformation. Let D denotes the differential transform operator and D^{-1} the inverse differential transform operator.

Definition : 1

If $u(x_1, x_2, x_3.....x_n, t)$ is analytic in the domain Ω, then its $(n+1)$-dimensional differential transform is given by

$$\begin{aligned} U(k_1, k_2, \ldots, k_n, k_{n+1}) &= \left(\frac{1}{k_1!, k_2!, ..., k_n!, k_{n+1}!}\right) \frac{\partial^{k_1+k_2+...+k_n+k_{n+1}}}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_n^{k_n} \partial t^{k_{n+1}}} \\ &. \quad u(x_1, x_2, ...x_n, t) \left.\right|_{x_1=0, x_2=0, \ldots, x_n=0, t=0} \end{aligned} \tag{3}$$

where

$$\begin{aligned} u(x_1, x_2, \ldots, x_n, t) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} ... \sum_{k_n=0}^{\infty} \sum_{k_{n+1}=0}^{\infty} U(k_1, k_2, ...k_n, k_{n+1}).(x_1^{k_1}.x_2^{k_2}.....x_n^{k_n}.t^{k_1}) \\ &= D^{-1}\left[U(k_1, k_2, ...k_n, k_{n+1})\right] \end{aligned} \tag{4}$$

Definition : 2

If $u(x_1, x_2, \ldots, x_n, t) = D^{-1}[U(k_1, k_2, ..., k_n, k_{n+1})]$,
$v(x_1, x_2, \ldots, x_n, t) = D^{-1}[V(k_1, k_2, ..., k_n, k_{n+1})]$
and $\otimes$ denotes the convolution,then the fundamental operations of the (n+1)-dimensional differential transform are expressed as follows:

$$\begin{aligned} &(a) D[u(x_1, x_2, \ldots, x_n, t).v(x_1, x_2, x_3, \ldots, x_n, t)] = U(k_1, k_2, ..., k_n, k_{n+1}) \otimes V(k_1, k_2, ..., k_n, k_{n+1}) \\ &= \sum_{a_1=0}^{k_1} \sum_{a_2=0}^{k_2} ... \sum_{a_n=0}^{k_n} \sum_{a_{n+1}=0}^{k_{n+1}} U(a_1, k_2 - a_2, ..., k_n - a_n, k_{n+1} - a_{n+1}).V(k_1 - a_1, a_2, ..., a_n, a_{n+1}). \end{aligned} \tag{5}$$

$$\begin{aligned} (b) D\left[\alpha u(x_1, x_2, \ldots, x_n, t) \pm \beta v(x_1, x_2, \ldots, x_n, t)\right] &= \alpha U(k_1, k_2, \ldots, k_n, k_{n+1}) \\ &\pm \beta V(k_1, k_2, \ldots, k_n, k_{n+1}). \end{aligned} \tag{6}$$

$$
\begin{aligned}
(c) D\left[\frac{\partial^{r_1+r_2+\ldots r_n+r_{n+1}}}{\partial x_1^{r_1} \partial x_2^{r_2} \ldots \partial x_n^{r_n} \partial t^{r_{n+1}}} . u(x_1, x_2, \ldots, x_n, t)\right] &= (k_1+1)(k_1+2) \ldots (k_1+r_1) \\
&. \quad (k_2+1)(k_2+2) \ldots (k_2+r_2) \\
&\ldots \quad (k_{n+1}+1)(k_{n+1}+2) \ldots (k_{n+1}+r_{n+1}) \\
&. \quad U(k_1+r_1, k_2+r_2, \ldots, k_{n+1}+r_{n+1}) . (7)
\end{aligned}
$$

3 Computational Illustrations of $(n+1)$ Dimensional Equal Width Wave Equation:

In this section, we describe the method explained in the previous section 2 by the following examples to validate the efficiency of the DTM.

3.1 *Example:*

Consider the (n+1)-dimensional Equal Width wave equation by assuming μ's and $\gamma = 1$

$$
\frac{\partial u}{\partial t} = \mu_1 \frac{\partial^3 u}{\partial x_1^2 \partial t} + \mu_2 \frac{\partial^3 u}{\partial x_2^2 \partial t} + \cdots + \mu_n \frac{\partial^3 u}{\partial x_n^2 \partial t} + \gamma u \frac{\partial u}{\partial x_1} \tag{8}
$$

subject to the initial condition

$$
\begin{aligned}
u(x_1, x_2, \ldots\ldots x_n, 0) &= u_0(x_1, x_2, \ldots\ldots x_n) \\
&= x_1 + x_2 + \cdots + x_n .
\end{aligned} \tag{9}
$$

Differential Transform Method for $(n+1)$-dimensional Equal Width wave equation

Taking the differential transform of eqn(8) we have

$$
\begin{aligned}
(k_{n+1}+1) U(k_1, k_2, \ldots k_n, k_{n+1}+1) &= (k_1+2)(k_1+1)(k_{n+1}+1) U(k_1+2, k_2, k_3, .., k_{n+1}+1) \\
&+ \quad (k_2+2)(k_2+1)(k_{n+1}+1) U(k_1, k_2+2, k_3, \ldots, k_{n+1}+1) + \ldots \\
&+ \quad (k_n+2)(k_n+1)(k_{n+1}+1) U(k_1, k_2, \ldots, k_n+2, k_{n+1}+1) \\
&+ \quad \sum_{a_1=0}^{k_1} \sum_{a_2=0}^{k_2} \ldots \sum_{a_n=0}^{k_n} \sum_{a_{n+1}=0}^{k_{n+1}} (k_1+1-a_1) \\
&. \quad U(k_1+1-a_1, a_2, a_3, \ldots, a_n, a_{n+1}) \\
&\times \quad U(a_1, k_2-a_2, k_3-a_3, \ldots, k_n-a_n, k_{n+1}-a_{n+1}) .
\end{aligned} \tag{10}
$$

From the initial condition eqn(9), it can be seen that

$$\begin{aligned} u(x_1, x_2, ...x_n, 0) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty} U(k_1, k_2, ..., k_n, 0).(x_1^{k_1}x_2^{k_2}...x_n^{k_n}) \\ &= x_1 + x_2 + ... + x_n. \end{aligned} \tag{11}$$

where

$$U(k_1, k_2, ..., k_n, 0) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, ..., n \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

using eq(12) into (11) one can obtain the values of $U(k_1, k_2, ...k_n, k_{n+1})$ as follows

$$U(k_1, k_2, ..., k_n, 1) = \begin{cases} 1, & if k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, ..., n \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

$$U(k_1, k_2, ..., k_n, 2) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, ..., n \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

$$U(k_1, k_2, ..., k_n, 3) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, ..., n \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

$$U(k_1, k_2, ..., k_n, n) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, ..., n \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

Then from eqn(4) we have

$$\begin{aligned} u(x_1, x_2, ..., x_n, t) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty}\sum_{k_{n+1}=0}^{\infty} U(k_1, k_2, ...k_n, k_{n+1})x_1^{k_1}x_2^{k_2}...x_n^{k_n}t^{k_{n+1}} \\ &= (x_1 + x_2 + ... + x_n) + (x_1 + x_2 + ... + x_n)t + (x_1 + x_2 + ... + x_n)t^2 + ... \end{aligned} \tag{17}$$

Thus the exact solution is given by

$$u(x_1, x_2, ...x_n, t) = \frac{(x_1 + x_2 + ... + x_n)}{1 - t}, \text{provided that } 0 \leq t < 1 \tag{18}$$

$$\tag{19}$$

3.2 Example

Consider the (3+1)-dimensional Equal Width wave equation as

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^3 u}{\partial y^2 \partial t} + \frac{\partial^3 u}{\partial z^2 \partial t} + u\frac{\partial u}{\partial x} \tag{20}$$

with the initial condition

$$u(x, y, z, 0) = u_0(x, y, z) = x + y + z. \tag{21}$$

Taking the differential transform of (20) we have

$$\begin{aligned}(k_4+1)U(k_1,k_2,k_3,k_4+1) &= (k_1+2)(k_1+1)(k_4+1)U(k_1+2,k_2,k_3,k_4+1)\\ &+ (k_2+2)(k_2+1)(k_4+1)U(k_1,k_2+2,k_3,k_4+1)\\ &+ (k_3+2)(k_3+1)(k_4+1)U(k_1,k_2,k_3+2,k_4+1)\\ &+ \sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\sum_{a_3=0}^{k_3}\sum_{a_4=0}^{k_4}(k_1+1-a_1)\\ &. \ U(k_1+1-a_1,a_2,a_3,a_4)\\ &\times \ U(a_1,k_2-a_2,k_3-a_3,a_4)\end{aligned} \tag{22}$$

From the initial condition (20) We see that

$$u(x, y, z, t, 0) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} U(k_1, k_2, k_3, 0)x^{k_1}y^{k_2}z^{k_3} = x + y + z, \tag{23}$$

where

$$U(k_1, k_2, k_3, 0) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

using eq(24) in (22) we obtain the values of $U(k_1, k_2, k_3, k_4)$ as follows

$$U(k_1, k_2, k_3, 1) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases} \tag{25}$$

$$U(k_1, k_2, k_3, 2) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

$$U(k_1, k_2, k_3, 3) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

Then from eqn(4) we have

$$\begin{aligned} u(x,y,z,t) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty} U(k_1,k_2,k_3,k_4)x^{k_1}y^{k_2}z^{k_3}t^{k_4} \\ &= (x+y+z)+(x+y+z)t+(x+y+z)t^2+.... \end{aligned} \tag{28}$$

The exact solution is

$$u(x,y,z,t) = \frac{x+y+z}{1-t}, \text{ provided that } 0 \leq t < 1. \tag{29}$$

3.3 Example

Consider the (2+1)-dimensional Equal Width wave equation as

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^3 u}{\partial y^2 \partial t} + u\frac{\partial u}{\partial x} \tag{30}$$

with the initial condition

$$u(x,y,0) = u_0(x,y) = x+y. \tag{31}$$

Taking the differential transform method for (30) we have

$$\begin{aligned} (k_3+1)(k_1,k_2,k_3+1) &= (k_1+2)(k_1+1)(k_3+1)U(k_1+2,k_2,k_3+1) \\ &+ (k_2+2)(k_2+1)(k_3+1)U(k_1,k_2+2,k_3+1) \\ &+ \sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\sum_{a_3=0}^{k_3}(k_1+1-a_1) \\ &. \ U(k_1+1-a_1,a_2,a_3)\times U(a_1,k_2-a_2,k_3-a_3). \end{aligned} \tag{32}$$

From the initial condition (31) We see that

$$u(x,y,t,0) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} U(k_1,k_2,0)x^{k_1}y^{k_2} = x+y \tag{33}$$

where

$$U(k_1,k_2,0) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i,j = 1,2 \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

using (34) in (32) we obtain the values of $U(k_1, k_2, k_3)$ as follows

$$U(k_1,k_2,1) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2 \\ 0\,, & \text{otherwise} \end{cases} \tag{35}$$

$$U(k_1,k_2,3) = \begin{cases} 1, & \text{if } k_i = 1, k_j = 0, i \neq j, i, j = 1, 2 \\ 0, & \text{otherwise} \end{cases} \tag{36}$$

Then from eqn(4) we have

$$\begin{aligned} u(x,y,t) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} U(k_1,k_2,k_3)x^{k_1}y^{k_2}t^{k_3} \\ &= (x+y) + (x+y)t + (x+y)t^2 + ... \end{aligned}$$

The exact solution is

$$u(x,y,t) = \frac{x+y}{1-t}, \text{provided} \quad \text{that} \quad 0 \leq t < 1. \tag{37}$$

is the exact solution of eqn(30).

4 Conclusions

1. The differential transform method have been successfully applied for solving the (n+1) - dimensional Equal Width wave equation.

2. The solutions obtained by this method is an infinite power series for the appropriate initial condition ,which can in turn expressed in a closed form, the exact solution.

3. The results reveal that this method is very effective,convenient and quite accurate mathematical tools for solving the (n+1)-dimensional Equal Width wave equation.

4. This method can be used without any need to complex computations except the simple and elementary operations are also promising technique for solving the other nonlinear problems.

References

[1] Morrison . P.J., Meiss.J.D., Carey .J.R : *Scattering of RLW Solitary Waves,* Physica D 11.

[2] Zhou.J.K.,1986, *Differential transform and its application for electrical circuits,* Huazhong university Press,china.

[3] Jang.M.J., C.L.Chen and Y.CLiu,2001, *Two dimensional Differential transform for Partial Differential Equations,* Applied Mathematics and Computation,121:261-270

[4] Ayaz, Fatma,2003, *On the Two Dimensional Differential Transform Method .* Applied Mathematics and Computation,143:361-314

[5] Ayaz, Fatma,2004 , *Solution of the System of the Differential Equations using differential transform method .* Applied Mathematics and Computation,147:547-567

[6] H O,S.H and C.K.Chen,1998, *Analysis Of General Elastically and restrained Non-Uniform BeamsUsing Differential Transform Method .* Applied Mathematical Modelling,22:219-234

[7] Ozdemir. O and Kaya,2006 ,*Flapwise bending Vibration analysis of a rotating tapered Cantilever Bernouulli-Euler beam by Differential transform method .* Journal of Sound and Vibration,289:413-420

[8] Ozgumus. O and Kaya,2006, *Flapwise bending Vibration analysis of double tapered rotating Bernouulli-Euler beam by Differential transform method .* Meccanica, 41:661-670

[9] Seval Catal,2008, *Solution of free vibration equations of beam on elastic soil by using Differential transform method .* Applied Mathematical Modelling ,32:1744-1757

[10] Mei.C, 2006, *Application of differential transformation technique to free vibration analysis of a centrifugally stiffened beam .* Computers and Structures, 86:1280-1284