Certain Fractional Q-Derivative Integral Formulae for the Basic Analogue of Multivariable H-Function

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ABSTRACT
In the present paper, we derive two theorem involving fractional q-integral operators of Erdélyi-Kober type and a basic analogue of multivariable H-function. Corresponding assertions for the Riemann-Liouville and Weyl fractional q-integral transforms are also presented. Several special cases of the main results have been illustrated in the concluding section.

Keywords : fractional q-integral operator, basic integration, basic analogue of multivariable H-function.

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1.Introduction.

The fractional q-calculus is the q-extension of the ordinary fractional calculus. The subject deals with the investigations of q-integral and q-derivatives of arbitrary order, and has gained importance due to its various applications in the areas like ordinary fractional calculus, solutions of the q-difference (differential) and q-integral equations, q-transform analysis see ([1] and [8]). Motivated by these avenues of applications, a number of workers have made use of these operators to evaluate fractional q-calculus formulae for various special function, basic analogue of Fox's H-function, general class of q-polynomials etc. One may refer to the recent paper [4]-[5], [15] and [14]-[16] on the subject.

In this paper, we have established two theorems involving the fractional q-integral operator of Erdélyi-Kober type, which generalizes the Riemann-Liouville and Weyl fractional q-integral operators concerning the basic analogue of multivariable H-function. Several special cases of the main results have been illustrated in the concluding section.

In the theory of q-series, for real or complex a and $|q| < 1$, the q-shifted factorial is defined as :

$$(a;q)_n = \prod_{i=1}^{n-1}(1-aq^i) = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \qquad (n \in \mathbb{N}) \tag{1.1}$$

so that $(a;q)_0 = 1$,

or equivalently

$$(a,q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)} \qquad (a \neq 0,-1,-2,\cdots). \tag{1.2}$$

The q-analogue of the familiar Riemann-Liouville fractional integral operator of a function $f(x)$ due to Agarwal [2], is given by

$$I_q^\alpha\{f(x)\} = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-tq)_{\alpha-1}f(t)d_qt \qquad (Re(\alpha)>0, |q|<1). \tag{1.3}$$

Also, the basic analogue of the Kober fractional integral operator, see Agarwal [2] is defined by

$$I_q^{\eta,\alpha}\{f(x)\} = \frac{x^{-\eta-\alpha}}{\Gamma_q(\alpha)}\int_0^x (x-tq)_{\alpha-1}t^\eta f(t)d_qt \quad (Re(\alpha)>0, \eta\in\mathbb{R}, |q|<1). \tag{1.4}$$

a q-analogue of the Weyl fractional integral operator due to Al-Salam [3] is given by

$$K_q^{\alpha}\{f(x)\}=\frac{q^{\alpha-1)/2}}{\Gamma_q(\alpha)}\int_x^{\infty}(t-x)_{\alpha-1}f(tq^{1-\alpha})d_qt\ (Re(\alpha)>0,|q|<1). \tag{1.5}$$

In the same paper Al-Salam [3] intoduced the q-analogue of the generalized Weyl fractional integral operator in the following manner

$$K_q^{\eta,\alpha}\{f(x)\}=\frac{q^{-\eta}x^{\eta}}{\Gamma_q(\alpha)}\int_x^{\infty}(t-x)_{\alpha-1}t^{-\eta-\alpha}f(t)d_qt\,(Re(\alpha)>0,\eta\in\mathbb{R},|q|<1). \tag{1.6}$$

Also the basic integral, see Gasper and Rahman [4,5]) are given by

$$\int_0^x f(t)d_qt=x(1-q)\sum_{k=0}^{\infty}q^kf(xq^k) \tag{1.7}$$

$$\int_x^{\infty} f(t)d_qt=x(1-q)\sum_{k=1}^{\infty}q^{-k}f(zq^{-k}) \tag{1.8}$$

$$\int_0^{\infty} f(t)d_qt=x(1-q)\sum_{k=-\infty}^{\infty}q^kf(zq^k) \tag{1.9}$$

2.Basic analogue of multivariable H-function.

In this section, we introduce the basic analogue of multivariable H-function defined by Srivastava and Panda [12,13].

We note

$$G(q^a)=\left[\prod_{n=0}^{\infty}(1-q^{a+n})\right]^{-1}=\frac{1}{(q^a;q)_{\infty}} \tag{2.1}$$

$$H(z_1,\cdots,z_r;q)=H^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q':p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha'_j,\cdots,\alpha_j^{(r)})_{1,p}:(c'_j,\gamma')_{1,p_1},\cdots,(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ \cdot & \cdot \\ \cdot \quad ;\mathrm{q} & \cdot \\ \mathrm{z}_r & (\mathrm{b}_j;\beta'_j,\cdots,\beta_j^{(r)})_{1,q'}:(d'_j,\delta')_{1,q_1},\cdots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{array}\right)$$

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\phi(s_1,\cdots,s_r;q)\prod_{i=1}^{r}\theta_i(s_i;q)x_1^{s_1}\cdots x_r^{s_r}d_qs_1\cdots d_qs_r \tag{2.2}$$

where

$$\phi(s_1,\cdots,s_r;q)=\frac{\prod_{j=1}^{n}G(q^{1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i})}{\prod_{j=n+1}^{p}G(q^{a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i})\prod_{j=1}^{q'}G(q^{1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i})} \tag{2.3}$$

$$\theta_i(s_i;q)=\frac{\prod_{j=1}^{m_i}G(q^{d_j^{(i)}-\delta_j^{(i)}s_i})\prod_{j=1}^{n_i}G(q^{1-c_j^{(i)}+\gamma_j^{(i)}s_i})}{\prod_{j=m_i+1}^{q_i}G(q^{1-d_j^{(i)}+\delta_j^{(i)}s_i})\prod_{j=n_i+1}^{p_i}G(q^{c_j^{(i)}-\gamma_j^{(i)}s_i})G(q^{1-s_i})\sin\pi s_i} \tag{2.4}$$

$i=1,\cdots,r$

where the integers n,p,q,m_i,n_i,p_i,q_i are constrained by the inequalities $0\leq n\leq p, 0\leq q', 1\leq m_i\leq q_i$ and $0\leq n_i\leq p_i, i=1,\cdots,r$. The poles of the integrand are assumed to be simple.

The quantities, $a_j, j=1,\cdots,p; c_j^{(i)}, j=1,\cdots,p_i; b_j, j=1,\cdots,q'; d_j^{(i)}, j=1,\cdots,q_i, i=1,\cdots,r$ are complex numbers and the following quantities $\alpha_j^{(i)}, j=1,\cdots,p; \gamma_j^{(i)}, j=1,\cdots,p_i; \beta_j^{(i)}, j=1,\cdots,q'; \delta_j^{(i)}, j=1,\cdots,q_i$, $i=1,\cdots,r$ are positive real numbers.

The contour L_i in the complex s_i-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $\omega\infty$ with indentations, if necessary to ensure that all the poles of $G(q^{d_j^{(i)}+\delta_j^{(i)}s_i}), j=1,\cdots,m_i$ are separated from those of $G(q^{1-c_j^{(i)}+\gamma_j^{(i)}s_i})$, $i=1,\cdots,n_i$, $G(q^{1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i})$, $j=1,\cdots,n, i=1,\cdots,r$. For large values of $|s_i|$ the integrals converge if $Re(slog(z_i)-log\sin\pi s_i)<0, i=1,\cdots,r$.

3. Main results.

Let

$$U=m_1,n_1;\cdots;m_r,n_r\,;V=p_1,q_1;\cdots;p_r,q_r \tag{3.1}$$

$$\mathrm{A}=(\mathrm{a}_j;\alpha_j',\cdots,\alpha_j^{(r)})_{1,p}:B=(c_j',\gamma')_{1,p_1},\cdots,(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \tag{3.2}$$

$$\mathrm{C}=(\mathrm{b}_j;\beta_j',\cdots,\beta_j^{(r)})_{1,q'}:D=(d_j',\delta')_{1,q_1},\cdots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \tag{3.3}$$

In this section, we will establish two fractional q-integral formulae about the basic analogue of multivariable H-function.

Theorem 1.

Let $Re(\mu)>0, |q|<1, \eta\in\mathbb{R}$ and $I_q^{\eta,\alpha}\{.\}$ be the Kober fractional q-integral operator [11], then the following result holds :

$$I_q^{\eta,\mu}\left\{x^{\lambda-1}H_{p,q':V}^{0,n:U}\left(\begin{array}{c}\mathrm{z}_1x^{\rho_1}\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{\rho_r}\end{array};\mathrm{q}\,\middle|\,\begin{array}{c}\mathrm{A:B}\\ \cdot\\ \cdot\\ \mathrm{C:D}\end{array}\right)\right\}=(1-q)^{\mu}x^{\lambda-1}$$

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c}\mathrm{z}_1x^{\rho_1}\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{\rho_r}\end{array};\mathrm{q}\,\middle|\,\begin{array}{c}(1\text{-}\lambda-\eta;\rho_1,\cdots,\rho_r),A:B\\ \cdot\\ \cdot\\ \mathrm{C},\ (1\text{-}\lambda-\eta-\mu;\rho_1,\cdots,\rho_r):D\end{array}\right) \tag{3.4}$$

where $\rho_i\in\mathbb{N}$, $Re(slog(z_i)-log\sin\pi s_i)<0\,,i=1,\cdots,r.$

Proof

To prove the above theorem, we consider the left hand side of equation (3.4) (say I) and make use of the definitions (1.4) and (2.2), we obtain

$$I=\frac{x^{-\eta-\alpha}}{\Gamma_q(\alpha)}\int_0^x(x-yq)_{\alpha-1}y^{\eta}\frac{y^{\lambda-1}}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\phi(s_1,\cdots,s_r;q)\prod_{i=1}^{r}\theta_i(s_i;q)$$

$$z_1^{s_1}\cdots z_r^{s_r}x^{\rho_1s_1+\cdots\rho_rs_r}d_qs_1\cdots d_qs_rd_qy$$

interchanging the order of integrations which is justified under the conditions mentioned above, we obtain

$$I=\frac{x^{-\eta-\alpha}}{\Gamma_q(\alpha)(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\phi(s_1,\cdots,s_r;q)\prod_{i=1}^{r}\theta_i(s_i;q)z_1^{s_1}\cdots z_r^{s_r}$$

$$\int_0^x (x-yq)_{\alpha-1} y^{\eta}\{y^{\rho_1 s_1+\cdots \rho_r s_r+\lambda-1}\}\ d_q y d_q s_1 \cdots d_q s_r$$

The above equation writes

$$I = \frac{x^{-\eta-\alpha}}{(2_1 pi\omega)^r}\int_{L_1}\int_{L_2} \pi^r \phi(s,t;q)\theta_1(s;q)\theta_2(s;q) z_1^{s_1}\cdots z_r^{\rho_r} I_q^{\eta,\mu}\{x^{\rho_1 s_1+\cdots+\rho_r s_r+\lambda-1}\} d_q s d_q t \tag{3.5}$$

Now using the result due to Yadav and Purohit ([14], p. 440, eq. (19))

$$I_q^{\eta,\mu}\{x^{\lambda-1}\} = \frac{\Gamma_q(\lambda+\eta)}{\Gamma_q(\lambda+\eta+\mu)} x^{\lambda-1}, (Re(\lambda+\mu)>0). \tag{3.6}$$

Subsituting (3.5) in the above equation, we obtain

$$I = \frac{x^{-\eta-\alpha}}{(2\pi\omega)^r}\int_{L_1}\int_{L_2} \pi^r \phi(s,t;q)\theta_1(s;q)\theta_2(s;q) z_1^{s_1}\cdots z_r^{\rho_r} \frac{\Gamma_q(\rho_1 s_1+\cdots \rho_r s_r+\lambda+\eta)}{\Gamma_q(\rho_1 s_1+\cdots \rho_r s_r+\lambda+\eta+\mu)} x^{\rho_1 s_1+\cdots \rho_r s_r+\lambda-1} \tag{3.7}$$

Now, interpreting the q-Mellin-Barnes multiple integrals contour in terms of the basic analogue of multivariable H-function, we get the desired result (3.4).

If $Re(\mu)>0, |q|<1, \eta\in\mathbb{R}$, then the generalized Weyl q-integral operator for the basic analogue of multivariable H-function is given by

$$K_q^{\eta,\mu}\left\{x^{\lambda-1} H_{p,q':V}^{0,n:U}\left(\begin{array}{c} z_1 x^{-\rho_1} \\ \cdot \\ \cdot \\ z_r x^{-\rho_r}\end{array} ; q \ \middle|\ \begin{array}{c} \text{A: B} \\ \cdot \\ \cdot \\ \text{C: D}\end{array}\right)\right\} = (1-q)^{\mu} x^{\lambda} q^{-\mu\lambda}$$

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c} z_1 (xq^{-\mu})^{-\rho_1} \\ \cdot \\ \cdot \\ z_2 (xq^{-\mu})^{-\rho_r}\end{array} ; q \ \middle|\ \begin{array}{c} (1+\lambda-\eta;\rho_1,\cdots,\rho_r), A : C \\ \cdot \\ \cdot \\ \text{B},(1+\lambda-\mu-\eta;\rho_1,\cdots,\rho_r) : D\end{array}\right) \tag{3.8}$$

where $\rho_i\in\mathbb{N},\ Re(s log(z_i) - log\sin\pi s_i) < 0\,, i=1,\cdots,r.$

Proof
Using the definition of the generalized Weyl fractional q-integral operator (1.6) in the left hand side of (3.8), writing the function in the form by (2.2), interchanging the order of integrations which is justified under the conditions mentioned above, using the result (3.6) and interpreting the multiple q-Mellin-Barnes integrals contour in terms of the basic analogue of multivariable H-function, we get the desired result (3.8).

4. Special cases.

In this section, we shall see several corollaries.

Corollary 1.

Let $Re(\mu)>0, |q|<1, \eta\in\mathbb{R}$ and $I_q^{\mu}\{.\}$ be the Riemann-Liouville fractional q-integral operator (1.3), then the following result holds :

$$I_q^{\mu}\{x^{\lambda-1}H_{p,q':V}^{0,n:U}\left(\begin{array}{c} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ z_rx^{\rho_r} \end{array} ; q \left| \begin{array}{c} A: B \\ \cdot \\ \cdot \\ C: D \end{array}\right.\right)\Bigg\} = (1-q)^{\mu}x^{\lambda+\mu-1}$$

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ z_rx^{\rho_r} \end{array} ; q \left| \begin{array}{c} (1\text{-}\lambda;\rho_1,\cdots,\rho_r), A : B \\ \cdot \\ \cdot \\ C,\ (1\text{-}\lambda-\mu;\rho_1,\cdots,\rho_r):D \end{array}\right.\right) \tag{4.1}$$

where $\rho_i \in \mathbb{N},\ Re(slog(z_i) - log\sin\pi s_i) < 0\,, i = 1,\cdots,r.$

Corollary 2

$$K_q^{\mu}\{x^{\lambda-1}H_{p,q':V}^{0,n:U}\left(\begin{array}{c} z_1x^{-\rho_1} \\ \cdot \\ \cdot \\ z_rx^{-\rho_r} \end{array} ; q \left| \begin{array}{c} A: B \\ \cdot \\ \cdot \\ C: D \end{array}\right.\right)\Bigg\} = (1-q)^{\mu}x^{\lambda+\mu}q^{-\mu\lambda-\mu(\mu+1)/2}$$

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{array}{c} z_1(xq^{-\mu})^{-\rho_1} \\ \cdot \\ \cdot \\ z_2(xq^{-\mu})^{-\rho_r} \end{array} ; q \left| \begin{array}{c} (1\text{+}\lambda-\mu;\rho_1,\cdots,\rho_r), A : C \\ \cdot \\ \cdot \\ \text{B},(1\text{+}\lambda;\rho_1,\cdots,\rho_r):D \end{array}\right.\right) \tag{4.2}$$

where $\rho_i \in \mathbb{N},\ Re(slog(z_i) - log\sin\pi s_i) < 0\,, i = 1,\cdots,r.$

Remark : We obtain the same relations with basic analogue of H-function of two variables defined by Saxena et al. [10], the basic analogue of Fox's H-function defined by Saxena et al. [9].

5. Conclusion.

The importance of our all the results lies in their manifold generality. By specialising the various parameters as well as variables in the basic analogue of multivariable H-function, we obtain a large number of results involving remarkably wide variety of useful basic functions (or product of such basic functions) which are expressible in terms of basic H-function [13], Basic Meijer's G-function, Basic E-function, basic hypergeometric function of one and two variables and simpler special basic functions of one and two variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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