On Integration of Certain Products Involving General Polynomials, Aleph-Function and the Multivariable Aleph-Function

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ABSTRACT
The integrals evaluated here involve the product of Jacobi polynomials, Aleph-function, general polynomials and the multivariable Aleph-function. The main results of this paper are quite general in nature and capable of yielding a very large number of integrals involving polynomials and various special function occurring in the problems of mathematical analysis and mathematical physics.

Keywords : Jacobi polynomials, Aleph-function, general polynomials, multivariable Aleph-function, integrals.

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1. Introduction.

The generalized polynomials defined by Srivastava ([7],p. 251, Eq. (C.1)), is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \quad (1.1)$$

we shall note

$$a_1 = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \quad (1.2)$$

where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex. I we take $s=1$ in the (1.3) and denote $A[N,K]$ thus obtained by $A_{N,K}$, we arrive at general class of polynomials $S_N^M(x)$ study by Srivastava ([6],p. 1, Eq. 1).

The Aleph- function , introduced by Südland et al. [10,11], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z \;\middle|\; \begin{matrix}(a_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r'}\\(b_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r'}\end{matrix}\right) = \frac{1}{2\pi\omega}\int_L \Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)z^{-s}\mathrm{d}s \quad (1.3)$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r'}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \quad (1.4)$$

With $|argz| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M}B_j + \sum_{j=1}^{N}A_j - c_i\left(\sum_{j=M+1}^{Q_i}B_{ji} + \sum_{j=N+1}^{P_i}A_{ji}\right) > 0, i=1,\cdots,r'$

For convergence conditions and other details of Aleph-function , see Südland et al [10,11]. The serie representation of Aleph-function is given by Chaurasia and Singh [4].

$$\aleph^{M,N}_{P_i,Q_i,c_i;r'}(z)=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)}{B_G g!}z^{-s} \tag{1.5}$$

With $s=\eta_{G,g}=\dfrac{b_G+g}{B_G}, P_i<Q_i, |z|<1$ and $\Omega^{M,N}_{P_i,Q_i,c_i;r'}(s)$ is given in (1.2) (1.6)

We will use the contracted form about the multivariable aleph-fonction $\aleph(z_1,\cdots,z_u)$ by :

$$\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathrm{n}:V}_{p_i,q_i,\tau_i;R:W}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\ ;\ \mathrm{C} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B};\ \mathrm{D}\end{array}\right)=\frac{1}{(2\pi\omega)^u}\int_{L_1}\cdots\int_{L_u}\psi'(s_1,\cdots,s_u)\prod_{k=1}^{u}\theta'_k(s_k)z_k^{s_k}\mathrm{d}s_1\cdots\mathrm{d}s_u \tag{1.7}$$

with $\omega=\sqrt{-1}$

See Ayant [1], concerning the definition of the following quantities $V,W,\psi'(s_1,\cdots,s_r),A,B,C,D$ and $\theta'_k(s_k)$.

The complex numbers z_i are not zero. Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

The following known result will be used throughout this paper ([2], p. 945, p. 946 and [5], p. 172)

$$P_k^{(\beta,\upsilon)}(t+\sigma)P_k^{(\beta,\upsilon)}(t-\sigma)=\frac{(-)^k(1+\upsilon)_k(1+\beta)_k}{(k!)^2}\sum_{n=0}^{k}(-k)_n\frac{(1+\upsilon+\beta+k)_n}{(1+\upsilon)_n(1+\beta)_r}P_n^{(\beta,\beta)}(x)t^n \tag{1.8}$$

$$\sigma^k P_k^{(\beta,\beta)}\left(\frac{1-xt}{\sigma}\right)=\frac{(1+\beta)_k}{k!}\sum_{n=0}^{k}\frac{(-k)_n}{(1+\beta)_n}P_n^{(\beta,\beta)}(x)t^n \tag{1.9}$$

$$\frac{1}{\sigma}(1-t+\sigma)^{-\beta}(1+t+\sigma)^{-\upsilon}=2^{-\upsilon-\beta}\sum_{n=0}^{\infty}P_n^{(\beta,\upsilon)}(x)t^n \tag{1.10}$$

where $\sigma=(1-2xt+t^2)^{\frac{1}{2}}$

2. Required integrals.

Lemma 1.

$$\int_{-1}^{1}(1-x)^{\beta}(1+x)^{\eta}P_t^{(\beta,\upsilon)}(x)\,S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left[(1+x)^{\rho_1},\cdots,(1+x)^{\rho_s}\right]\aleph^{M,N}_{P_i,Q_i,c_i;r'}(z(1+x)^h)$$

$$\aleph\left[z_1(1+x)^{h_1},\cdots,z_r(1+x)^{h_r}\right]\mathrm{d}x=2^{\beta+\eta+\sum_{j=1}^{s}K_j\rho_j+h\eta_{G,g}+1}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}a_1\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(u)}{B_G g!}z^{-u}\quad\Gamma(\beta+t+1)\aleph^{0,\mathrm{n}+2:V}_{p_i+2,q_i+2,\tau_i+2;R:W}$$

$$\left(\begin{array}{c|c} 2^{h_1}z_1 & (\text{-}\eta-\sum_{j=1}^{s}\rho_js_j-h\eta_{G,g}:h_1,\cdots,h_r),(\upsilon-\eta-\sum_{j=1}^{s}\rho_js_j-h\eta_{G,g}:h_1,\cdots,h_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r}z_r & \mathrm{B},(\text{-}\beta-\eta-\sum_{j=1}^{s}\rho_js_j-h\eta_{G,g}-t-1:h_1,\cdots,h_r),(-\upsilon-\eta-\sum_{j=1}^{s}\rho_js_j-h\eta_{G,g}+t:h_1,\cdots,h_r);D\end{array}\right) \tag{2.1}$$

Provided that

$Re(\beta) > -1, Re(\eta) > -1, \rho_j > 0, h_i, h > 0, j = 1, \cdots, s; i = 1, \cdots, r$

$|argz(1+x)^h| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} A_j - c_i \left(\sum_{j=M+1}^{Q_i} B_{ji} + \sum_{j=N+1}^{P_i} A_{ji} \right) > 0, i = 1, \cdots, r'$

$|\arg(z_i(1+x)^{h_i}])< \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by Ayant [2] and

$Re(\eta + h\eta_{G,g}) + \sum_{i=1}^{r} h_i \min_{1 \leqslant j \leqslant m_i} Re \left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > -1$

Proof

We use the series form concerning the polynomial of several variables and Aleph-function of one variable with help of (1.1) and (1.5) respectively and expressing the multivariable Aleph-function in multiple Mellin-Barnes integrals. Interchange the series and Mellin-Barnes integrals due to absolute convergence of the series and integrals involved in the process. Now evaluate the inner x-integral and interpret the expression in multivariable Aleph-function, after algebraic manipulations, we obtain the result (2.1).

Lemma 2.

$$\int_{-1}^{1} (1-x)^{\eta}(1+x)^{\rho} P_t^{(\beta,\upsilon)}(x)\, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1-x)^{\alpha_1}(1+x)^{\beta_1}, \cdots, (1-x)^{\alpha_s}(1+x)^{\beta_s}\right]$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)\, \aleph\left[z_1(1-x)^{h_1}(1+x)^{k_1}, \cdots, z_r(1-x)^{h_r}(1+x)^{k_r}\right] \mathrm{d}x =$$

$$2^{\rho+\eta+\sum_{j=1}^{s} K_j(\alpha_j+\beta_j)+h\eta_{G,g}+1} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{n=0}^{t} a_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!} z^{-u}$$

$$\frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!} z^{-u} \frac{(-t)^n (\beta+\upsilon+t+1)_n}{(\beta+1)_n n!} \aleph_{p_i+2,q_i+1,\tau_i+2;R:W}^{0,\mathfrak{n}:V}$$

$$\left(\begin{array}{c|c} 2^{h_1+k_1} z_1 & (-\rho - \sum_{j=1}^{s} \beta_j s_j - h\eta_{G,g} : k_1, \cdots, k_r), (-n-\eta-\sum_{j=1}^{s} \alpha_j s_j, h_1, \cdots, h_r), A; C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r+k_r} z_r & \text{B,(-1-n-}\eta - \sum_{j=1}^{s}(\alpha_j+\beta)s_j - h\eta_{G,g}; h-1+k_1; \cdots; h_r+k_r); D \end{array} \right) \quad (2.2)$$

Provided that

$Re(\rho) > -1, Re(\eta) > -1, \alpha_j, \beta_j > 0, h_i, k_i, h > 0, j = 1, \cdots, s; i = 1, \cdots, r$

$|argz(1+x)^h| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} A_j - c_i \left(\sum_{j=M+1}^{Q_i} B_{ji} + \sum_{j=N+1}^{P_i} A_{ji} \right) > 0, i = 1, \cdots, r'$

$|\arg(z_i(1-x)^{h_i}(1+x)^{k_i}])< \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by Ayant [2] and

$Re(\rho + h\eta_{G,g}) + \sum_{i=1}^{r} k_i \min_{1 \leqslant j \leqslant m_i} Re \left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > -1, Re(\eta) + \sum_{i=1}^{r} h_i \min_{1 \leqslant j \leqslant m_i} Re \left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > -1$

The proof of (2.2) is similar that (2.1).

3. Main integrals.

Theorem 1.

$$\int_{-1}^{1}(1-x)^{\beta}(1+x)^{\eta}P_k^{(\beta,\upsilon)}(t+\sigma)P_k^{(\beta,\upsilon)}(t-\sigma)\,S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1+x)^{\rho_1},\cdots,(1+x)^{\rho_s}\right]\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)$$

$$\aleph\left[z_1(1+x)^{h_1},\cdots,z_r(1+x)^{h_r}\right]\mathrm{d}x = 2^{\beta+\eta+\sum_{j=1}^{s}K_j\rho_j+h\eta_{G,g}+1}\frac{(-)^k\Gamma(1+\beta+k)\Gamma(1+\upsilon+k)}{(k!)^2}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{k}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!}z^{-u}\quad(-k)_n(1+\beta+\upsilon+k)_n t^n\aleph_{p_i+2,q_i+2,\tau_i+2;R:W}^{0,\mathfrak{n}+2:V}$$

$$\left(\begin{array}{c|c}2^{h_1}z_1 & (\text{-}\eta-\sum_{j=1}^{s}\rho_js_j-h\eta_{G,g}:h_1,\cdots,h_r),(\upsilon-\eta-\sum_{j=1}^{s}\rho_js_j-h\eta_{G,g}:h_1,\cdots,h_r),A;C\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{h_r}z_r & \mathrm{B},(\text{-}\beta-\eta-\sum_{j=1}^{s}\rho_js_j-h\eta_{G,g}-t-1:h_1,\cdots,h_r),(-\upsilon-\eta-\sum_{j=1}^{s}\rho_js_j-h\eta_{G,g}+t:h_1,\cdots,h_r);D\end{array}\right)\tag{3.1}$$

under the same conditions that lemma 1

Proof

We multiply both the sides of (1.8) by

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1+x)^{\rho_1},\cdots,(1+x)^{\rho_s}\right]\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)\aleph\left[z_1(1+x)^{h_1},\cdots,z_r(1+x)^{h_r}\right]$$

and integrating both sides with respect to x between the limits -1 to 1, we obtain

$$\int_{-1}^{1}(1-x)^{\beta}(1+x)^{\eta}P_k^{(\beta,\upsilon)}(t+\sigma)P_k^{(\beta,\upsilon)}(t-\sigma)\,S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1+x)^{\rho_1},\cdots,(1+x)^{\rho_s}\right]\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)$$

$$\aleph\left[z_1(1+x)^{h_1},\cdots,z_r(1+x)^{h_r}\right]\mathrm{d}x =$$

$$\int_{-1}^{1}\frac{(-)^k(1+\upsilon)_k(1+\beta)_k}{(k!)^2}\sum_{n=0}^{k}(-k)_n\frac{(1+\upsilon+\beta+k)_n}{(1+\upsilon)_n(1+\beta)_n}P_n^{(\beta,\beta)}(x)t^n$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1+x)^{\rho_1},\cdots,(1+x)^{\rho_s}\right]\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)$$

$$\aleph\left[z_1(1+x)^{h_1},\cdots,z_r(1+x)^{h_r}\right]\tag{3.3}$$

Now, we interchange the order of integration and summation on the right hand side of (3.3), which is justified due to the absolute convergent of the integral involved in the process, then we evaluate the inner integral with the help of lemma 1.

Theorem 2.

$$\int_{-1}^{1}(1-x)^{\beta}(1+x)^{\eta}P_k^{(\beta,\beta)}\left(\frac{1-xt}{\sigma}\right)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1+x)^{\rho_1},\cdots,(1+x)^{\rho_s}\right]\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)$$

$$\aleph\left[z_1(1+x)^{h_1},\cdots,z_r(1+x)^{h_r}\right]\mathrm{d}x = 2^{\beta+\eta+\sum_{j=1}^{s}K_j\rho_j+h\eta_{G,g}+1}\frac{\Gamma(1+\beta+k)}{k!}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{k}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!}z^{-u}\quad(-k)_n t^n\aleph_{p_i+2,q_i+2,\tau_i+2;R:W}^{0,\mathfrak{n}+2:V}$$

$$\left(\begin{array}{c|c} 2^{h_1}z_1 & (-\eta-\sum_{j=1}^s\rho_js_j-h\eta_{G,g}:h_1,\cdots,h_r),(\upsilon-\eta-\sum_{j=1}^s\rho_js_j-h\eta_{G,g}:h_1,\cdots,h_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r}z_r & \mathrm{B},(-\beta-\eta-\sum_{j=1}^s\rho_js_j-h\eta_{G,g}-t-1:h_1,\cdots,h_r),(-\upsilon-\eta-\sum_{j=1}^s\rho_js_j-h\eta_{G,g}+t:h_1,\cdots,h_r);D \end{array}\right) \quad (3.4)$$

under the same conditions that lemma 1

Proof
the proof is similar that theorem 1 but we use the formula (1.9) and lemma 1.

Theorem 3.

$$\int_{-1}^{1}\frac{1}{\sigma}(1-t+\sigma)^{-\beta}(1+t+\sigma)^{-\upsilon}\,(1-x)^{\beta}(1+x)^{\eta}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1+x)^{\rho_1},\cdots,(1+x)^{\rho_s}\right]\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)$$

$$\aleph\left[z_1(1+x)^{h_1},\cdots,z_r(1+x)^{h_r}\right]\mathrm{d}x=2^{\eta-\beta+1+\sum_{j=1}^s K_j\rho_j+h\eta_{G,g}}\sum_{n=0}^{\infty}\Gamma(1+\beta+n)t^n$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{k}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!}z^{-u}\,\aleph_{p_i+2,q_i+2,\tau_i+2;R:W}^{0,\mathfrak{n}+2:V}$$

$$\left(\begin{array}{c|c} 2^{h_1}z_1 & (-\eta-\sum_{j=1}^s\rho_js_j-h\eta_{G,g}:h_1,\cdots,h_r),(\upsilon-\eta-\sum_{j=1}^s\rho_js_j-h\eta_{G,g}:h_1,\cdots,h_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r}z_r & \mathrm{B},(-\beta-\eta-\sum_{j=1}^s\rho_js_j-h\eta_{G,g}-n-1:h_1,\cdots,h_r),(-\upsilon-\eta-\sum_{j=1}^s\rho_js_j-h\eta_{G,g}+n:h_1,\cdots,h_r);D \end{array}\right) \quad (3.5)$$

under the same conditions that lemma 1

To prove the theorem 3, we use the formula (1.10) and lemma 1.

Theorem 4.

$$\int_{-1}^{1}(1-x)^{\eta}(1+x)^{\rho}P_k^{(\beta,\upsilon)}(t+\sigma)P_k^{(\beta,\upsilon)}(t-\sigma)\,S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1-x)^{\alpha_1}(1+x)^{\beta_1},\cdots,(1-x)^{\alpha_s}(1+x)^{\beta_s}\right]$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)\,\aleph\left[z_1(1-x)^{h_1}(1+x)^{k_1},\cdots,z_r(1-x)^{h_r}(1+x)^{k_r}\right]\mathrm{d}x=2^{\rho+\eta+\sum_{j=1}^s K_j(\alpha_j+\beta_j)+h\eta_{G,g}+1}$$

$$\frac{(-)^k\Gamma(1+\beta+k)\Gamma(1+\upsilon+k)}{(k!)^2}\sum_{n=0}^{k}\sum_{Q=0}^{n}(-k)_n\frac{(1+\upsilon+\beta+k)_n}{\Gamma(1+\upsilon+n)(\Gamma(1+\beta+n)}\frac{(-n)_Q(1+\beta+\upsilon+n)_Q}{(1+\beta)_Q Q!}t^n$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!}z^{-u}\quad\aleph_{p_i+2,q_i+1,\tau_i+2;R:W}^{0,\mathfrak{n}:V}$$

$$\left(\begin{array}{c|c} 2^{h_1+k_1}z_1 & (-\rho-\sum_{j=1}^s\beta_js_j-h\eta_{G,g}:k_1,\cdots,k_r),(-\eta-\sum_{j=1}^s\alpha_js_j-h\eta_{G,g}-Q,h_1,\cdots,h_r),A;C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 2^{h_r+k_r}z_r & \mathrm{B},(-\eta-\rho-\sum_{j=1}^s(\alpha_j+\beta)s_j-h\eta_{G,g}-Q-1;h_1+k_1;\cdots;h_r+k_r);D \end{array}\right) \quad (3.6)$$

under the same conditions that lemma 2

Proof
To prove (3.6) we use the formula (1.8) and the lemma 2.

Theorem 5.

$$\int_{-1}^{1}(1-x)^{\beta}(1+x)^{\eta}P_k^{(\beta,\beta)}\left(\frac{1-xt}{\sigma}\right)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1-x)^{\alpha_1}(1+x)^{\beta_1},\cdots,(1-x)^{\alpha_s}(1+x)^{\beta_s}\right]$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)\,\aleph\left[z_1(1-x)^{h_1}(1+x)^{k_1},\cdots,z_r(1-x)^{h_r}(1+x)^{k_r}\right]\mathrm{d}x=2^{\beta+\rho+\sum_{j=1}^{s}K_j(\alpha_j+\beta_j)+h\eta_{G,g}+1}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!}z^{-u}\frac{\Gamma(1+\beta+k)}{k!}$$

$$\sum_{n=0}^{k}(-k)_n\frac{1}{\Gamma(1+\beta+n)}t^n\sum_{Q=0}^{n}\frac{(2\beta+n+1)_Q}{(\beta+1)_Q Q!}\aleph_{p_i+2,q_i+1,\tau_i+2;R:W}^{0,\mathfrak{n}:V}$$

$$\left(\begin{array}{c|c} 2^{h_1+k_1}z_1 & (\text{-}\rho-\sum_{j=1}^{s}\beta_j s_j-h\eta_{G,g}:k_1,\cdots,k_r),(-\eta-\sum_{j=1}^{s}\alpha_j s_j-h\eta_{G,g}-Q,h_1,\cdots,h_r),A;C\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{h_r+k_r}z_r & \text{B},(\text{-}\eta-\rho-\sum_{j=1}^{s}(\alpha_j+\beta)s_j-h\eta_{G,g}-Q-1;h_1+k_1;\cdots;h_r+k_r);D \end{array}\right) \quad (3.7)$$

under the same conditions that lemma 2

Proof
To prove (3.7) we use the formula (1.9) and the lemma 2.

Theorem 6.

$$\int_{-1}^{1}\frac{1}{\sigma}(1-t+\sigma)^{-\beta}(1+t+\sigma)^{-\upsilon}(1-x)^{\beta}(1-x)^{\eta}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[(1-x)^{\alpha_1}(1+x)^{\beta_1},\cdots,(1-x)^{\alpha_s}(1+x)^{\beta_s}\right]$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z(1+x)^h)\,\aleph\left[z_1(1-x)^{h_1}(1+x)^{k_1},\cdots,z_r(1-x)^{h_r}(1+x)^{k_r}\right]\mathrm{d}x=$$

$$2^{-\beta-\upsilon-\eta+\rho+\sum_{j=1}^{s}K_j(\alpha_j+\beta_j)+h\eta_{G,g}+1}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{Q=0}^{n}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!}z^{-u}$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(u)}{B_G g!}z^{-u}\frac{(-n)_Q(\beta+\upsilon+n+1)_Q}{(\beta+1)_Q Q!}(n)_Q\aleph_{p_i+2,q_i+1,\tau_i+2;R:W}^{0,\mathfrak{n}:V}$$

$$\left(\begin{array}{c|c} 2^{h_1+k_1}z_1 & (\text{-}\rho-\sum_{j=1}^{s}\beta_j s_j-h\eta_{G,g}:k_1,\cdots,k_r),(-\eta-\sum_{j=1}^{s}\alpha_j s_j-h\eta_{G,g}-Q,h_1,\cdots,h_r),A;C\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ 2^{h_r+k_r}z_r & \text{B},(\text{-}\eta-\rho-\sum_{j=1}^{s}(\alpha_j+\beta)s_j-h\eta_{G,g}-Q-1;h_1+k_1;\cdots;h_r+k_r);D \end{array}\right) \quad (3.8)$$

under the same conditions that lemma 2

Proof

To prove (3.8) we use the formula (1.10) and the lemma 2.

Remark :

If the Aleph-function of one variable and the multivariable Aleph-function reduce in the H-function of one variable and the multivariable H-function defined by Srivastava and Panda [8,9] respectively, we obtain the results due to Chaurasia and Patni [3].

4.Conclusion.

By specializing the various parameters as well as variables in the multivariable Aleph-function, the aleph-function and the multivariable polynomial, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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